CHAPTER VI.

CIRCULAR RIB WITH HINGED ENDS.

80. **Circular Rib to be of Uniform Section.** — Passing next to the consideration of the arch whose curve is the arc of a circle, we shall assume that the rib is of uniform section, and not, as before, of increasing breadth from the crown to the springing. As the rib is of uniform section, it can no longer be compared to a horizontal beam, as regards its change of inclination and deflection under bending moments, and the length along the arch, instead of its projection on a horizontal line, must be used in spacing off and in summing up the usual quantities; that is, the sum of the changes of inclination between any two points will be made up from the change of inclination at each successive point *along the rib*. We must therefore use ds for dx in our integration, where s denotes the length of an arc; and polar co-ordinates will, in the more complex cases, be used in place of rectangular ones. In spacing off the rib for equal divisions, or for summing the ordinates arithmetically, the measurements will be made along the curve, and each division will subtend the same angle at the centre of the circle.

We stated, it will be remembered, that a segmental arch of the circular type, if the rise did not exceed one-tenth of the span, might, without serious error, be treated as if it were parabolic. In discussing circular arches, there will be so many points similar to those we have already explained, that we shall

not go into much detail on some points, but leave the reader to make the extended application as examples come up in his own practice.

81. **Experimental Verification.** — The values to be obtained for y_0, for a rib of uniform section, curved to the arc of a circle, and hinged or free to turn at the ends, can be readily verified or illustrated experimentally as follows: — Take a piece of moderately stiff iron wire, and bend it accurately into the desired shape, A C B, Fig. 19; suspend the wire from a horizontal bar E F by means of strings fastened at A and B, and then attach a weight at any point C. It will be convenient to stretch a thread from A to B, which, as the span is to be unchanged, will not interfere with the reactions. If the point E is now moved horizontally, the length of the string E A being at the same time changed, the line A B can be brought parallel with E F, as can be readily ascertained with a scale. Then E A and F B prolonged will meet at D on C D, and D G will equal y_0. E A and F B will actually intersect on the vertical through the centre of gravity of the wire and weight combined; but if the weight of the wire is as small as is consistent with stiffness, while the weight at C is large in comparison, the centre of gravity will practically be in C D. If A B becomes slack, it shows that E and F are not sufficiently far apart. By fastening two long threads independently to E and F, the lines E A and F B can be easily prolonged to an intersection.

82. **Semicircular Arch with Hinged Ends; Value of y_0.** — If the rib with hinged ends is first taken up for discussion, the value of y_0 for a load at any point on a semicircular arch is easily obtained by a simple device. Recurring again to the usual formula in its modified form, we must satisfy the condition

$$\Sigma\, DE^2 = \Sigma\, DE \,.\, DF.$$

If we let D E, Fig. 20, $= z$; D F $= y$; A D $= x$; and represent a small portion of arc by ds, this equation becomes, for the entire semicircle,

$$\int_0^\pi z^2\, ds = \int_0^\pi yz\, ds.$$

If we draw a radius from any point E of the rib to the centre O, and also draw the infinitesimal triangle whose sides are ds, dx, and dz, we shall have, from similarity of triangles,

$$r : z = ds : dx, \text{ or } z\,ds = r\,dx;$$

substituting this value in the above equation, we get

$$r\int_0^{2c} z\,dx = r\int_0^{2c} y\,dx.$$

The integral of $z\,dx$ between the given limits is the area of the semicircle, while that of $y\,dx$ is the triangle A C B. Substitute the value $\frac{1}{2}\pi r^2$ for the former, and $r y_0$ for the latter, and we obtain

$$\tfrac{1}{2}\pi r^3 = r^2 y_0; \text{ or } y_0 = \tfrac{1}{2}\pi r = 1.5708\,r.$$

The ordinate y_0, for a load at any point, on a semicircular rib with hinged ends, is therefore a constant quantity, equal to the length of the half rib. If we draw a horizontal line at this height above the springing, it will contain the vertices of all the equilibrium polygons for single loads.

83. **Segmental Arch; Value of y_0.** — If the arch is segmental, that is, less than a semicircle, we shall use the following notation: Let the angle N O B, Fig. 21, subtended at the centre of the circle by the half arch, be denoted by β; the angle N O I, from the crown to the point where the weight is placed, be denoted by α; and the angle N O E, from the crown to any point where the ordinates D E and E F are measured, be θ. The radius of the arch $= r$. If, then, A C B is the desired curve of equilibrium, C K $= y_0$. The value of this ordinate will be proved to be

$$y_0 = r\,\frac{(\sin^2\beta - \sin^2\alpha)\left(\beta\,\dfrac{1 + 2\cos^2\beta}{\sin\beta} - 3\cos\beta\right)}{(\sin^2\beta - \sin^2\alpha) + 2\cos\beta\,(\alpha\sin\alpha + \cos\alpha - \beta\sin\beta - \cos\beta)}.$$

If the arch is a semicircle, $\beta = 90° = \frac{1}{2}\pi$, and this value reduces to $y_0 = \frac{1}{2}\pi r$, as previously obtained.

The work of computing y_0 for different values of α is not great; as, for a given arch, β is constant, and the second factor

of the numerator is a constant quantity. Since a segmental arch may subtend any angle, it is not worth while to go into the computation here of values of y_0 for a given value of β; but, as examples of y_0, we will give

If $\beta = 45°$ and $\alpha = 0°$, then $y_0 = .39\,r$ nearly.
" 45° " 30°, " .42 r "
" 60° " 30°, " .71 r "

All that one needs for the calculation from this formula is an ordinary table of natural sines and cosines. The angles or arcs β and α are to be expressed in lengths of arc, which subtend the given number of degrees, to radius unity. The arc for one degree being $\frac{\pi}{180}$, or 0.017453, any other arc will be obtained by multiplying this quantity by the number of degrees which the arc subtends, minutes being expressed as a decimal part of a degree.

84. **Proof.** — From Fig. 21 we have D E $= r(\cos\theta - \cos\beta)$.

$$\text{D F} : \text{C K} = \text{A D} : \text{A K} = r(\sin\beta + \sin\theta) : r(\sin\beta + \sin\alpha)$$

on the left of K, or $\text{D F} = \dfrac{\sin\beta + \sin\theta}{\sin\beta + \sin\alpha}\,y_0$;

on the right of K, $\text{D F} = \dfrac{\sin\beta - \sin\theta}{\sin\beta - \sin\alpha}\,y_0$.

Substituting these values in the usual equation, § 39, $\Sigma\,\text{D E}^2 = \Sigma\,\text{D E} \cdot \text{D F}$, we obtain for the first member of the equation, remembering to use $ds = r\,d\theta$ in place of dx, and considering angles to the left of O N as negative,

$$r^3\int_{-\beta}^{+\beta}(\cos\theta - \cos\beta)^2\,d\theta = r^3\int_{-\beta}^{+\beta}(\cos^2\theta - 2\cos\beta\cos\theta + \cos^2\beta)\,d\theta^*$$
$$= r^3(\beta + 2\beta\cos^2\beta - 3\sin\beta\cos\beta). \quad (a.)$$

For the integral of the second member between α and $-\beta$ we have

$$\frac{r^2 y_0}{\sin\beta + \sin\alpha}\int_{-\beta}^{\alpha}(\sin\beta\cos\theta + \sin\theta\cos\theta - \sin\beta\cos\beta - \cos\beta\sin\theta)\,d\theta\,\dagger$$

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* $\int \cos^2\theta\,d\theta = \frac{1}{2}(\theta + \sin\theta\cos\theta)$; $\cos -\beta = \cos\beta$; $\sin -\beta = -\sin\beta$.

† $\int \sin\theta\cos\theta\,d\theta = -\frac{1}{2}\cos^2\theta$.

$$= \frac{r^2 y_0}{\sin\beta + \sin a}\left(\sin a \sin\beta - \tfrac{1}{2}\cos^2 a - a \sin\beta\cos\beta\right.$$
$$\left. + \cos a \cos\beta + \sin^2\beta - \tfrac{1}{2}\cos^2\beta - \beta\sin\beta\cos\beta\right).$$

Likewise for the integral of the second member between a and $+\beta$ we have

$$\frac{r^2 y_0}{\sin\beta - \sin a}\int_a^\beta (\sin\beta\cos\theta - \sin\theta\cos\theta - \sin\beta\cos\beta + \cos\beta\sin\theta)\, d\theta$$
$$= \frac{r^2 y_0}{\sin\beta - \sin a}\left(\sin^2\beta - \tfrac{1}{2}\cos^2\beta - \beta\sin\beta\cos\beta - \sin a \sin\beta\right.$$
$$\left. - \tfrac{1}{2}\cos^2 a + a\sin\beta\cos\beta + \cos a\cos\beta\right).$$

These two quantities are to be reduced to a common denominator, added together and equated with the first member (*a.*). Upon making simple cancellations, dividing through by $\sin\beta$, and factoring, we get the form of y_0 given in the last section.

85. **Formula for H; Value of Ordinates.** — When the value, of y_0 is computed, we can readily draw the stress diagram of Fig. 21, and scale the value of H; or the formula proved before, § 40, may be applied here, and is easily converted into the third form,

$$\mathrm{H} = \frac{\mathrm{W}}{y_0}\cdot\frac{c^2 - b^2}{2c} = \mathrm{W}\frac{\mathrm{AK}\,.\,\mathrm{KB}}{\mathrm{CK}\,.\,\mathrm{AB}} = \frac{r(\sin^2\beta - \sin^2 a)}{y_0\,.\,2\sin\beta}\,\mathrm{W}. \quad (1.)$$

If calculations have already been made for y_0, the quantities desired for this formula are at hand.

Then the ordinate at each point of division, by which H is to be multiplied to give M for that point, will be, from § 84, if θ is the angle between the two radii from the crown and the point E,

$$\mathrm{EF} = \mathrm{DF} - \mathrm{DE} = y_0\frac{\sin\beta \pm \sin\theta}{\sin\beta \pm \sin a} - r(\cos\theta - \cos\beta). \quad (2.)$$

The plus sign is to be used for points between the weight and the farther abutment, and the minus sign between the weight and the nearer abutment. We must remember, however, that, if θ is measured from the crown to the right as the positive direction, all angles θ on the left of the crown will be negative, and their *sines* will be minus. If E F is plus, it gives a positive bending moment, tending to make the arch less convex, and *vice versa.*

86. **Numerical Computation of M.** — In any practical case *we should much prefer*, as more easy and sufficiently accurate, *to scale all of these quantities from a good-sized diagram;* but it may be well to compute one set

of values of M as an example, for fear the signs may give some readers trouble. Taking the case of Fig. 22, let $\beta = 45°$ and $a = 20°$. Then the arc $\beta = .7854$ and $a = .3491$; $\sin\beta = \cos\beta = .7071$; $\sin a = .3420$, $\cos a = .9397$. These values, substituted in the equation of § 83, give

$$y_0 = r\frac{(.5 - .1170)\left(.7854\frac{2}{.7071} - 2.1213\right)}{.5 - .1170 + 1.4142\,(.1194 + .9397 - .5554 - .7071)} = \frac{.0384}{.0954}r = .403\,r;$$

(1.), § 85, will then become

$$\mathrm{H} = \frac{(.5 - .1170)\,r}{1.4142 \times .403\,r}\mathrm{W} = \frac{.383}{.570}\mathrm{W} = .672\,\mathrm{W}.$$

$$\mathrm{Sin}\,\beta + \sin a = 1.0491;\quad \sin\beta - \sin a = .3651;$$

$$\frac{y_0}{\sin\beta + \sin a} = \frac{.403\,r}{1.0491} = .384\,r;\quad \frac{y_0}{\sin\beta - \sin a} = \frac{.403\,r}{.3651} = 1.104\,r.$$

VALUES OF M.

							W.			
θ	−40°	−30°	−20°	−10°	0°	10°	20°	30°	40°	
$\sin\beta$ =	.7071								.7071	$\sin\beta$
$+\sin\theta$	−.6428	−.5	−.3420	−.1736	0	+.1736	+.3420	.5	.6428	$-\sin\theta$
Mult. by	.0643	.2071	.3651	.5335	.7071	.8807	1.0491	.2071	.0643	Mult. by
.384 =	.0247	.0795	.1402	.2049	.2715	.3382	.4029	.2286	.0710	1.104 =
$-\cos\theta$	.7660	.8660	.9397	.9848	1.0	.9848	.9397	.8660	.7660	$-\cos\theta$
	−.7413	−.7865	−.7995	−.7799	−.7285	−.6466	−.5368	−.6374	−.6950	
$+\cos\beta$	.7071								.7071	$+\cos\beta$
	−.0342	−.0794	−.0924	−.0728	−.0214	+.0605	+.1703	+.0697	+.0121	
× .672 W	−.0230	−.0534	−.0621	−.0489	−.0144	+.0407	+.1144	+.0468	+.0081	r W = M

87. **Shear at any Right Section.** — Suppose that the rib of Fig. 22 carries a single weight under the point C, and that the curve of equilibrium is A C B. If 012 is the stress diagram, 2–3 will be the vertical component of the reaction at A, and 3–1 that at B. To find the shear on a right section near A, as at E, lay off 2–3, or P_1 in Fig. 23, and draw H so that the arrows may follow one another; then from 0 draw a line 0–4 parallel to the tangent at E; the perpendicular distance 4–2 will be the

shear in the web. For we see by the direction of the arrows that these forces last drawn balance P_1 and H, and, as in Fig. 18, no matter how much the bending moment, and hence the flange stress, may be, the perpendicular distance 4–2 is unchanged. The line 0–4 will be the magnitude of the direct thrust. Both of these forces are given on the right of the section, and this shear is therefore negative. In the same way, for the point E near B, draw 1–3 = —P_2 and 3–0 = H; draw 0–8 parallel to the tangent at E; 8–1, perpendicular to it, will be the shear on the right of the section, again negative, and 0–8 will be the direct thrust. It is noticeable that the normal shear in the web near the left abutment is opposite in sign to P_1, while near the right abutment it agrees in sign with P_2. For the kind of brace needed, see Fig. 10. It is evident that these figures may at once be drawn on the stress diagram, where 0–4 and 4–2 are already sketched. Such a way will answer well for a few points on a large figure, especially if we have applied such loads as give the maximum shear at any particular point. If, however, we desire to see the variation of the shear across the span, we may draw a different diagram.

88. **Shear Diagram.** — As the tangent is perpendicular to the radius at the point of contact, we may at once see that the angles marked θ in Fig. 23 correspond with the angle θ made by the radius to the crown and that to the point E. Hence we get a value for the normal shear, $P \cos \theta - H \sin \theta$. As the point E is distant horizontally from the middle of the span an amount $r \sin \theta$, the last term of this expression for shear varies directly as the distance from the centre; and if we draw 3–7, in the stress diagram of Fig. 22, parallel to the radius at A, cutting 0–6 which is parallel to the tangent at A, 3–7 will be $H \sin \theta$ for A, and may be laid off at $a\,w$ and $b\,r$ of Fig. 23. The vertical ordinate $e\,d$ will then represent $H \sin \theta$ at any point. P_1 is laid off at $c\,l$, and P_2 at $c\,m$; with c as centre, and these two distances as radii, draw the dotted arcs seen in the figure; lay off several angles θ at c, as, for instance, $l\,c\,g$ and $m\,c\,n$ for the points E; project g and n horizontally to f under the respective points E;

$d\,f$ will be $P \cos \theta$, and from several similarly located points the curves $s\,l\,t$ and $v\,f\,r$ are found. Then at any point the *vertical* distance $d\,f - e\,d$ or $e\,f$ will be the *normal* shear in the web on the left of the section, positive if above the inclined line, negative if below it.

From the formula $P \cos \theta - H \sin \theta$, a table of shears may be easily computed for any given arch. $P \sin \theta + H \cos \theta$ will give the direct thrust.

89. **Distribution of Load to produce Equilibrium.** — A series of lines drawn in the stress diagram from 0, parallel to the tangents at a number of equidistant points in a circular rib, will cut off such portions of the load line as represent the loads necessary to make the successive sides of the equilibrium polygon parallel to these tangents, or, in short, coincident with the rib. But the lines radiating from 0 will successively intercept increasing lengths of load line. Hence the load which will keep a circular arch in equilibrium must increase in intensity per horizontal foot from the crown to the springing, and must become infinite at the springing of a semicircular arch. Hence it follows that no amount and distribution of vertical load can make a semicircular arch a true equilibrium curve, that is, one which has no bending moment at any point. In fact, no curve which starts vertically from the abutment can be an equilibrium curve under vertical loads. This may be seen in a more simple manner if we consider that no arrangement of weights will cause a cord, attached at two points, to hang in a funicular polygon whose first side is vertical.

90. **Effect of Change of Temperature.** — The horizontal thrust or tension, due to a change of temperature, in a circular rib hinged at the ends, is found by a similar method to that pursued for the parabolic rib. Referring, to avoid repetition, to what was said at that time, §§ 71–73, the equation may be written, as given in § 74,

$$H_t \,.\, \Sigma\, D\, E^2 = \pm\, 2\, E\, I \,.\, t\, e\, c.$$

Fig. 16 will answer for this case, if we imagine the arc to be circular. As we saw, in § 82, that $\Sigma \mathrm{D\,E}^2$ for a semicircular arch was $\frac{1}{2}\pi r^3$, a substitution in the above equation gives at once

$$\mathrm{H}_t = \pm \frac{4\,\mathbf{E\,I}\,.\,tec}{\pi r^3} = \pm\, 1.264\, \frac{\mathbf{E\,I}\,te}{r^2}$$

for a semicircular rib. The bending moment at the crown, where it is a maximum, will be

$$\mathrm{M\,(max.)} = \frac{4\,\mathbf{E\,I}\,te}{\pi r}.$$

If the arch is less than a semicircle, (*a.*), § 84, gives

$$\Sigma\,\mathrm{D\,E}^2 = r^3(\beta + 2\beta\cos^2\beta - 3\sin\beta\cos\beta),$$

and $c = r\sin\beta$; therefore, substituting, we obtain

$$\mathrm{H}_t = \pm \frac{2\,\mathbf{E\,I}\,te\sin\beta}{r^2(\beta + 2\beta\cos^2\beta - 3\sin\beta\cos\beta)},$$

and the bending moment at the crown will be

$$\mathrm{M\,(max.)} = \frac{2\,\mathbf{E\,I}\,te\sin\beta\,(1-\cos\beta)}{r(\beta + 2\beta\cos^2\beta - 3\sin\beta\cos\beta)}.$$

91. **Shear from Change of Temperature.** — If a load of the proper amount and distribution were imposed on the rib to place it entirely in equilibrium, and cause it to exert against the abutments the desired value of H due to temperature, such a load would supply the amount of shear needed at each section, and, when the load is absent, the bracing must supply such shear. The line *wecer* of the shear diagram of Fig. 23 will therefore limit the ordinates for shear at right sections of the web under changes of temperature, when 0–3 is the amount of H_t. A reference to § 78 and § 87 will aid the reader in recalling these points.

CHAPTER VII.

CIRCULAR RIB WITH FIXED ENDS.

92. **Values of Equations of Condition.** — When the circular rib is fixed at the ends, we apply the three equations of condition which were developed in §§ 17–19, summing up the ordinates, however, along the arch, as has just been done in the preceding case, in place of the horizontal line. When the arch is a complete semicircle, or, as it is often called, a complete arch, as distinguished from a segmental one, the value of y_0, y_1, and y_2 may be obtained by a device similar to the one employed in § 82. The equation to satisfy the first condition is easily derived, but the two others present more difficulty; it is therefore not expedient to take up the semicircle as a special case, but rather to work out the general equations, and make the necessary substitutions.

In the arch of Fig. 24, let $\mathrm{A\,N} = y_1$, $\mathrm{C\,K} = y_0$, and $\mathrm{B\,R} = y_2$; $\mathrm{M\,O\,B} = \mathrm{M\,O\,A} = \beta$, $\mathrm{M\,O\,I} = \alpha$, and $\mathrm{M\,O\,E}$, to any point E, $= \theta$, angles to the right of M being positive. The notation agrees with that just used. Then it may be proved that the three equations of condition will reduce to

$$\sin\beta\, y_0 + \tfrac{1}{2}(\sin\beta + \sin\alpha)\, y_1 + \tfrac{1}{2}(\sin\beta - \sin\alpha)\, y_2 = (\beta - \sin\beta\cos\beta)\, r; \quad (1.)$$

$$\begin{aligned} &-\sin\beta\,(\cos\alpha - \cos\beta + \alpha\sin\alpha - \beta\sin\beta)\, y_0 \\ &+ \tfrac{1}{2}(\sin\beta - \sin\alpha)(\cos\alpha - \cos\beta + \alpha\sin\alpha + \beta\sin\alpha)\, y_1 \\ &+ \tfrac{1}{2}(\sin\beta + \sin\alpha)(\cos\alpha - \cos\beta + \alpha\sin\alpha - \beta\sin\alpha)\, y_2 \\ &= (\sin\beta - \beta\cos\beta)(\sin^2\beta - \sin^2\alpha)\, r; \quad (2.) \end{aligned}$$