weight of the arch will be found at each of these points, and one-thirty-sixth at A and B; for A and B will each carry directly one-half of the adjacent division. Therefore, beginning and closing with one-thirty-sixth, space off the load-line into eighteenths; from the middle of the load-line lay off $H' = 3.68\ W = 3\text{–}0$, where $W =$ weight of one division, or $H' = \frac{3.68}{18} = .204$ of the whole weight of the rib. One-half of this load-line is 1–3. Lay off y_1' and $y_2' = .17\ r$, at A and B, and draw the sides of the equilibrium polygon parallel to the lines which radiate from the extremity of H′ to the points of division of the load-line, thus obtaining the curve E G D. The second half of the curve was obtained by spacing off 0′– 3 to the left.

101. **Practical Application.** — Having at hand a wooden model of an arch-ring, representing the voussoirs, or stones, of a semicircular arch, we tried some experiments as tests of the accuracy of this method of analysis and of the correctness of these results. The arch is represented by Fig. 26, and consisted of forty-two independent voussoirs. The span, A B, of the middle line of the ring, 18 inches, was 13.09 times the thickness of the ring, and the structure would apparently just stand alone when left to itself: a slight additional weight at the crown would cause that part to sink, the haunches to move outwards, and the ring to fall in pieces. Considering that this arch, so long as it rested squarely on the faces at A and B, was fixed in direction, or not free to turn at the ends, we laid off at A E and B D the value of y_1 obtained in the last section, and drew the equilibrium polygon, as just described, on the centre line of the ring, beginning at D with a line parallel to 0–4. It will be noted that no line is used from 0 to 1; for the weight represented by 1–4 is directly supported at B; while the amount 4–5 is the weight concentrated on the first vertical just above D.

As the arch is a continuous ring, the weights may properly be concentrated at a greater number of points; so that finally the true equilibrium curve will pass through the vertices of the polygon we have just constructed: the difference between the two is unimportant, however, and is only appreciable near the crown. The bending moment at any point has been proved to be equal to H multiplied by the vertical ordinate between the centre line and the equilibrium curve, or, by § 10, also equal to T, the thrust along the tangent to the equilibrium curve, multiplied by the perpendicular from a point on the centre line to this tangent: therefore if we draw E F as this tangent, the bending moment at A will equal either H. E A, or the thrust along E F multiplied by the perpendicular from A. The direction of the thrust E F, if prolonged, cuts the springing joint very close to the outside edge: it will also be noticed that the equilibrium curve approaches quite near to the edge of the voussoirs at the crown G. Now, as we reminded the reader in § 11 that the force T, or 0′–1, at the distance F A from the centre line of the rib, is equal to the same force at the centre line and the couple which produces bending moment, conversely, the resultant of the pressure of this rib at the end A must cut the base in the prolongation of the line E F: in short, the tangent to the equilibrium curve at each point gives the direction and point of application of the resultant thrust at that right section of the rib to which it belongs, as ascertained by erecting a vertical from the middle point of the section.

102. **Limiting Position of Equilibrium Curve.** — If, as is usually the case, the intensity of the resisting force of the abutment at A is assumed to vary uniformly from one edge to the other, then, in case the resistance is zero at the inside edge and a maximum at the outside edge, the intensity at all points can be represented, as shown in the small sketch marked A′, by the ordinates of a triangle whose base is the breadth of a voussoir, and whose longest ordinate is the intensity of the pressure at the edge near F. The total pressure will be equal to the area of the triangle, and the resultant will pass through the centre of gravity of the triangle, cutting the base at one-third of its length from the outer edge. If there existed any tension near the inner edge, we should have two triangles, as shown in the

other sketch, the inclined line cutting the base at the point where the stress changed from tension to compression; and the resultant of the two stresses must, since they are of opposite kinds, lie outside of their separate resultants, and on the side of the greater one. This fact as to the position of the resultant of two opposite parallel forces was indicated in § 11, Fig. 2, and is one of the well-known properties of the lever, as proved in Mechanics.

Since, then, the resultant force, or the thrust on a section of the rib of Fig. 26, at A, B, and C, passes near the edge of the section, or, as it is often stated, outside of the *middle third* of the cross-section, we should expect to find tension at the inside edge of the joint at these points. As this model consists simply of wooden blocks placed in juxtaposition, a voussoir cannot exert tension on its neighbor at any point of contact, and movement must immediately take place when the weight of the rib is allowed to act freely, rotation being set up about the outside edges at F, G, and Q. The crown will sink, the haunches will move outwards, and the arch may be expected to fall. The reader will remember that it was explained, in § 12, that an arch tends to move away from the equilibrium curve.

Since any material is compressible, it is probable that the assumption of a uniform variation of intensity of stress at any section will not be strictly true; that the stress may not be exerted over the entire surface of the *originally* plane joint; and that therefore the equilibrium curve may pass somewhat outside of the middle third of the joint without causing the arch to fall, although the joint will then open slightly at the edge where no pressure is exerted, by reason of the compression causing the joint to be no longer plane. But such an assumption gives an additional element of safety to a design, when the engineer so proportions his rib of rectangular section that the equilibrium curve of the load at any time shall never leave the limits of the middle third, and the tensile strength of the cement will not then be relied upon to assure stability.

103. **Model as hinged at Three Points.** — The arch of Fig. 26 stood when the string which at first passed around the exterior was removed, although a slight change of shape was observable. A close inspection, however, showed that the voussoirs at the crown and the two springings were then in contact only at the outer edges. The rotation at these joints, indicated in the last section as probable, had commenced; but, as soon as the rib became thus hinged at three points, it was in equilibrium. It is desirable, then, as a further test, to draw the equilibrium curve for this rib hinged at the crown and springings. As the change of shape and curvature was very little, the supposition that the weight of the voussoirs is concentrated along the arc K Q will be sufficiently near the truth for our purpose.

The half-weight being represented by 1–3, the first step is to find the value of H for this case, when the load is concentrated at intervals of ten degrees along the outer semicircle. We can avail ourselves of the formula of § 23, finding the different values of b by measurement, or from tables of sines, since $b = r \sin \theta$, and summing up the several amounts of H for the whole semicircle; or, as is done in this figure, we may use the principle explained in § 30, that any two sides of the funicular polygon, or two tangents to the equilibrium curve, will meet, when prolonged, on the vertical through the centre of gravity of the included weight. Since the arch is symmetrically loaded, the thrust at the crown will be horizontal, and therefore lie in the line K L; the centre of gravity of the quadrant arc K Q will be on the vertical line P L, drawn at such a distance, K L, from the crown as to satisfy the value for the ordinate from the centre of a circle to the centre of gravity of a circular arc, viz., $\frac{\text{radius} \times \text{chord}}{\text{length of arc}}$; and therefore the thrust at the springing will lie in the line Q L, drawn from Q to the intersection of the other two forces. As 1–3 represents the weight of one-half the arch, and the thrust at the crown is parallel to 3–0, a line from 1, parallel to Q L, will complete the triangle of forces, and cutting the horizontal line at 9, will determine 3–9 to be the desired value of H. The equilibrium polygon can now be drawn from Q to K, its sides being successively parallel to lines radiating from 9, the first line being 9–4 and the last one 9–6. These lines are not drawn in the stress diagram. The other half of the polygon may be added, if desired.

It will now be seen, that, excepting the hinged points, the nearest approach of the equilibrium curve to the edge of a voussoir is at P, where it is still well within the rib, and consequently no further movement of the rib is to be expected. Another model, somewhat thinner than the one here illustrated, was experimented with, and would not stand. If the arch of Fig. 26 is slightly weighted at K, the joint at P begins to open on the outside, confirming the result, that the equilibrium curve here passes nearest to the inner edge. If it be objected that the change of outline previously referred to carries the portion of the rib near P farther from the centre, so that the equilibrium curve may run nearer the edge than we have plotted it, we rejoin, that such a movement, carrying the centre of gravity, and hence the line P L, in the same direction, will cause Q L to make a slightly less angle with the vertical, diminishing the value of H, and moving the equilibrium curve also a little away from P.

104. **Model as hinged at Abutments.** — For the purpose of making an additional test of our results, we finally placed a small wire at A and B, thus hinging the rib on its centre line at these points. The equilibrium curve for one-half of the arch is A N K. The amount of H is determined by computation from the formula of § 85, which becomes, for a semicircular rib, $H = \frac{\cos^2 \alpha}{\pi} W$; and the summation for the whole arch, carrying W at intervals of ten degrees along the centre line, is $H = 2.86\,W$, laid off at 3–8. Radiating lines between 8–4 and 8–6 will enable one to draw A N K. The arch, when released, fell in ruins, and the first joint to open, on the outside at the haunch, was near N, lower than P in the former case.

We have dwelt on these curves at some length, as they give so good a confirmation of previous deductions and results, and as they will aid the reader in assuring himself that he understands the method of treatment. Such diagrams must, for accuracy, be drawn to quite a large scale, and the results will then be very satisfactory.

105. **Effect of Change of Temperature.** — It remains to find the effect of change of temperature on the circular rib with fixed ends. As was previously indicated in § 76, we must find the height $A G = B I = y_1$, at which the equilibrium line shall be drawn in Fig. 27, by the condition that the change of inclination at the abutments, or $\Sigma E F = 0$. If the notation of the angles subtended by portions of the arch is as before, and as marked in the figure, we have $E F = D E - y_1$, and

$$\Sigma E F = \int_{-\beta}^{+\beta} r(r\cos\theta - r\cos\beta - y_1)\,d\theta = 2r(r\sin\beta - r\beta\cos\beta - y_1\beta) = 0,$$

or

$$y_1 = r\left(\frac{\sin\beta}{\beta} - \cos\beta\right),$$

which becomes, for a semicircle,

$$y_1 = \frac{2r}{\pi} = 0.632\,r.$$

The first term of (1.), § 76, therefore becomes $\Sigma D E^2 - y_1 . \Sigma D E$. From § 84, $\Sigma D E^2 = r^3(\beta + 2\beta\cos^2\beta - 3\sin\beta\cos\beta)$, while $y_1 \cdot \Sigma D E$ gives, as above, $r^3\left(\frac{\sin\beta}{\beta} - \cos\beta\right)(2\sin\beta - 2\beta\cos\beta)$; so that the first term reduces to $r^3\left(\beta + \sin\beta\cos\beta - \frac{2\sin^2\beta}{\beta}\right)$, and (1.), § 76, takes the form of

$$H_t \cdot r^3\left(\beta + \sin\beta\cos\beta - \frac{2\sin^2\beta}{\beta}\right) = \pm\, 2 \mathbf{E}\mathbf{I} t e r \sin\beta.$$

$$H_t = \pm \frac{2 \mathbf{E}\mathbf{I} t e}{r^2\left(\frac{\beta}{\sin\beta} + \cos\beta - 2\frac{\sin\beta}{\beta}\right)}.$$

For a semicircle, the formula for horizontal thrust simplifies into

$$H_t = \pm \frac{2\,E\,I\,t\,e}{r^2\left(\frac{\pi}{2} - 2\frac{2}{\pi}\right)} = \pm\, 6.45\,\frac{E\,I\,t\,e}{r^2}.$$

The bending moments at the crown and springing can now be readily written, and compared with the values of § 90. The horizontal thrust for the semicircular rib fixed at the ends is five times as great as when the ends are hinged. The remarks of § 91 in regard to shear will apply equally well here.

For the Elliptic Rib, see § 153.

106. **Maximum Stress determined by Length of Ordinate; Rib of Rectangular Section.** — It may sometimes be convenient to have the means of determining from a simple inspection of a diagram, by noting the position of the equilibrium polygon, how much the maximum intensity of stress at any section exceeds the mean intensity. As the mean intensity $f = T \div S$ where T is the direct thrust and S is the area of cross-section, and is obtained at any point from the known value of the thrust in the side of the equilibrium polygon, the maximum intensity of stress will be readily found by multiplying by the proper ratio. The stress arising from bending moment in a solid section is always taken as uniformly varying (see Fig. 2). The combination of direct stress with that from bending moment will also give a uniformly varying stress.

Considering, first, the rib of rectangular cross-section, Fig. 28, we see, that if we call the intensity, A C, of direct stress unity, a bending moment which will produce a compression, D E, of unity at the upper extreme fibre, and a tension, C A, of unity at the lower extreme fibre, will bring the resultant stress at all points to the amounts indicated in the left-hand sketch, twice the mean intensity at one edge, and zero at the other. If the cross-section is treated by the method of Part I., "Roofs," p. 57, Fig. 24, in order to make an equivalent area of uniform stress equal to the maximum, we get the shaded area of the left, which is evidently one-half of the whole section. The centre of gravity of this area, lying at one-third the height from the upper edge, will be the point of application of the resultant force on the cross-section. If the bending moment is reversed, the sketch will be inverted: hence, when the line of thrust, or the side of the equilibrium polygon, passes at *one-sixth* of the depth above or below the *axis* of the rib, the intensity of stress at that edge of the rib which is nearer the line of thrust will be twice the mean intensity.

If, again, the maximum intensity is to be thrice the mean, the line F G, starting at a distance B F = 3 B D, still cuts C D at its middle point in order to make the total tension from bending moment equal to the total compression from the same cause. Noting where F G cuts A B, we have the point of no stress at $\frac{3}{4}h$ from the upper edge of the section: hence the shaded areas are drawn as given in the section on the right, the upper one for compression, the lower one for tension. The area of the upper one is $\frac{1}{2}b \cdot \frac{3}{4}h = \frac{3}{8}bh$: the lower one, being similar, but of one-third the altitude, has one-ninth the area of the other, or $\frac{1}{24}bh$. The difference is $\frac{1}{3}bh$, or one-third the area of the cross-section, as required if the maximum intensity is to be three times the mean. Letting these areas represent the forces, and taking moments about the upper edge, each force being applied at the centre of gravity of its triangle, we have for the position of the resultant, measured from the upper edge,

$$\frac{\frac{3}{8}bh \cdot \frac{1}{4}h - \frac{1}{24}bh \cdot 1\frac{1}{12}h}{\frac{1}{3}bh} = \frac{1}{6}h.$$

If, therefore, the line of thrust passes at $\frac{1}{6}h$ from the edge, or one-third the depth from the axis, the intensity of compression on the outside fibre nearer the line will be three times the mean compression, and at the other edge there will be a tension equal in magnitude to the mean stress.

In the same way it may be shown, that, when the line of thrust cuts the edge, the compression there will be B I, four times the mean, and the tension at the other edge will be A K, twice the magnitude of the mean stress. Thus it will be seen, that, for every one-sixth h that the line of thrust is distant from the axis, the compression on the square inch will be increased by unity on the side to which the line deviates, and diminished by unity on the other side, the mean compression being denoted by unity. This is indicated by the numerals marked on the sketches of Fig. 29.

107. **Rib of Two Flanges.** — If the rib is composed of two flanges and an open-work web, the stress in either flange is easily determined. If the line of thrust is in the axis, each flange will carry one-half of the direct stress. If the line of thrust passes through one flange, Fig. 30, that flange may be considered to carry all of the compression uniformly distributed, and the other flange to be under no stress; for the depth of the flange is so small, compared with the whole depth of the rib, that no error of importance is involved in considering the stress as uniformly distributed over the section of one flange. If the line of thrust passes without the rib a distance equal to its depth, we get, by taking moments at A, Fig. 30,

Thrust at C $\times$ 2 A B = Compression at B $\times$ A B;
or, *Compression* at B = 2 $\times$ direct stress.

If moments are taken at B, we find,

Tension at A = direct stress.

In the same way, if $B'C' = 2A'B'$,

Compression at $B' = 3 \times$ direct stress; Tension at $A' = 2 \times$ direct stress.

Hence we may draw a sketch for this rib similar to the one for the rectangular rib. The numerals here denote that one flange carries once, twice, &c., the *entire* direct stress. If the rib has a plate web, or is an I beam, the above method will give a good approximation to the true stresses. If the web is heavy, the method of the next section may be applied.

108. **Rib of Circular Section; General Construction.** — When the rib is of less simple section, we must return to the

graphical construction first referred to. As an instance, suppose the cross-section of the rib to be a circle. The variation of stress on a diameter, in the direction of deviation, is indicated by the left-hand sketch of Fig. 31, when the intensity of stress is twice the mean at one edge, and zero at the other. By constructing, according to the principles already laid down, Part I., "Roofs," the equivalent area of maximum intensity, we obtain the shaded area of the figure, and then we determine its centre of gravity by cutting out the area, and balancing it over a knife-edge. The deviation of the line of thrust from the centre of the circle, to make the maximum intensity twice the mean, and the minimum zero, is thus found, and proves to be one-fourth the radius.

By the construction of the other sketch, taking moments as in § 106, or reasoning by analogy, we find that the deviation, in order that the maximum shall be thrice the mean intensity of compression, and the tension at the other end of the diameter shall equal the mean stress, must be one-half the radius from the centre: hence, when the line of thrust cuts the edge, the maximum compression equals five times the mean, and the tension at the other extreme of the diameter is three times the mean compression. Thus we get the numerals and their positions, as given in the figure.

In a thin tube of circular, elliptical, or oval section, the maximum compression is nearly three times the mean intensity of direct stress where the equilibrium polygon cuts the surface of the tube; and a tensile stress equal in magnitude to the mean will then be found at the other end of the extremity of the diameter: hence proportionate distances of the side of the equilibrium polygon from the axis of the rib will give twice, four times, &c., the mean stress.