CHAPTER V.

CHANGE OF TEMPERATURE.

71. **Action of Change of Temperature.** — If the arch, when either fixed or hinged at the ends, is exposed to a change of temperature, it will tend to change its shape. If the rib were perfectly free, its expansion or contraction would be uniform in all directions, so that the new arch would be the old arch on a slightly altered scale. In a bowstring girder, the tie expands and contracts with the bow, so that the horizontal projection of the change of length of the bow is the same as the elongation or contraction of the horizontal member. But as the abutments of the arch are considered as fixed, its span must remain unchanged; and the alteration of the arch by a change of temperature will be manifested by a rise or fall of the crown of the arch, which movement, in the case of a metal rib, may be a marked quantity.

It is manifest, that, if we imagine the rib at its normal temperature to be placed upon its springing points or skewbacks, it will have a horizontal thrust against the abutments due to its form and weight. If the temperature changes, the structure endeavors to expand or contract in equal proportion in all directions; and hence, if possible, the span would be lengthened just in proportion to the rise of temperature t, the coefficient of expansion e, and the span $2c$, or the change of span would equal $2tec$. If t expresses the number of degrees of fall in temperature, it may be called *minus*, and the quantity $2tec$

will denote the shortening of the span. But this attempted change of length, being resisted at the points of attachment, cannot take place, but must cause a horizontal force, either tension or compression, which keeps the span invariable. This $+H$ or $-H$ must exert a bending moment upon all parts of the rib, as well as a direct thrust, which moment is too important to be neglected. It being recollected that the condition $\Sigma EF . DE = 0$ denoted that the change of span equalled zero, it will be sufficient in this case to still make it zero, when we have added or subtracted a quantity proportional to $2tec$.

72. **Change of Span influenced by Material and Cross-section of Arch.** — The bending moment M at any point has been demonstrated, § 4, to be equal to the product of H from the stress diagram multiplied by the vertical ordinate from that point to the equilibrium polygon. Then it was shown, § 18, that, if all these ordinates were summed up, that is, if we took ΣEF between two points, this sum would be *proportional* to the change of inclination between those two points; but it was not stated that this quantity was *equal* to the change of inclination, for neither the material nor the form of cross-section of the rib was taken into account. As the amount of flexure was stated, in Part II., "Bridges," §§ 85 and 86, to vary inversely as the modulus of elasticity and the moment of inertia, we must write $\frac{\Sigma M}{\mathbf{EI}}$ or $\frac{H . \Sigma EF}{\mathbf{EI}}$ to obtain a quantity which shall *equal* the change of inclination. The same thing is true of the expressions for deflection and change of span. When, however, the summation is made from one abutment to the other, and then put equal to zero, if $\mathbf{E}$ and $\mathbf{I}$ are constant, as well as H, it must be true that $\Sigma EF = 0$, as heretofore stated; and likewise of the other equations. Now $\mathbf{E}$ is constant, as the material of the rib is the same throughout; and since the parabolic rib, of cross-section varying with the secant of the inclination of the rib to the horizon, has been demonstrated, § 36, to deflect vertically like a straight beam of uniform section equal to that of the rib at the crown, $\mathbf{I}$ is likewise constant in these formulæ,

and represents the moment of inertia of the section at the crown. In short, where one quantity is directly proportional to another, if one is equal to zero, the other is also; consequently we can deal with areas, area moments, &c., as if they were the changes of inclination, deflections, &c., themselves.

73. **Formula for H from Change of Temperature.** — But now we wish to introduce the distance $2\,t\,e\,c$, the change of span which would occur from change of temperature, were it unchecked. As this is an absolute and not a proportional quantity, we must divide our original quantity for change of span, § 7, by **E I**. We shall, therefore, have for the new condition,

$$\frac{H_t \cdot \Sigma\,E\,F \,.\, D\,E}{\mathbf{E\,I}} \pm 2\,t\,e\,c = 0,$$

where H_t is used to signify the horizontal force (thrust or tension) which is occasioned by the change of temperature; or, if we clear of fractions, we get the more convenient expression

$$H_t \cdot \Sigma\,E\,F \,.\, D\,E \pm 2\,\mathbf{E\,I}\,t\,e\,c = 0.$$

A rise of temperature will make H a thrust or positive, while a fall of temperature will make H a tension or negative. The double sign is not needed in the above equation if the sign is contained in the symbol t, that is, if t is negative for a diminution of temperature below the one at which the rib is constructed or laid out. The bending moments exerted on the rib will be of the contrary kind when H_t is minus, while the ordinates are unchanged.

74. **Application to Parabolic Rib, Hinged at Ends.** — To take up first the case of the parabolic rib hinged at ends. The amount of H_t is to be determined. As there can be no bending moment at either abutment, and H_t at each abutment is the only applied force, the equilibrium polygon or line of thrust, Fig. 16, must be in the line joining the two springings. The bending moment at any point will, therefore, be equal to the ordinate to the rib at that point, multiplied by the desired value of H_t. The expression $\Sigma\,E\,F \,.\, D\,E$ therefore becomes for

this case $\Sigma\,D\,E^2$; and we have, transposing the second term of the equation of the previous section,

$$H_t \cdot \Sigma\,D\,E^2 = 2\,\mathbf{E\,I}\,t\,e\,c.$$

The value of $\Sigma\,D\,E^2$ was shown in § 39 (2.), to be $\frac{16}{15}\,k^2\,c$; therefore, substituting and transposing, we see that

$$H_t = \tfrac{15}{8} \cdot \frac{t\,e\,\mathbf{E\,I}}{k^2},$$

a value which is independent of the span.

The maximum bending moment, which occurs at the middle of the span, where the ordinate will be k, is

$$M\ (\text{max.}) = \tfrac{15}{8} \cdot \frac{t\,e\,\mathbf{E\,I}}{k}.$$

The ordinates at all the usual points of division will be the values of z, used repeatedly before; and, by multiplying H_t by these several values of z, the bending moments at all points are obtained for a given change of temperature t. An additional line can be placed below the table of M to contain these quantities, so as to have them convenient for use. All of these moments will be positive for a fall of temperature below, and negative for a rise above, that at which the rib was designed. The worst effect of either change must be provided for.

75. **Formula for Change of Span deduced analytically.** — If one likes to prove this value for change of span analytically, he may proceed as follows: Let any ordinate to the arch be denoted by y, and the abscissa measured horizontally from one abutment by x. Then, if $v =$ the vertical deflection ordinate, that is, the deflection of any point from its original position, we may write the usual equations for curvature, slope, and deflection of beams, recollecting that this arch acts like a beam of uniform section in deflecting vertically,

$$\frac{d^2 v}{d x^2} = \frac{M}{\mathbf{E\,I}};\ \frac{d v}{d x} = \int \frac{M}{\mathbf{E\,I}}\,d x;\ \text{and } v = \int\!\!\int \frac{M}{\mathbf{E\,I}}\,d x^2.$$

Now $M = H\,y = H\,\frac{k}{c^2}\,(2\,c\,x - x^2)$; therefore

$$\frac{d v}{d x} = \frac{H}{\mathbf{E\,I}} \cdot \frac{k}{c^2} \int (2\,c\,x - x^2)\,d x = \frac{H}{\mathbf{E\,I}} \cdot \frac{k}{c^2}\left(c\,x^2 - \frac{x^3}{3} + C\right).$$

$\frac{dv}{dx} = 0$, for $x = c$; therefore $C = -\frac{2}{3} c^3$. Then

$$\frac{dv}{dx} = \frac{H}{\mathbf{EI}} \cdot \frac{k}{c^2} (c x^2 - \tfrac{1}{3} x^3 - \tfrac{2}{3} c^3). \quad (a.)$$

If u = horizontal displacement of any point, the infinitesimal horizontal displacement $d u$, due to the movement of the portion of arc $d s$, will give, as may be seen to the right of Fig. 16,

$$d u : d v = d y : d x.$$

Since $y = \frac{k}{c^2}(2 c x - x^2)$, $d y = \frac{k}{c^2}(2 c - 2 x)\, d x$, and we have

$$d u = \frac{2 k}{c^2}(c - x)\, d v.$$

Substitute the value of $d v$ from $(a.)$, and it becomes

$$d u = \frac{H}{\mathbf{EI}} \cdot \frac{2 k^2}{c^4} (c^2 x^2 - \tfrac{4}{3} c x^3 - \tfrac{2}{3} c^4 + \tfrac{1}{3} x^4 + \tfrac{2}{3} c^3 x)\, d x.$$

If this equation is integrated between the limits 0 and $2 c$, we obtain $u = -\frac{H}{\mathbf{EI}} \cdot \frac{16}{15} k^2 c$, which will be seen to correspond with the value of $2 t e c$ in the preceding section.

76. **Application to Fixed Parabolic Rib.** — If we turn next to the rib with fixed ends, it will be manifest, that, since there will be bending moments at the springings, the line which corresponds to the equilibrium polygon and limits the ordinates for bending moments cannot now pass through those points. As the resistance to expansion or contraction is the only cause of those moments, the two abutment moments will be equal, and the line will be horizontal. In order also to satisfy the condition that the change of inclination at the abutments shall equal zero, or, as expressed in § 18, Σ E F = 0, the horizontal line must be so drawn as to make the areas within and without the arch equal to one another, which will occur when the line is drawn at a height of $\frac{2}{3} k$ above the springing, as seen in Fig. 17. To prove the equality of areas it is only necessary to recall the fact that the area of a parabolic segment equals two-thirds of the enclosing rectangle. The area included within the whole arch will therefore be $\frac{2}{3} k \,.\, 2 c = \frac{4}{3} k c$. The rectangle of height $\frac{2}{3} k$ has the same area. Therefore the portions of the arch area and of the rectangle which do not coincide must be equal to one another. The third condition, of § 19, that Σ E F . D B = 0, or the equality of area moments, is also satisfied by this construction; for the rectangle multiplied by the half span, which is the distance of its centre of gravity from one abutment, is equal to the area included by the whole arch multiplied by the same distance.

To deduce in this case the value of H_t: as before,

$$H_t \,.\, \Sigma\, \mathrm{EF} \,.\, \mathrm{DE} \pm 2 \mathbf{EI}\, t e c = 0. \quad (1.)$$

From what has just been stated,

$$\Sigma\, \mathrm{EF} \,.\, \mathrm{DE} = \Sigma (\mathrm{DE} - \tfrac{2}{3} k)\, \mathrm{DE} = \Sigma\, \mathrm{DE}^2 - \tfrac{2}{3} k \,.\, \Sigma\, \mathrm{DE}. \quad (2.)$$

The first term, as before, amounts to $\frac{16}{15} k^2 c$; since Σ D E = area enclosed by the arch, $= \frac{4}{3} k c$, the second term is $\frac{8}{9} k^2 c$; therefore

$$H_t \,.\, \tfrac{8}{45} k^2 c = 2 \mathbf{EI}\, t e c, \quad \text{or} \quad H_t = \tfrac{45}{4} \frac{t e \mathbf{EI}}{k^2}.$$

The bending moment at the crown will therefore be

$$M = H_t \,.\, \tfrac{1}{3} k = \tfrac{15}{4} \,.\, \frac{t e \mathbf{EI}}{k},$$

and at the springing,

$$M = H_t \,.\, \tfrac{2}{3} k = \tfrac{15}{2} \,.\, \frac{t e \mathbf{EI}}{k},$$

or double the former amount, but of the opposite kind. Whether the bending moment at either point is positive or negative, depends upon whether H_t is tension or compression. These moments also can be conveniently added to the proper table for M, as explained for the first case.

77. **Comparison of Arches under Change of Temperature.** — The bending moments for temperature, in both the arch with hinged ends and that with fixed ends, will vary like those of a beam uniformly loaded, and either simply supported or fixed at the ends. Part II., "Bridges," §§ 95, 99.

It may be well to notice the comparative straining effect of the same change of temperature in the two classes of parabolic arches, for ribs of the same rise. H_t is six times as great when the arch is fixed as when it is hinged at the ends, and the direct stress in the ribs will therefore vary in the same proportion. The maximum moment, at the springing, for the rib with fixed ends, is four times as great as at the crown of the rib with hinged ends, and of the opposite kind; while the value of M at the two crowns is as two to one against the rib with fixed ends.

78. **Shear from Change of Temperature.** — The shear on a right section can be shown by the accompanying Fig. 18. If $a\,b$ represents the amount of H caused by a change of temperature, we may draw $a\,d$ and $b\,c$ parallel to the upper and lower flange at any right section S of the rib, when $e\,a$ will be the value of the direct stress at the section, one-half in each flange, and $b\,e$ will be the shear.* The bending moment will have any magnitude, depending upon the length of the ordinate from the equilibrium line to the point on the centre line of the arch where this section is taken. As $a\,e$ and $g\,b$ are parallel, the perpendicular distance $b\,e, = c\,d$, between them is constant, so that $f\,d$ may be taken, for our purpose, to represent the stress in one chord, and $g\,c$ that in the other due to bending moment, the resultant stresses being $a\,d$ and $c\,b$, while the shear on the right of a right section of the web will be $d\,c$. Since the resultant stress at any section must be H, the directions of the forces, shown by the arrows, in this closed polygon, are at once fixed. As the inclination of the arch changes, the value of $c\,d$ will change, being zero at the crown and a maximum, at the springings. The arrows denote the case where H is a thrust. The bending moment will be negative, if the rib is hinged at the ends, the bottom chord will be compressed, the top chord will have a force exerted upon it amounting to the difference between the direct thrust and the tension due to the moment, and consequently $c\,b$ will be the stress exerted by the top chord against the right side of the cross-section in the accompanying sketch.

79. **Diagram for Vertical Shear.** — Let us suppose a fall of temperature to take place; the rib will have a tendency to come down at the crown. We recall the fact that a uniform load has a parabola for its equilibrium curve, and a load of the proper intensity on any parabolic arch will produce the value of H which is now supposed to exist. It is evident, then, as is also shown by the sign of M, that the rib may be imagined to be loaded uniformly horizontally with a weight sufficient to produce this deflection or these values of M. This imaginary weight will be just sufficient at all points to balance the component of an opposite kind which is required in combination with the value of H_t (in this case a horizontal tension), in order to give a resultant stress in the direction of the tangent to the rib. And, further, if this weight were not just sufficient to balance the above component, a remainder, of one sign or the other, would be found at the abutments, as a vertical component of the reaction there; but we know that no such vertical component exists. If a bent spring is placed with its two ends on a horizontal line, and compression or tension is applied in that line, no vertical force is needed for equilibrium. As the uniform weight was entirely imaginary, the vertical components must be supplied by the web and flanges, and hence we conclude that the diagram for *vertical* shear in the arch affected by a change of temperature, will be that of an ordinary truss, supported or fixed at its two ends, and carrying a complete uniform load, and that the normal component will be carried by the web. For a fall of temperature, therefore, the shear on a vertical section will be of the same kind as, and, for a rise of temperature, will be of the opposite kind to, that produced by a load on a truss with horizontal chords.

*In Fig 18, the point f should bisect $e\,a$.