circular. As we saw, in § 82, that $\Sigma\, D\, E^2$ for a semicircular arch was $\frac{1}{2} \pi r^3$, a substitution in the above equation gives at once

$$H_t = \pm \frac{4\,\mathbf{E\,I}\,.\,t e c}{\pi r^3} = \pm\ 1.264\ \frac{\mathbf{E\,I}\,t e}{r^2}$$

for a semicircular rib. The bending moment at the crown, where it is a maximum, will be

$$M\ (\text{max.}) = \frac{4\,\mathbf{E\,I}\,t e}{\pi r}.$$

If the arch is less than a semicircle, (*a.*), § 84, gives

$$\Sigma\, D\, E^2 = r^3\,(\beta + 2\,\beta \cos^2 \beta - 3 \sin \beta \cos \beta),$$

and $c = r \sin \beta$; therefore, substituting, we obtain

$$H_t = \pm \frac{2\,\mathbf{E\,I}\,t e \sin \beta}{r^2\,(\beta + 2\,\beta \cos^2 \beta - 3 \sin \beta \cos \beta)},$$

and the bending moment at the crown will be

$$M\ (\text{max.}) = \frac{2\,\mathbf{E\,I}\,t e \sin \beta\,(1 - \cos \beta)}{r\,(\beta + 2\,\beta \cos^2 \beta - 3 \sin \beta \cos \beta)}.$$

91. **Shear from Change of Temperature.** — If a load of the proper amount and distribution were imposed on the rib to place it entirely in equilibrium, and cause it to exert against the abutments the desired value of H due to temperature, such a load would supply the amount of shear needed at each section, and, when the load is absent, the bracing must supply such shear. The line *w e c e r* of the shear diagram of Fig. 23 will therefore limit the ordinates for shear at right sections of the web under changes of temperature, when 0–3 is the amount of H_t. A reference to § 78 and § 87 will aid the reader in recalling these points.

CHAPTER VII.

CIRCULAR RIB WITH FIXED ENDS.

92. **Values of Equations of Condition.** — When the circular rib is fixed at the ends, we apply the three equations of condition which were developed in §§ 17–19, summing up the ordinates, however, along the arch, as has just been done in the preceding case, in place of the horizontal line. When the arch is a complete semicircle, or, as it is often called, a complete arch, as distinguished from a segmental one, the value of y_0, y_1, and y_2 may be obtained by a device similar to the one employed in § 82. The equation to satisfy the first condition is easily derived, but the two others present more difficulty; it is therefore not expedient to take up the semicircle as a special case, but rather to work out the general equations, and make the necessary substitutions.

In the arch of Fig. 24, let $A\,N = y_1$, $C\,K = y_0$, and $B\,R = y_2$; $M\,O\,B = M\,O\,A = \beta$, $M\,O\,I = \alpha$, and $M\,O\,E$, to any point E, $= \theta$, angles to the right of M being positive. The notation agrees with that just used. Then it may be proved that the three equations of condition will reduce to

$$\sin \beta\, y_0 + \tfrac{1}{2}(\sin \beta + \sin \alpha)\, y_1 + \tfrac{1}{2}(\sin \beta - \sin \alpha)\, y_2 = (\beta - \sin \beta \cos \beta)\, r; \quad (1.)$$

$$\begin{aligned} &- \sin \beta\,(\cos \alpha - \cos \beta + \alpha \sin \alpha - \beta \sin \beta)\, y_0 \\ &+ \tfrac{1}{2}(\sin \beta - \sin \alpha)(\cos \alpha - \cos \beta + \alpha \sin \alpha + \beta \sin \alpha)\, y_1 \\ &+ \tfrac{1}{2}(\sin \beta + \sin \alpha)(\cos \alpha - \cos \beta + \alpha \sin \alpha - \beta \sin \alpha)\, y_2 \\ &= (\sin \beta - \beta \cos \beta)(\sin^2 \beta - \sin^2 \alpha)\, r; \quad (2.) \end{aligned}$$

$$[(\beta - \cos\beta\sin\beta)\sin\alpha - (\alpha + \sin\alpha\cos\alpha - 2\sin\alpha\cos\beta)\sin\beta]\,y_0$$
$$+ \tfrac{1}{2}(\sin\beta - \sin\alpha)(\alpha + \sin\alpha\cos\alpha + \beta - \sin\beta\cos\beta - 2\sin\alpha\cos\beta)\,y_1$$
$$+ \tfrac{1}{2}(\sin\beta + \sin\alpha)(\alpha + \sin\alpha\cos\alpha - \beta + \sin\beta\cos\beta - 2\sin\alpha\cos\beta)\,y_2 = 0. \quad (3.)$$

It will be easier to solve the numerical equations after the values of α and β, with their sines and cosines, are introduced, than to deduce independent values of y_1, &c., at present. They may be written more briefly, for convenience in substitution, if

$$\sin\beta - \sin\alpha = a;\ \sin\beta + \sin\alpha = b;\ \alpha + \sin\alpha\cos\alpha - 2\sin\alpha\cos\beta = c;$$
$$\beta - \sin\beta\cos\beta = d;\ \cos\alpha - \cos\beta + \alpha\sin\alpha = e;$$

$$\sin\beta\, y_0 + \tfrac{1}{2}b\,y_1 + \tfrac{1}{2}a\,y_2 = d\,r. \quad (4.)$$

$$-(e - \beta\sin\beta)\sin\beta\,y_0 + \tfrac{1}{2}a(e + \beta\sin\alpha)\,y_1 + \tfrac{1}{2}b(e - \beta\sin\alpha)\,y_2$$
$$= a\,b(\sin\beta - \beta\cos\beta)\,r. \quad (5.)$$

$$(d\sin\alpha - c\sin\beta)\,y_0 + \tfrac{1}{2}a(c + d)\,y_1 + \tfrac{1}{2}b(c - d)\,y_2 = 0. \quad (6.)$$

93. **Special Values for Semicircular Rib.** — If the arch is a semicircle, $\beta = \frac{1}{2}\pi$; $\sin\beta = 1$; $\cos\beta = 0$; and the three equations of the last section reduce to

$$y_0 + \tfrac{1}{2}(1 + \sin\alpha)\,y_1 + \tfrac{1}{2}(1 - \sin\alpha)\,y_2 = \tfrac{1}{2}\pi r; \quad (1.)$$

$$(\tfrac{1}{2}\pi - \cos\alpha - \alpha\sin\alpha)\,y_0 + \tfrac{1}{2}(1 - \sin\alpha)(\cos\alpha + \alpha\sin\alpha + \tfrac{1}{2}\pi\sin\alpha)\,y_1$$
$$+ \tfrac{1}{2}(1 + \sin\alpha)(\cos\alpha + \alpha\sin\alpha - \tfrac{1}{2}\pi\sin\alpha)\,y_2 = (1 - \sin^2\alpha)\,r; \quad (2.)$$

$$(\tfrac{1}{2}\pi\sin\alpha - \alpha - \sin\alpha\cos\alpha)\,y_0 + \tfrac{1}{2}(1 - \sin\alpha)(\alpha + \sin\alpha\cos\alpha + \tfrac{1}{2}\pi)\,y_1$$
$$+ \tfrac{1}{2}(1 + \sin\alpha)(\alpha + \sin\alpha\cos\alpha - \tfrac{1}{2}\pi)\,y_2 = 0. \quad (3.)$$

If equation (1.) is multiplied by α, equation (3.) may be added to it, and then (2.) may be multiplied by $\sin\alpha$, and subtracted from their sum, when there will result

$$(\alpha + \tfrac{1}{4}\pi - \tfrac{1}{4}\pi\sin\alpha)\,y_1 + (\alpha - \tfrac{1}{4}\pi - \tfrac{1}{4}\pi\sin\alpha)\,y_2 = (\tfrac{1}{2}\pi\alpha - \sin\alpha)\,r. \quad (4.)$$

If (1.) is multiplied by $\frac{1}{2}\pi - \cos\alpha - \alpha\sin\alpha$, and equation (2.) is subtracted from it, we shall get, upon dividing by the common coefficient of y_1 and y_2,

$$\tfrac{1}{2}(y_1 + y_2) = \frac{\frac{1}{2}\pi(\frac{1}{2}\pi - \cos\alpha - \alpha\sin\alpha) - \cos^2\alpha}{\frac{1}{2}\pi - 2\cos\alpha - 2\alpha\sin\alpha + \frac{1}{2}\pi\sin^2\alpha}\,r,$$

which, if the quantity in the parentheses be represented by g, may be written,

$$\tfrac{1}{2}(y_1 + y_2) = \frac{\frac{1}{2}\pi g - \cos^2\alpha}{2g - \frac{1}{2}\pi\cos^2\alpha}\,r. \quad (5.)$$

Upon multiplying this equation by $2\alpha - \frac{1}{2}\pi\sin\alpha$, and subtracting it from (4.), we obtain, by factoring the second member,

$$\tfrac{1}{2}(y_1 - y_2) = \frac{\left(\frac{4}{\pi} - \frac{\pi}{2}\right)(\alpha\cos^2\alpha - g\sin\alpha)}{2g - \frac{1}{2}\pi\cos^2\alpha}\,r. \quad (6.)$$

The sum of (5.) and (6.) will give y_1; their difference will give y_2; and these values, inserted in (1.), will readily give us y_0.

94. **First Equation of Condition.** — Many of the following expressions are similar to those of § 84, and a remembrance of the relation between y_1 and y_2 will, in a measure, prevent the ensuing work from seeming so involved as it otherwise may appear. Generally, coefficients of y_1 and y_2 will differ only in the signs of the terms which contain α and sine α. The first condition is

$$\Sigma\,DE^2 = \Sigma\,DF\,.\,DE.$$

From § 84, we have

$$\Sigma\,DE^2 = r^3(\beta + 2\beta\cos^2\beta - 3\sin\beta\cos\beta).$$

It will be seen, from Fig. 24, that $DF = DL + LF = y_1$ (or y_2) $+ LF$, DL in the sketch being negative on the right of K, and that, therefore, in place of the values of the section just referred to, we shall write

$$DF = y_1 + \frac{\sin\beta + \sin\theta}{\sin\beta + \sin\alpha}(y_0 - y_1), \text{ on the left of K;}$$

$$DF = y_2 + \frac{\sin\beta - \sin\theta}{\sin\beta - \sin\alpha}(y_0 - y_2), \text{ on the right of K.}$$

For the value of the second member of the above equation of condition between α and $-\beta$ we have then, since $DE = r(\cos\theta - \cos\beta)$,

$$r^2\int_{-\beta}^{\alpha}\Big[y_1(\cos\theta - \cos\beta) + \frac{y_0 - y_1}{\sin\beta + \sin\alpha}(\sin\beta\cos\theta + \sin\theta\cos\theta - \sin\beta\cos\beta$$
$$- \cos\beta\sin\theta)\Big]^{*}d\theta = r^2\Big[y_1(\sin\alpha - \alpha\cos\beta + \sin\beta - \beta\cos\beta)$$
$$+ \frac{y_0 - y_1}{\sin\beta + \sin\alpha}(\sin\alpha\sin\beta - \tfrac{1}{2}\cos^2\alpha - \alpha\sin\beta\cos\beta + \cos\alpha\cos\beta$$
$$+ \sin^2\beta - \tfrac{1}{2}\cos^2\beta - \beta\sin\beta\cos\beta)\Big].$$

Likewise, for the value of the second member between α and $+\beta$

* Compare § 84.

$$r^2\int_a^\beta [y_2(\cos\theta - \cos\beta) + \frac{y_0 - y_2}{\sin\beta - \sin a}(\sin\beta\cos\theta - \sin\theta\cos\theta - \sin\beta\cos\beta + \cos\beta\sin\theta)]^* d\theta = r^2 [y_2(\sin\beta - \beta\cos\beta - \sin a + a\cos\beta) + \frac{y_0 - y_2}{\sin\beta - \sin a}(\sin^2\beta - \tfrac{1}{2}\cos^2\beta - \beta\sin\beta\cos\beta - \sin a\sin\beta - \tfrac{1}{2}\cos^2 a + a\sin\beta\cos\beta + \cos a\cos\beta)].$$

Equating the sum of these two quantities which make up the second member, with the first member, we obtain the first equation of condition, which, when cleared of fractions, becomes

$$y_0(2\sin^3\beta - \sin\beta\cos^2\beta - 2\beta\sin^2\beta\cos\beta - \cos^2 a\sin\beta + 2\cos a\sin\beta\cos\beta - 2\sin^2 a\sin\beta + 2a\sin a\sin\beta\cos\beta) + y_1(\tfrac{1}{2}\sin\beta\cos^2\beta - \sin^3 a + a\sin^2 a\cos\beta + \beta\sin^2 a\cos\beta + \tfrac{1}{2}\cos^2 a\sin\beta - \cos a\sin\beta\cos\beta - \tfrac{1}{2}\sin a\cos^2 a - a\sin a\sin\beta\cos\beta + \sin a\cos a\cos\beta + \sin a\sin^2\beta - \tfrac{1}{2}\sin a\cos^2\beta - \beta\sin a\sin\beta\cos\beta) + y_2(\tfrac{1}{2}\sin\beta\cos^2\beta + \sin^3 a - a\sin^2 a\cos\beta + \beta\sin^2 a\cos\beta + \tfrac{1}{2}\cos^2 a\sin\beta - \cos a\sin\beta\cos\beta + \tfrac{1}{2}\sin a\cos^2 a - a\sin a\sin\beta\cos\beta - \sin a\cos a\cos\beta - \sin a\sin^2\beta + \tfrac{1}{2}\sin a\cos^2\beta + \beta\sin a\sin\beta\cos\beta) = r(\sin^2\beta - \sin^2 a)(\beta + 2\beta\cos^2\beta - 3\sin\beta\cos\beta).$$

95. **Second Equation of Condition.** — The next condition to be satisfied is $\Sigma\,DE = \Sigma\,DF$, or, introducing the values of these quantities from the preceding section,

$$r^2\int_{-\beta}^{+\beta}(\cos\theta - \cos\beta)\,d\theta = r\int_{-\beta}^{a}[y_1 + \frac{y_0 - y_1}{\sin\beta + \sin a}(\sin\beta + \sin\theta)]\,d\theta + r\int_a^\beta [y_2 + \frac{y_0 - y_2}{\sin\beta - \sin a}(\sin\beta - \sin\theta)]\,d\theta.$$

Performing the indicated integration, and clearing of fractions, we obtain

$$y_0(2\beta\sin^2\beta - 2\cos a\sin\beta + 2\sin\beta\cos\beta - 2a\sin a\sin\beta) + y_1(-\beta\sin^2 a - a\sin^2 a + \cos a\sin\beta - \sin\beta\cos\beta + a\sin a\sin\beta + \beta\sin a\sin\beta - \sin a\cos a + \sin a\cos\beta) + y_2(-\beta\sin^2 a + a\sin^2 a + \cos a\sin\beta - \sin\beta\cos\beta + a\sin a\sin\beta - \beta\sin a\sin\beta + \sin a\cos a - \sin a\cos\beta) = 2r(\sin^2\beta - \sin^2 a)(\sin\beta - \beta\cos\beta).$$

* Compare § 84.

96. **Third Equation of Condition.** — The third condition, in the modified from of § 59, is $\Sigma\,DE\,.\,DB = \Sigma\,DF\,.\,DB$. Since $DB = r(\sin\beta - \sin\theta)$, this condition becomes, by multiplying the previous condition by DB,

$$r^3\int_{\beta-}^{+\beta}(\sin\beta\cos\theta - \sin\theta\cos\theta - \sin\beta\cos\beta + \cos\beta\sin\theta)\,d\theta = r^2\int_{-\beta}^{a}[y_1(\sin\beta - \sin\theta) + \frac{y_0 - y_1}{\sin\beta + \sin a}(\sin^2\beta - \sin^2\theta)\,d\theta + r^2\int_a^\beta [y_2(\sin\beta - \sin\theta) + \frac{y_0 - y_2}{\sin\beta - \sin a}(\sin^2\beta - 2\sin\beta\sin\theta + \sin^2\theta)]\,d\theta,^*$$

which, when integrated and cleared of fractions, gives

$$y_0(2\beta\sin^3\beta - a\sin\beta - \sin a\cos a\sin\beta + 2\sin^2\beta\cos\beta - 2a\sin a\sin^2\beta + \beta\sin a + \sin a\sin\beta\cos\beta - 2\cos a\sin^2\beta) + y_1(-\tfrac{3}{2}\sin^2\beta\cos\beta + \cos a\sin^2\beta - \beta\sin^2 a\sin\beta - a\sin^2 a\sin\beta + \sin^2 a\cos\beta - \tfrac{1}{2}\sin^2 a\cos a + \tfrac{1}{2}a\sin\beta - \tfrac{1}{2}\sin a\cos a\sin\beta + \tfrac{1}{2}\beta\sin\beta + \beta\sin a\sin^2\beta + a\sin a\sin^2\beta - \tfrac{1}{2}a\sin a - \tfrac{1}{2}\beta\sin a + \tfrac{1}{2}\sin a\sin\beta\cos\beta) + y_2(-\tfrac{1}{2}\sin^2\beta\cos\beta + \cos a\sin^2\beta - \beta\sin^2 a\sin\beta + a\sin^2 a\sin\beta - \sin^2 a\cos\beta + \tfrac{1}{2}\sin^2 a\cos a + \tfrac{1}{2}a\sin\beta + \tfrac{3}{2}\sin a\cos a\sin\beta - \tfrac{1}{2}\beta\sin\beta - \beta\sin a\sin^2\beta + a\sin a\sin^2\beta + \tfrac{1}{2}a\sin a - \tfrac{1}{2}\beta\sin a - \tfrac{3}{2}\sin a\sin\beta\cos\beta) = 2r\sin\beta(\sin^2\beta - \sin^2 a)(\sin\beta - \beta\cos\beta).$$

97. **Reduction of Equations.** — If the second equation of condition is multiplied by $\cos\beta$, and added to the first, there results an equation in which, as soon as we write $1 - \sin^2 a$ for $\cos^2 a$, and $1 - \sin^2\beta$ for $\cos^2\beta$, there will be found a common factor $(\sin^2\beta - \sin^2 a)$. This being cancelled out, there results (1.), § 92. The second equation again may be divided by 2, and then factored, by simple inspection, into (2.), § 92. Finally, the second equation of condition may be multiplied by $\sin\beta$, and subtracted from the third, when, upon factoring, we obtain (3.), § 92.

It will be seen that the solution of (4.), (5.), and (6.), § 92, for any given arch, and for several values of a, will not involve much work, owing to the recurrence of the known factors denoted by a, b, c, d, and e. As the arch may subtend any angle, it will not be expedient to go into calculations here for any special values of β. One case will be taken up later.

98. **Values of H, &c.** — When the desired ordinates for any arch are computed, we have the option of obtaining the values

* $\int\sin^2\theta\,d\theta = \tfrac{1}{2}(\theta - \sin\theta\cos\theta)$. See also note to § 84.

of H, of the vertical components of the abutment reactions, and of the ordinates for bending moment, either by graphical construction, or by formulæ similar to those applied to the parabolic rib. By noticing the expressions to be substituted for b, c, and k in the case of the circular arch with hinged ends, one can readily adapt the formulæ of § 63 and § 65 to the computations for this case. The ordinates to the circular arch will be the same as in § 85.

99. **Table of y_0, y_1, and y_2 for Semicircle.** — We may, however, obtain the ordinates y_0, &c., for a semicircle with comparative ease, and, as such a rib is sometimes used for large roofs, these values may be convenient. Semicircular masonry arches, having backing above the abutments, present a different case.

If α is taken as 20° or .3491, $\sin\alpha = .3420$, $\cos\alpha = .9397$, and $\frac{1}{2}\pi = 1.5708$; hence, in § 93, $g = .5117$, and (5.) and (6.) become

$$\tfrac{1}{2}(y_1 + y_2) = \frac{-.0792}{-.3646} r = .2172\, r;$$

$$\tfrac{1}{2}(y_1 - y_2) = \frac{-.2977 \times .1333}{-.3646} r = .1088\, r;$$

whence $y_1 = .326\, r$, and $y_2 = .108\, r$. By substitution in (1.), § 93, $y_0 = (1.5708 - .2187 - .0357)\, r = 1.316\, r$.

If similar computations are carried out for other values of α, we shall complete the following table for a semicircular rib with fixed ends:

a	y_1.	y_0.	y_2.
0°	.241 r	1.330 r	.241 r
10	.288	1.326	.183
20	.326	1.316	.108
30	.360	1.298	.011
40	.387	1.275	— .125
50	.413	1.245	— .330
60	.434	1.210	— .665
70	.455	1.170	— 1.333
80	.475	1.125	— 3.319

Other intermediate values can be obtained, if desired, by the

formula for interpolation, § 45. The number of decimals it is desirable to use in any particular case will depend upon the value of r. The equilibrium polygons for these ordinates have been drawn in Fig. 25, and from them we get the different values of H, for a weight W at the several divisions, as shown in the accompanying stress diagram.

100. **Example.** — As an application of these results, let us draw the equilibrium curve for a semicircular arch of uniform section carrying only its own weight. As this weight is symmetrically disposed, $y_1' = y_2'$. By drawing the stress diagram of Fig. 25 to a sufficiently large scale, we shall find by measurement, that H, for a weight at the crown, 10°, 20°, &c., from the crown, will be .46, .44, .39, .31, .23, .14, .07, .02, and .01 W respectively. If we double all of these values except the one for a weight at the crown, and take the sum of the whole, we shall obtain for the horizontal thrust, $H' = 3.68$ W for 17 loads, each equal to W, at the 17 points of division in the whole arch.

To find y_1', multiply each y_1 by its H, remembering, that, when the weights are on the left of the crown, the values of y_2 in the table of § 99 become y_1, and that we may, therefore, before multiplying by H, add together y_1 and y_2 for each point except the crown, and then divide the sum of these products by H′, just obtained. (Compare § 67.) For example, for a load W on each of the two points distant 30° from the crown, $H\, y_1 + H\, y_2 = .31\, W\, (.360 + .011)\, r = .115\, r\, W$, the value of M at the abutments. Performing the operations, and taking the algebraic sum of the products, we get $.6225\, r\, W$ for the total moment at either abutment, and $\dfrac{.6225\, r\, W}{3.68\, W} = 0.17\, r = y_1' = y_2'$.

To construct the equilibrium curve, we divide the semicircle A C B, Fig. 26, into eighteen equal parts, each subtending 10°, and draw verticals through the points of division. Assume the weight of the arch to be represented by a vertical line of any convenient length. Since the loads are supposed to be concentrated at the points of division, one-eighteenth of the gross