CHAPTER VIII.

ARCHED RIBS UNDER WIND PRESSURE: HORIZONTAL FORCES.

109. **Wind Pressure on an Inclined Surface.** — When arched ribs are used, as is often the case, for the support of a roof, the pressure of the wind, being normal to the surface, will have a different effect upon the arch from that caused by a simple weight or vertical force. While referring to Part I., "Roofs," p. 31, for some remarks about the action of wind on a roof, we will repeat here, that, if P equals the horizontal force of the wind on a square foot of a vertical plane, the perpendicular pressure on a square foot of a surface inclined at an angle i to the horizon may be expressed by the empirical formula, —

$$P \sin i^{1.84 \cos i - 1}.$$

If, then, the maximum force of the wind be taken as forty pounds per square foot, which is an amount sufficiently great for the purposes of a design, the perpendicular or normal pressure per square foot, on surfaces inclined at different angles to the horizon, will be: —

Angle of Roof.	Normal Pressure.	Angle of Roof.	Normal Pressure.
5°	5.2 lbs.	35°	30.1 lbs.
10	9.6	40	33.4
15	14.0	45	36.1
20	18.3	50	38.1
25	22.5	55	39.6
30	26.5	60	40.0

For steeper pitches, the pressure may be taken as forty pounds.

The resultant pressure at each of the joints in the rafter which is on the side of the wind is then ascertained as in the case of any roof. If the roof surface is curved, any short portion between two points where braces abut, or purlins rest, may be considered as straight, and the wind force will then be perpendicular to such portion; this pressure being the only force exerted by the wind. If the resultant pressure at each joint is then found, either graphically or otherwise, and is resolved into vertical and horizontal components, we may include the vertical component in the analysis already carried out in detail. The effect of the horizontal component remains to be considered.

110. **Form of the Equilibrium Polygon; Vertical Component of Reaction.** — The tendency of such a force to distort the arch being resisted by the stiffness of the rib, the equilibrium polygon for a single horizontal force H, applied at any point I on the rib, Fig. 32, must, if the arch is hinged at the ends, be two straight lines, which start from the two springing points, and meet on the prolongation of the line of action of H; for the rib must be in equilibrium under H and the two forces at the abutments. In the case of the arch A C B of Fig. 32, the reactions at A and B must lie in the lines A G and B G, the point G being found on the horizontal line I G, but its location on that line being at present unknown. It will be evident, when we conceive H to be applied to the equilibrium polygon at G, that the side A G will be in *tension*, while G B is compressed: therefore the reaction at B will be a thrust, as usual, but that at A will be a tension; and, if H were the only applied force, the arch would tend to rise from the abutment A, and would require fastening down.

As H acts at a vertical distance I L above the springing line, the moment which tends to overturn the frame is H . I L. If we take either abutment as the axis of moments, the condition of equilibrium that the moments of exterior forces must balance gives H . I L = P . A B; and consequently the vertical component of the reaction at either abutment is, —

$$P = H \frac{IL}{AB},$$

being tension at the side nearer to I, and compression on the other side. H will be partially resisted at each abutment. The stress diagram will be a figure like 1 2 3, in which 3–4 and 4–1 are — P and H_1 for A, while 2–4 and 4–3 are H_2 and + P for B, 1–2 being equal to H.

111. **Rib hinged at Three Points.** — As was the case with arches under vertical forces only, so also with ribs under a wind pressure: the hinging of the rib at three points makes the analysis at once very simple. If the arch of Fig. 32 is pivoted or jointed at A, C, and B, C being usually taken at the crown of the rib, and the external horizontal force H is applied at I, the line of thrust for the right-hand portion of the arch must be B C. This will be plainly seen, if we consider that the part B E C of the rib is supported by a reaction at B and the thrust of the other half of the arch at C, while there is no other force exerted upon it: for equilibrium, therefore, these two forces must lie in one straight line, which can be no other than B C, drawn through the two points of application. Then, as proved before, the reaction at A must lie in A G, drawn to the intersection of H with B C. It may be noted that 1–4, or H_1, is always greater that one-half of H.

112. **Value of Bending Moments.** — If we make a section at any point E on the right of C, the only force acting on the right of the section is the inclined reaction at the abutment B. The bending moment at E is, therefore, equal to (3–2) E N, or to either of the equal products H_2 . E F and P . E K. The bending moment at any point between C and I, for the same reason, will still be expressed by H_2 . E F or P . E K, but will be of the opposite kind, since we passed a point of no bending moment at C, and E F or E K is drawn in a reverse direction. For sections between I and A it will be easier to take the force on the left

of the plane of section, which will be the tension of the left abutment, as this is the only force on that side: the bending moment will therefore be H_1 . E F or P . E K. It will be perceived, on a little reflection, that these moments will agree in kind with those between C and I; the reversal of the ordinate E F from the outside to the inside of the rib offsetting the change from H_2, compression, to H_1, tension. The application of H at I to a moderately flexible wire of the shape A C B would flatten the left portion, and make the right portion more convex.

We may very simply consider the bending moment at any point of the rib to be represented by the product P . E K, where E K is the horizontal distance or abscissa from E to the equilibrium polygon. We thus have an evident analogy between the equilibrium polygons for horizontal and for vertical forces, if the ordinate for bending moment is taken parallel to the applied force, and is then multiplied by a constant, P in this case, H in the other. The point of contraflexure is where the polygon meets the rib, and one point of maximum flexure is at I, the point of application of the external force.

The insertion of pivots at three points of the rib enables one to draw the equilibrium polygon at once for one or all of the forces to which the roof may be at one time subjected, and we will therefore proceed, without further delay, to consider the case of the parabolic rib hinged at the abutments only.

113. **Parabolic Rib hinged at Abutments; Formula for** x_0. — If Fig. 33 represents a parabolic rib hinged at A and B, with a horizontal force H applied at I, the point of intersection of A G and B N must be determined. Since it will lie upon the horizontal line drawn through I, the distance of G horizontally from the middle of the span will be denoted by x_0, positive when measured from the middle away from I. The well-known condition that change of span shall be zero may be put either

$$\Sigma H_2 . EF . DE \text{ (from B to I)} + \Sigma H_1 . EF . DE \text{ (from A to I)} = 0,$$

or

$$P . \Sigma EK . DE = 0, \quad (1.)$$

in which latter expression P, being constant, may be omitted. If b, as usual, denotes the horizontal distance of I, the point of application of the force, from the middle of the span, and c equals the half-span, we can find that

$$x_0 = \frac{b^3}{4\,c^4}(5\,c^2 - b^2) = \tfrac{1}{4}\,n^3\,(5 - n^2)\,c, \qquad (2.)$$

when $b = n\,c$. We shall see that x_0, depending for its sign upon that of b, will always be laid off on the opposite side of the centre from b, since it is so first taken in the figure, and hence that H_1, the horizontal *tension*, is always greater than one-half of H. The value of x_0 is independent of k.

114. **Proof of Formula.** — Retaining the usual notation, we have $A\,L = c - b$, $L\,B = c + b$; and $G\,Q = I\,L = \frac{k}{c^2}(c^2 - b^2)$. If x denotes the horizontal distance, B D, to the abutment, from any ordinate, D E, on the right of I we have

$$D\,E = \frac{k}{c^2}(2\,c\,x - x^2), \text{ and } D\,F : D\,B = G\,Q : Q\,B, \text{ or } D\,F = \frac{k}{c^2}(c^2 - b^2)\,\frac{x}{c - x_0}.$$

As E K : E F = Q B : G Q, and E F = D E − D F, we have

$$E\,K = (D\,E - D\,F)\,\frac{Q\,B}{G\,Q}, \text{ and } E\,K\,.\,D\,E = (D\,E^2 - D\,E\,.\,D\,F)\,\frac{Q\,B}{G\,Q}.$$

Substituting the values of these quantities, we get

$$\Sigma\,E\,K\,.\,D\,E = \int \frac{k}{c^2}\left[(2\,c\,x - x^2)^2 - (2\,c\,x - x^2)\,x\,\frac{c^2 - b^2}{c - x_0}\right]\frac{c - x_0}{c^2 - b^2}\,d\,x$$

as the expression which is applicable from B to I. From A to I the abscissa E K will be limited by the line A G, which differs in inclination from B C. If x, however, is now reckoned from A to the right, and A Q, denoted by $c + x_0$, is used in place of Q B, we have an expression for the space from A to I. This expedient was used in previous sections. As A G is in tension while B C is compressed, these two portions of (1.), § 113, will have opposite signs, and, when integrated, must be equal: we may, therefore, in equating, strike out the common constant quantities, obtaining

$$(c - x_0)\int_0^{c+b}(4\,c^2\,x^2 - 4\,c\,x^3 + x^4)\,d\,x - (c^2 - b^2)\int_0^{c+b}(2\,c\,x^2 - x^3)\,d\,x$$
$$= (c + x_0)\int_0^{c-b}(4\,c^2\,x^2 - 4\,c\,x^3 + x^4)\,d\,x - (c^2 - b^2)\int_0^{c-b}(2\,c\,x^2 - x^3)\,d\,x.$$

Performing the indicated integration, we get

$$(c - x_0)\left[\tfrac{4}{3}\,c^2\,(c+b)^3 - c\,(c+b)^4 + \tfrac{1}{5}\,(c+b)^5\right] - (c^2 - b^2)\left[\tfrac{2}{3}\,c\,(c+b)^3 - \tfrac{1}{4}\,(c+b)^4\right]$$
$$= (c + x_0)\left[\tfrac{4}{3}\,c^2\,(c-b)^3 - c\,(c-b)^4 + \tfrac{1}{5}\,(c-b)^5\right] - (c^2 - b^2)\left[\tfrac{2}{3}\,c\,(c-b)^3 - \tfrac{1}{4}\,(c-b)^4\right],$$

which at once reduces to

$$\tfrac{16}{15}\,c^5\,x_0 = \tfrac{4}{3}\,c^3\,b^3 - \tfrac{4}{15}\,c\,b^5,$$

or

$$x_0 = \frac{b^3}{4\,c^4}(5\,c^2 - b^2).$$

115. **Another Proof.** — We may, if we please, find the desired distance x_0 by another method. Imagine the roof of Fig. 34 to have two equal but opposite forces, H, applied at the two points C and G in the same horizontal line. These forces, if acting alone, will tend to diminish the span of the roof; there will be no vertical forces; and as the bending moments caused by them, in case the rib did not rest upon abutments, would be directly proportional to E F, the change of span would be proportional to Σ E F . D E from C to G. When the rib is retained by abutments, one H will give rise to H_1 at A, and H_2 at B: the other H will cause H_2 at A, and H_1 at B. As H_1 is always opposite in sign to H_2, the resultant force at each abutment will be $H_1 - H_2$, and is manifestly a tension exerted by the abutment on the rib. The change of span due to $H_1 - H_2$ will be proportional to $\Sigma\,D\,E^2$ from A to B (compare § 74), and this change of span must offset the one from H.

If D is at a distance x from the middle of the span, and C is distant b from the same point, we have $D\,E = \frac{k}{c^2}(c^2 - x^2)$, and $E\,F = \frac{k}{c^2}(b^2 - x^2)$. Since the rib is acted upon symmetrically, we need only integrate from the middle to one side; and we therefore have, when we drop the common factor $\frac{k}{c^2}$,

$$(H_1 - H_2)\int_0^c (c^2 - x^2)^2\,d\,x = H\int_0^b (b^2 - x^2)\,(c^2 - x^2)\,d\,x,$$

or

$$(H_1 - H_2)\tfrac{8}{15}c^5 = H\left(\tfrac{2}{3}b^3c^2 - \tfrac{2}{15}c^5\right). \quad (a.)$$

From the stress diagram of Fig. 33 we see that

$$H_1 : H_2 : H = c + x_0 : c - x_0 : 2c;$$

whence

$$H_1 - H_2 = H\frac{c + x_0 - c + x_0}{2c} = H\frac{x_0}{c}.$$

Substituting this value in (*a.*) we get, as before, § 114,

$$x_0 = \frac{b^3}{4c^4}(5c^2 - b^2).$$

116. **Formulæ for H_1 and P.** — The value of H_1 is seen to be, from the above proportion,

$$H_1 = H\frac{c + x_0}{2c} = H\left(\tfrac{1}{2} + \frac{x_0}{2c}\right) = H\left[\tfrac{1}{2} + \frac{b^3}{8c^5}(5c^2 - b^2)\right].$$

We also have, from Fig. 33,

$$P : H = GQ : AB = \frac{k}{c^2}(c^2 - b^2) : 2c;$$

or

$$P = H\frac{k}{2c^3}(c^2 - b^2) = H\frac{k}{2c}(1 - n^2).$$

The reader may now calculate, if desirable, numerical values of x_0, H_1, and P, for different values of b, as was previously done for vertical forces. The several values of x_0 for four different positions of H are plotted in Fig. 33.

117. **Shear and Direct Stress.** — The shear will undergo some modification when the force applied to the arch acts horizontally, instead of vertically. The stress diagram is, as we have seen, a triangle, whose base is H, and whose altitude is P, represented by 0 1 2 of Fig. 36. At A of the parabolic rib the *thrust* is 1–0: if 1–4 is drawn parallel to the tangent at A, and 0–8 perpendicular to it, 1–8 will be the direct thrust, and 8–0 the negative shear, on a right section at A. This shear will diminish at successive sections until we reach a point where the tangent to the rib is parallel to A G, when the shear will be zero, and the direct thrust 1–0. Beyond this point the shear will be positive until we pass I. At the abutment B, there is a *tension* 2–0: if 2–7 is drawn parallel to the tangent at B, 2–9 will be the direct tension, and 9–0 the shear, again negative, on a right section at B. In the same way the shear just to the left of I will be 10–0, positive, and to the right of I, 11–0, negative. It will be remembered that positive shear acts upward on the left of any section.

118. **Shear Diagram.** — A shear diagram may be drawn for a rib under a horizontal force by a similar method to the one previously explained, showing the *vertical* shear which will be projected on each right section. Lay off at a the quantity $P = 3–0 = af$, which is the vertical component of the reaction at A, and as P is constant across the entire span, being, in fact, the only external vertical force, complete the rectangle $a f d b$. The vertical component which is required at A to produce 1–4 is 3–4, laid off at $a e$; and at B is 3–7, laid off above the line at $b l$, because 0–2 is a tension. A load of uniform intensity horizontally being required to put a parabolic rib in equilibrium, and H_1 being constant as far as I, draw $e c g$ through c, the middle point of $a b$, and draw $l n$ so as to pass through c, if prolonged. Then will the vertical ordinates between the inclined lines and $f d$ represent the shear on a *vertical* section, and the projection of these ordinates on the respective normal sections will be the shear in the web. Thus ef is 4–0, which gives by projection 8–0, $i g$ is 0–5, and $i n$ is 0–6. As in previous diagrams, the ordinates will be measured from the inclined lines, positive above and negative below, as marked. The shear will change sign at the point of maximum bending moment, and it will plainly be equal to P at the crown of the arch.

If it is remembered that the abutment reaction at B is of the opposite kind to that at A, or to the usual reaction for a weight W, the rotation of the diagram on the right of i, from the customary position below the line to its present place above

$a\,b$, will be accounted for. The force H has been assumed on the right in Fig. 36, in order that this shear diagram may be compared with that of Fig. 8. The vertical shear from a normal force may be found from an addition of these two figures. Moment diagrams cannot be added together in the same way, as the values of H and H_1 or H_2 will not be the same in the two cases.

119. **Circular Rib hinged at Ends.** — The method of finding x_0, introduced in § 115, is easily applied to the circular rib hinged at the ends; while the process of § 114 is considerably more involved. Let the angle subtended, in Fig. 35, by the half arch of radius r be denoted by β; the angle from the crown to the point of application of the external horizontal force, H, be α; and the variable angle from the crown to any point be θ. Let H be applied at two opposite points at the same level, as shown by the arrows in the figure, and let the abutment reactions be $H_1 - H_2$. Then, by parallel reasoning to that of § 115, we have, if y denotes any ordinate, and a the ordinate to the point of application of H,

$$(H_1 - H_2)\int_0^\beta y^2\,d\,s = H\int_0^a (y - a)\,y\,d\,s.$$

$$y = r\,(\cos\theta - \cos\beta)\,;\quad a = r\,(\cos\alpha - \cos\beta)\,;\ \therefore$$

$$(H_1 - H_2)\,r^3\int_0^\beta (\cos^2\theta - 2\cos\theta\cos\beta + \cos^2\beta)\,d\,\theta$$
$$= H\,r^3\int_0^\alpha (\cos^2\theta - \cos\theta\cos\beta - \cos\theta\cos\alpha + \cos\alpha\cos\beta)\,d\,\theta.$$

Performing the integration, we get

$$(H_1 - H_2)\,(\tfrac{1}{2}\beta - \tfrac{3}{2}\sin\beta\cos\beta + \beta\cos^2\beta)$$
$$= H\,(\tfrac{1}{2}\alpha - \tfrac{1}{2}\sin\alpha\cos\alpha - \sin\alpha\cos\beta + \alpha\cos\alpha\cos\beta).$$

As in § 115, $x_0 = \dfrac{H_1 - H_2}{H}\,c = \dfrac{H_1 - H_2}{H}\,r\sin\beta$: whence

$$x_0 = r\sin\beta\,\frac{\alpha - \sin\alpha\cos\alpha - 2\cos\beta\,(\sin\alpha - \alpha\cos\alpha)}{\beta - 3\sin\beta\cos\beta + 2\beta\cos^2\beta}.\quad (1.)$$

If the rib is a semicircle, $\beta = \frac{1}{2}\pi$; $\cos\beta = 0$; $\sin\beta = 1$; and (1.) becomes,

$$x_0 = \frac{2\,r}{\pi}\,(\alpha - \sin\alpha\cos\alpha).\quad (2.)$$

120. **Formulæ for H_1 and P.** — The value of H_1 will be, as in § 116,

$$H_1 = H\,\frac{c + x_0}{2\,c} = H\,(\tfrac{1}{2} + \frac{x_0}{2\,r\sin\beta})$$
$$= \tfrac{1}{2}H\left(1 + \frac{\alpha - \sin\alpha\cos\alpha - 2\cos\beta\,(\sin\alpha - \alpha\cos\alpha)}{\beta - 3\sin\beta\cos\beta + 2\beta\cos^2\beta}\right),$$

and

$$P = \frac{H\,a}{2\,c} = \frac{\cos\alpha - \cos\beta}{2\sin\beta}\,H;$$

or, for a complete semicircle,

$$H_1 = \frac{\frac{1}{2}\pi + \alpha - \sin\alpha\cos\alpha}{\pi}\,H;\quad P = \tfrac{1}{2}\cos\alpha\,H.$$

121. **Experimental Verification.** — The values of x_0, obtained above, can be readily shown to be true by turning the model previously referred to through an angle of ninety degrees. A moderately stiff wire carefully bent to a curve A G B, Fig. 37, symmetrical with regard to the point G (an arc of a circle being probably the easiest one to fashion), is suspended from points C and D by strings from A to C, and from B to D. If the string B D is doubled so as to pass on both sides of the wire above G, A G B will be prevented from swinging round. A thread from A to B will hinder the span from enlarging, and will indicate by its slackening when the span is narrowed. If, then, a weight is attached at E, and, the string at C remaining stationary, that at D is moved until B is vertically below A, as proved by plumbing the thread A B, C A, when prolonged, will be found to intersect B D at F in the vertical line E F, giving the desired value of x_0. The point of intersection will be slightly changed by the weight of the wire, as before suggested in § 81. It is worthy of note that, H now being an external pull on the rib, in place of the usual thrust, x_0 will, in Fig. 37, be found on the same side of the centre with H.

122. **Parabolic Rib fixed at Ends; Formulæ for x_0, x_1, and x_2.** — Referring to Fig. 38, we will suppose that the external force H is applied at I, on the left of this parabolic rib with fixed ends; that the desired equilibrium polygon is given by the lines L G and N G C; and that the abscissæ, at present unknown, are, $AL = x_1$, $BN = x_2$, and $OQ = x_0$, the latter being measured from the middle of the span, and all being considered as positive when laid off as shown in this figure. The rest of the notation agrees with that used before. It may be proved that the abscissæ have the following easily computed values:

$$x_1 = \tfrac{1}{3}\left(c + \frac{4b^2}{c-b}\right); \quad x_2 = \tfrac{1}{3}\left(c + \frac{4b^2}{c+b}\right); \quad x_0 = 2\frac{b^3}{c^2},$$

or

$$x_1 = \tfrac{1}{3}c\left(1 + \frac{4n^2}{1-n}\right); \quad x_2 = \tfrac{1}{3}c\left(1 + \frac{4n^2}{1+n}\right); \quad x_0 = 2n^3c.$$

Several of these values, for different positions of H, are plotted in Fig. 38.

If b is given successive values from $0.1\,c$ to $0.9\,c$, these quantities will be found to be

b.	x_1.	x_0.	x_2.
$0.1\,c$	$0.35\,c$	$0.002\,c$	$0.35\,c$
.2	0.40	0.016	0.38
.3	0.50	0.054	0.43
.4	0.69	0.128	0.49
.5	1.00	0.250	0.56
.6	1.53	0.432	0.63
.7	2.51	0.688	0.72
.8	4.60	1.024	0.81
.9	11.17	1.442	0.90

If b exceeds $0.7\,c$, the point of intersection falls without the rib.

123. **First Equation of Condition.** — If we remark that Q G, Fig. 38, the ordinate to the line of action of H, will be equal to I S, or to $\frac{k}{c^2}(c^2 - b^2)$, and that $RK = DE$, we may find the value of E K as follows:

$$EK = RN - DN; \quad RN : RK = QN : QG, \text{ or } RN = \frac{RK \cdot QN}{QG};$$

therefore

$$EK = \frac{DE \cdot QN}{QG} - DN.$$

These quantities, in the notation employed, may be written, if x is measured from the right abutment,

$$DE = \frac{k}{c^2}(2cx - x^2); \quad QN = c + x_2 - x_0; \quad DN = x_2 + x; \quad QG = \frac{k}{c^2}(c^2 - b^2).$$

As $\frac{k}{c^2}$ will be a common factor in the equations which follow, we shall omit it. Substituting these values, we shall get, as the expression to be summed from B to I, for the first condition,

$$\Sigma EK \cdot DE = \int_0^{c+b}\left[\frac{c + x_2 - x_0}{c^2 - b^2}(4c^2x^2 - 4cx^3 + x^4) - (x_2 + x)(2cx - x^2)\right]dx.$$

If x is measured from the left abutment, L Q substituted for Q N, and x_1 written for x_2, we get an expression which is applicable from A to I, or

$$\Sigma EK \cdot DE = \int_0^{c-b}\left[\frac{c + x_1 + x_0}{c^2 - b^2}(4c^2x^2 - 4cx^3 + x^4) - (x_1 + x)(2cx - x^2)\right]dx.$$

As in § 114, these two expressions will be equated to make the change of span zero, and upon performing the indicated integrations, and multiplying through by $c^2 - b^2$, we obtain

$$(c + x_2 - x_0)\left[\tfrac{4}{3}c^2(c+b)^3 - c(c+b)^4 + \tfrac{1}{5}(c+b)^5\right] - (c^2 - b^2)\left[cx_2(c+b)^2 - \tfrac{1}{3}x_2(c+b)^3 + \tfrac{2}{3}c(c+b)^3 - \tfrac{1}{4}(c+b)^4\right] = (c + x_1 + x_0)\left[\tfrac{4}{3}c^2(c-b)^3 - c(c-b)^4 + \tfrac{1}{5}(c-b)^5\right] - (c^2 - b^2)\left[cx_1(c-b)^2 - \tfrac{1}{3}x_1(c-b)^3 + \tfrac{2}{3}c(c-b)^3 - \tfrac{1}{4}(c-b)^4\right].$$

This equation, by reduction and factoring, may be written,

$$8c^5x_0 - (c^5 - 5c^3b^2 + 5c^2b^3 - b^5)x_1 + (c^5 - 5c^3b^2 - 5c^2b^3 + b^5)x_2 = 10c^3b^3 - 2cb^5. \quad (1.)$$

124. **Second and Third Equations of Condition.** — The second condition, that the change of inclination at the abutments shall equal zero, is $\Sigma EK = 0$, and the portion of this expression from B to I will be,

$$\Sigma EK = \int_0^{c+b}\left[\frac{c + x_2 - x_0}{c^2 - b^2}(2cx - x^2) - (x_2 + x)\right]dx,$$