while from A to I we may write, as just explained,

$$\Sigma \mathrm{E\,K} = \int_0^{c-b} \left[\frac{c + x_1 + x_0}{c^2 - b^2}(2\,c\,x - x^2) - (x_1 + x)\right] d\,x.$$

Equating, integrating, and reducing, we get

$$\begin{aligned}(c + x_2 - x_0)\,[c\,(c+b)^2 - \tfrac{1}{3}(c+b)^3] - (c^2 - b^2)\,[x_2\,(c+b) + \tfrac{1}{2}(c+b)^2]\\ = (c + x_1 + x_0)\,[c\,(c-b)^2 - \tfrac{1}{3}(c-b)^3]\\ - (c^2 - b^2)\,[x_1\,(c-b) + \tfrac{1}{2}(c-b)^2];\end{aligned}$$

or

$$4\,c^3 x_0 - (c^3 - 3\,c\,b^2 + 2\,b^3)\,x_1 + (c^3 - 3\,c\,b^2 - 2\,b^3)\,x_2 = 4\,c\,b^3. \quad (1.)$$

In writing the third condition, that the abutment deflection shall equal zero, or $\Sigma \mathrm{E\,K} \,.\, \mathrm{D\,B} \doteq 0$, we must, if we use the values of E K already adopted, make D B equal to x on the right of I, and equal to $2\,c - x$ on the left of I. We then have, from B to I,

$$\int_0^{c+b} \left[\frac{c + x_2 - x_0}{c^2 - b^2}(2\,c\,x^2 - x^3) - (x_2 + x)\,x\right] d\,x,$$

and from A to I,

$$\int_0^{c-b} \left[\frac{c + x_1 + x_0}{c^2 - b^2}(4\,c^2\,x - 4\,c\,x^2 + x^3) - (x_1 + x)\,(2\,c - x)\right] d\,x.$$

Equating these two expressions and integrating, we find that

$$\begin{aligned}(c + x_2 - x_0)\,[\tfrac{2}{3}c\,(c+b)^3 - \tfrac{1}{4}(c+b)^4] - (c^2 - b^2)\,[\tfrac{1}{2}x_2\,(c+b)^2 + \tfrac{1}{3}(c+b)^3]\\ = (c + x_1 + x_0)\,[2\,c^2\,(c-b)^2 - \tfrac{4}{3}c\,(c-b)^3 + \tfrac{1}{4}(c-b)^4]\\ - (c^2 - b^2)\,[2\,c\,x_1\,(c-b) + \tfrac{1}{2}(2\,c - x_1)\,(c-b)^2 - \tfrac{1}{3}(c-b)^3],\end{aligned}$$

which reduces to

$$\begin{aligned}16\,c^4 x_0 - (7\,c^4 - 18\,c^2 b^2 + 8\,c\,b^3 + 3\,b^4)\,x_1 + (c^4 - 6\,c^2 b^2 - 8\,c\,b^3 - 3\,b^4)\,x_2\\ = -2\,c^5 - 4\,c^3 b^2 + 16\,c^2 b^3 + 6\,c\,b^4. \quad (2.)\end{aligned}$$

From (1.), § 123, and (1.) and (2.) of the present section, we may readily eliminate x_0, obtaining

$$(c^3 - b^3)\,x_1 - (c^3 + b^3)\,x_2 = 2\,c\,b^3,$$

and

$$(c^2 - b^2)\,x_1 + (c^2 - b^2)\,x_2 = \tfrac{2}{3}c^3 + 2\,c\,b^2,$$

whence may be deduced the formulæ of § 122.

125. **Formulæ for $\mathbf{H_1}$ and P.** — The values of H_1, H_2, and P, can now be scaled from the stress diagram, which will also give, if preferred, the proportion

$$H_1 : H_2 : H = c + x_1 + x_0 : c + x_2 - x_0 : 2\,c + x_1 + x_2,$$

or

$$H_1 = H\,\frac{c + x_1 + x_0}{2c + x_1 + x_2} = H\left[\tfrac{1}{2} + (5\,c^2 - 3\,b^2)\,\frac{b^3}{4\,c^5}\right] = \tfrac{1}{2}H\,[1 + \tfrac{1}{2}n^3\,(5 - 3\,n^2)].$$

H_1 will therefore always be greater than $\frac{1}{2}$ H.

Likewise we have, for the vertical component of the abutment reactions,

$$P : H = \frac{k}{c^2}(c^2 - b^2) : 2\,c + x_1 + x_2,$$

or

$$P = H \,.\, \tfrac{3}{8}k\,\frac{(c^2 - b^2)^2}{c^5} = \tfrac{3}{8}H\,\frac{k}{c}(1 - n^2)^2.$$

The shear diagram for this case will follow the explanation given in § 118.

126. **Circular Arch fixed at Ends.** — There remains to be considered the circular rib, fixed at the ends, under the action of an external horizontal force. The notation of the angles is the same as that previously used for the circular arch. As H is here applied at a point on the right side, x_0, measured from the middle of the span, will now lie on the left of the centre O. Then we will prove that

$$x_1 = \left[\frac{f}{a} - \frac{a\,b - d\,e}{a\,c}\left(\frac{f}{a} + \sin\beta\right)\right] r; \quad (1.)$$

$$x_2 = \left[\frac{f}{a} + \frac{a\,b - d\,e}{a\,c}\left(\frac{f}{a} + \sin\beta\right)\right] r; \quad (2.)$$

in which equations

$$a = \cos\alpha - \cos\beta, \qquad d = \beta\sin\alpha - \alpha\sin\beta,$$
$$b = \alpha\beta - \sin\alpha\sin\beta, \qquad e = 1 - \cos\alpha\cos\beta,$$
$$c = \beta^2 - 2\sin^2\beta + \beta\sin\beta\cos\beta, \qquad f = \beta - \cos\alpha\sin\beta.$$

It will be noticed that c is constant for a given arch. The value of x_0 can then be obtained from the equation

$$2\,(\sin\beta - \beta\cos\beta)\,x_0 - [\sin\beta + \sin\alpha - (\beta + \alpha)\cos\alpha]\,x_1 + [\sin\beta - \sin\alpha - (\beta - \alpha)\cos\alpha]\,x_2 = 2\,r\sin\beta\,(\sin\alpha - \alpha\cos\alpha). \quad (3.)$$

The distance x_1 and x_2 will, in every case, be laid off outwards from the abutments, and x_0 will be plotted away from the side where the force is applied. In these formulæ, x_1 is on the opposite side of the arch from the applied force, as is also H_1. In any case it is easy to distinguish between numerical values of x_1 and x_2, or H_1 and H_2, if we notice that the larger value belongs to the abutment which is nearer to the point of application of the external force.

Several of the equilibrium polygons have been drawn in Fig. 39 for a horizontal force applied at different distances from the crown. The angle β of this rib is 60°; and the computed values of the abscissæ, for H at points distant 10° successively from one another, are

α.	x_1.	x_0.	x_2.
10°	.3704 r	.0186 r	.4212 r
20	.4755	.0762	.5860
30	.5892	.2547	1.0345
40	.7291	.5950	2.1559
50	.8749	1.1339	5.9953

127. **First Equation of Condition.** — The processes to be followed are akin to those already given: although the work is somewhat more tedious, it presents no difficulty. As in § 123, we shall find that, Fig. 39,

$$\mathrm{EK} = \mathrm{RN} - \mathrm{DN} = \frac{\mathrm{DE}\,.\,\mathrm{QN}}{\mathrm{QG}} - \mathrm{DN}.$$ In the usual notation

$$\mathrm{DE} = r\,(\cos\theta - \cos\beta), \qquad \mathrm{QN} = r\sin\beta + x_2 + x_0,$$
$$\mathrm{QG} = r\,(\cos\alpha - \cos\beta), \qquad \mathrm{DN} = r\sin\beta + x_2 - r\sin\theta.$$

We therefore have

$$\mathrm{EK} = \frac{r\sin\beta + x_2 + x_0}{\cos\alpha - \cos\beta}\,(\cos\theta - \cos\beta) - (r\sin\beta + x_2 - r\sin\theta)$$

on the right of I. Upon the left of I, since E′ K now equals D′ L — R L, this expression will change in sign; and, since we measure from L, we must substitute x_1 in place of x_2, must subtract x_0 in place of adding it, and must change the sign of $r\sin\theta$: hence, on the left of I,

$$\mathrm{EK} = -\frac{r\sin\beta + x_1 - x_0}{\cos\alpha - \cos\beta}\,(\cos\theta - \cos\beta) + (r\sin\beta + x_1 + r\sin\theta).$$

The first condition, invariability of span, will now give,

$$\Sigma_{\alpha}^{\beta}\,\mathrm{EK}\,.\,\mathrm{DE} + \Sigma_{-\beta}^{\alpha}\,\mathrm{EK}\,.\,\mathrm{DE} = 0,$$

or, multiplying by $\cos\alpha - \cos\beta$,

$$r\int_{\alpha}^{\beta}[(r\sin\beta + x_2 + x_0)\,(\cos^2\theta - 2\cos\theta\cos\beta + \cos^2\beta) - (\cos\alpha - \cos\beta)\,(r\sin\beta + x_2 - r\sin\theta)\,(\cos\theta - \cos\beta)]\,d\theta$$
$$+ r\int_{-\beta}^{\alpha}[(r\sin\beta + x_1 - x_0)\,(-\cos^2\theta + 2\cos\theta\cos\beta - \cos^2\beta) + (\cos\alpha - \cos\beta)\,(r\sin\beta + x_1 + r\sin\theta)\,(\cos\theta - \cos\beta)]\,d\theta = 0.$$

The integration is similar to that already given for the circular rib in the earlier sections. There results, upon bringing together common factors,

$$(\beta - 3\sin\beta\cos\beta + 2\beta\cos^2\beta)\,x_0 - (\tfrac{1}{2}\beta + \tfrac{1}{2}\alpha - \tfrac{1}{2}\sin\beta\cos\beta - \tfrac{1}{2}\sin\alpha\cos\alpha - \sin\alpha\cos\beta - \cos\alpha\sin\beta + \beta\cos\alpha\cos\beta + \alpha\cos\alpha\cos\beta)\,x_1$$
$$+ (\tfrac{1}{2}\beta - \tfrac{1}{2}\alpha - \tfrac{1}{2}\sin\beta\cos\beta + \tfrac{1}{2}\sin\alpha\cos\alpha + \sin\alpha\cos\beta - \cos\alpha\sin\beta + \beta\cos\alpha\cos\beta - \alpha\cos\alpha\cos\beta)x_2 = r\sin\beta\,(\alpha - \sin\alpha\cos\alpha - 2\sin\alpha\cos\beta + 2\alpha\cos\alpha\cos\beta). \quad (1.)$$

128. **Second and Third Equations of Condition.** — The second condition, that $\Sigma_{\alpha}^{\beta}\,\mathrm{EK} + \Sigma_{-\beta}^{\alpha}\,\mathrm{EK} = 0$, similarly gives,

$$\int_{\alpha}^{\beta}[(r\sin\beta + x_2 + x_0)\,(\cos\theta - \cos\beta) - (\cos\alpha - \cos\beta)(r\sin\beta + x_2 - r\sin\theta)]\,d\theta$$
$$+ \int_{-\beta}^{\alpha}[(r\sin\beta + x_1 - x_0)\,(-\cos\theta + \cos\beta) + (\cos\alpha - \cos\beta)\,(r\sin\beta + x_1 + r\sin\theta)]\,d\theta = 0.$$

From this equation we obtain, by integrating and factoring,

$$(2\sin\beta - 2\beta\cos\beta)\,x_0 - (\sin\beta + \sin\alpha - \beta\cos\alpha - \alpha\cos\alpha)\,x_1 + (\sin\beta - \sin\alpha - \beta\cos\alpha + \alpha\cos\alpha)\,x_2 = r\sin\beta\,(2\sin\alpha - 2\alpha\cos\alpha). \quad (1.)$$

The third condition, that $\Sigma_{a}^{\beta} \mathrm{EK} \,.\, \mathrm{DB} + \Sigma_{-\beta}^{a} \mathrm{EK} \,.\, \mathrm{DB} = 0$, will give, when we introduce the value of $\mathrm{DB} = r(\sin\beta - \sin\theta)$,

$$r\int_{a}^{\beta} [(r \sin\beta + x_2 + x_0)(\cos\theta - \cos\beta)(\sin\beta - \sin\theta) - (\cos a - \cos\beta)(r\sin\beta + x_2 - r\sin\theta)(\sin\beta - \sin\theta)]\,d\theta + r\int_{-\beta}^{a}[(r\sin\beta + x_1 - x_0)(-\cos\theta + \cos\beta)(\sin\beta - \sin\theta) + (\cos a - \cos\beta)(r\sin\beta + x_1 + r\sin\theta)(\sin\beta - \sin\theta)]\,d\theta = 0.$$

Operating upon this equation also, we find that

$$(2\sin^2\beta - 2\beta\sin\beta\cos\beta)\,x_0 - (\sin^2\beta + \sin a\sin\beta - \tfrac{1}{2}\cos^2\beta - \tfrac{1}{2}\cos^2 a + \cos a\cos\beta - \beta\cos a\sin\beta - a\cos a\sin\beta)\,x_1 + (\sin^2\beta - \sin a\sin\beta + \tfrac{1}{2}\cos^2\beta + \tfrac{1}{2}\cos^2 a - \cos a\cos\beta - \beta\cos a\sin\beta + a\cos a\sin\beta)\,x_2 = r\sin\beta\,(2\sin a\sin\beta - \cos^2 a + \cos a\cos\beta - 2a\cos a\sin\beta) + r\beta(\cos a - \cos\beta). \quad (2.)$$

129. **Reduction.** — From (1.), § 127, and (1.) and (2.), § 128, we can determine the desired quantities x_0, x_1, and x_2, by any of the usual steps for elimination. If the second equation of condition is multiplied by $\sin\beta$, and then subtracted from the third, there will result

$$(\tfrac{1}{2}\cos^2\beta - \cos a\cos\beta + \tfrac{1}{2}\cos^2 a)(x_1 + x_2) = r\sin\beta(\cos a\cos\beta - \cos^2 a) + r\beta(\cos a - \cos\beta),$$

which, upon being divided by $\tfrac{1}{2}(\cos a - \cos\beta)$, becomes

$$(\cos a - \cos\beta)(x_1 + x_2) = 2r(\beta - \cos a\sin\beta). \quad (a.)$$

Again: the second equation may be multiplied by $\cos\beta$, and added to the first, after which the values of x_0 from the new equation and from the second equation of condition may be equated. If we then clear of fractions, and factor the resulting equation, it may be written

$$[a(b-c) - de]\,x_2 + [a(b+c) - de]\,x_1 = -2r\sin\beta\,(ab - de), \quad (b.)$$

while equation (*a.*) will be

$$a(x_1 + x_2) = 2fr; \quad (c.)$$

in which equations the literal coefficients stand for the quantities already given in § 126.

From (*b.*) and (*c.*) it is easy for one to obtain the half sum and the half difference of the two unknown quantities, and thence equations (1.) and (2.), § 126. Equation (3.) is identical with (1.), § 128.

130. **Formulæ for H_1, &c.; Semicircular Arch.** — To find the values of H_1, H_2, and P by formula, we make use of similar expressions to those of § 125. The figure gives us

$$H_1 : H_2 : H = r\sin\beta + x_1 - x_0 : r\sin\beta + x_2 + x_0 : 2r\sin\beta + x_1 + x_2;$$

or

$$H_1 = H\frac{r\sin\beta + x_1 - x_0}{2r\sin\beta + x_1 + x_2} = \frac{a}{2r}\cdot\frac{r\sin\beta + x_1 - x_0}{\beta - \sin\beta\cos\beta}\,H.$$

$$P : H = r(\cos a - \cos\beta) : 2r\sin\beta + x_1 + x_2 = ar : 2r\sin\beta + \frac{2fr}{a};$$

or

$$P = \tfrac{1}{2}\frac{a^2}{a\sin\beta + f}H = \tfrac{1}{2}H\frac{(\cos a - \cos\beta)^2}{\beta - \sin\beta\cos\beta}.$$

If the arch subtends a semicircle, $\beta = \tfrac{1}{2}\pi$, $\sin\beta = 1$, $\cos\beta = 0$, and the preceding values are much simplified. Without writing them in detail, it will be sufficient to indicate that then

$$a = \cos a, \qquad c = \tfrac{1}{4}\pi^2 - 2, \qquad e = 1,$$
$$b = \tfrac{1}{2}\pi a - \sin a, \qquad d = \tfrac{1}{2}\pi\sin a - a, \qquad f = \tfrac{1}{2}\pi - \cos a.$$

131. **Sign of Bending Moment.** — In determining the sign of the bending moment at any point when the arch is acted upon by a horizontal force, it will be well for the reader to recollect, that, when there is a thrust along any portion of the equilibrium polygon, the arched rib tends to move away from the polygon, but, when there is tension in any portion, the arch moves towards the polygon. This tendency to move in one direction or the other is easily fixed in the mind, if one thinks of the alteration of curvature of a bent wire when a force is applied at each end in the line joining the two ends. The same thing was noticed in the suspended arch of Fig. 1 and in those under vertical forces. Therefore, in Fig. 32 and the following

ribs, the arch tends to approach the tension side of the equilibrium polygon, and to recede from the compression side. If then, as before, that moment which makes any portion of the rib less curved, or which, if exerted on a beam supported at both ends, would make it concave on the upper side, be called positive, the areas of — M will occur between B and C in Figs. 32 and 33, and those of + M will be found between C and A. Ribs fixed at the ends will be strained similarly. In Fig. 38, for example, the area to the right of B will give + M; from the point where N G crosses the rib to C there will be — M, which then changes to + M on the left of C, and to — M, when the polygon crosses the rib above A.

132. **Example of Normal Forces.** — As we have now ascertained the values of the abutment reactions when a rib is acted upon by a horizontal force, we will show, by an example, that the various horizontal and vertical forces which are exerted at one time at different points of the rib may be provided for in one polygon, without the necessity for separate treatment of the horizontal and vertical components into which the normal or oblique external forces can be decomposed. We will suppose that a parabolic rib of 100 feet span and 50 feet rise is to be used as a principal to carry a roof, and that it is desired to ascertain the bending moments arising from the action of the wind upon one side. We will take the case where the rib is fixed at the ends as being less simple. After this discussion, the reader will have no difficulty in applying a similar treatment to other ribs.

Let the rib be represented by A C B, Fig. 40, and let us suppose that the normal wind pressure is directly resisted by the flanges and bracing of the rib at points D, E, F, and G, at which purlins rest, and which are distant 40 feet, 30 feet, 20 feet, and 10 feet horizontally from the middle of the span. The amount of the pressure N_2 at E will be the total or resultant of the distributed pressure on *m n*, the points *m* and *n* being taken midway of the spaces on each side of E. There will be no error of consequence in assuming that the wind pressure on *m n* is

perpendicular to the straight line *m n*, or to the tangent of the parabola at E.* To find this tangent, draw E E′ horizontally, make C E′′ = C E′, and E E′′ will be the desired tangent. The tangents at the other points are found in the same way. The angle E′ E E′′ is very nearly 50°; the intensity of wind pressure, by the table of § 109, is 38 pounds on the square foot of roof; and if the principals are 10 feet apart, and *m n* is 15½ feet, the total normal force N_2 at this point will be $38 \times 10 \times 15\frac{1}{2} =$ 5,890 pounds. For the four points we therefore find in detail

						N.	V.	H.
1	58°	40	× 19	× 10	=	7,600 lbs.	4,000 lbs.	6,400 lbs.
2	50	38	15½	10		5,890	3,800	4,500
3	38½	32	13	10		4,160	3,200	2,600
4	22	20	11	10		2,200	2,000	900

These normal forces are plotted on the figure, and then decomposed graphically into their vertical and horizontal components, which, scaled to the nearest one hundred pounds, are found above in the columns headed V and H. The figure and diagrams are drawn to scales of forty feet and ten thousand pounds equal one inch.

133. **Finding the Reactions.** — The next step will be to find the values of H_1, H_2, P_1, and P_2, for the above forces. First, upon referring to § 64, we see that a vertical force at E, Fig. 40, 0.6 *c* from the middle of the span, will cause a vertical reaction of 0.896 V at A, one of 0.104 V at B, and will give rise to H, at each abutment, of the amount $0.192 \frac{c}{k} V = 0.192 V$. We also see, by the table of § 62, that the ordinate at A will be $-0.667 k$, and at B $+0.333 k$, for the same force at E; and we can then obtain the values of M at the abutments arising from V by multiplying these ordinates by $H = 0.192 V$, just ascertained. The computations for the four loaded points may be grouped together as follows:

* If preferred, analyze the wind pressures as in Part I., Roofs, p. 44.

	V.		P₁.		H.	
1	4,000	× 0.972 =	3,888 lbs.	V × .0607 =	243 lbs.	
2	3,800	0.896	3,405	.1920	730	$V = 13{,}000$
3	3,200	0.784	2,509	.3308	1,059	$P_1' = 11{,}098$
4	2,000	0.648	1,296	.4320	864	$P_2' = 1{,}902$ lbs.
	13,000		$P_1' = 11{,}098$ lbs.		$H' = 2{,}896$ lbs.	

	H.	y_1.	M_1.	y_2.	M_2.
1	243 ×	$-2.000\,k$ =	$-24{,}300$ ft. lbs.	$0.370\,k$	$+4{,}495$ ft. lbs.
2	730	-0.667	$-24{,}333$	0.333	12,167
3	1,059	-0.222	$-11{,}767$	0.286	15,144
4	864	0.000	000	0.222	9,600
Totals . . .			$M_1' = -60{,}400$ ft. lbs.		$M_2' = +41{,}406$ ft. lbs.

It is to be understood that y_1, P_1, and M_1 refer to the left abutment, the others, to the right abutment.

From § 122 and § 125 we now compute the reactions from the horizontal forces at the four loaded points, and the accompanying bending moments:

	H.		± P.			
1	6,400	× .0486 =	311 lbs.	H × 0.894 =	5,722 lbs.	
2	4,500	.1536	691	0.712	3,204	$H = 14{,}400$
3	2,600	.2646	688	0.572	1,487	$H_1' = 10{,}872$
4	900	.3456	311	0.510	459	$H_2' = +3{,}528$ lbs.
	14,400					
Totals, P′ from H's			= ± 2,001 lbs.		$H_1' = -10{,}872$ lbs.	

	P.	x_1.	M_1.	x_2.	
1	311 ×	$4.600\,c$ =	$-71{,}530$ ft. lbs.	$0.807\,c$	$+12{,}549$ ft. lbs.
2	691	1.533	$-52{,}976$	0.633	21,870
3	688	0.689	$-23{,}702$	0.486	16,718
4	311	0.400	$-6{,}220$	0.378	5,878
Totals			$M_1' = -154{,}428$ ft. lbs.		$M_2' = +57{,}015$ ft. lbs.

The final abutment moments will be

$$M_1' = -60{,}400 - 154{,}428 = -214{,}828 \text{ ft. lbs.}$$

$$M_2' = 41{,}406 + 57{,}015 = +98{,}421 \text{ ft. lbs.}$$

The components of the reaction at A are, if thrusts are considered positive,

$$P_1' = P_1 - P = 11{,}098 - 2{,}001 = +9{,}097 \text{ lbs.}$$

$$H_1' = H + H_1 = 2{,}896 - 10{,}872 = -7{,}976 \text{ lbs.}$$

The components at B will be

$$P_2' = P_2 + P = 1{,}902 + 2{,}001 = +3{,}903 \text{ lbs.}$$

$$H_2' = H + H_2 = 2{,}896 + 3{,}528 = +6{,}424 \text{ lbs.}$$

The arrows at A and B show these reactions. If the rib consists of chords and bracing, the stresses on the pieces can be found by a diagram like Fig. 21, Part I., "Roofs," care being taken to have the stresses in the two flanges at the abutment give the proper reaction (see § 195). If the equilibrium polygon is to be drawn, from which to find bending moments and chord stresses, we need the point of beginning for the polygon.

The abscissa, or ordinate to the equilibrium polygon at A, will be found by dividing the total M at that point by P_1' or H_1'; and similarly for the abutment B; thus,

$$x_1' = \frac{-214{,}828}{+9{,}097} = -23.6 \text{ ft.} \qquad x_2' = \frac{+98{,}421}{+3{,}903} = +25.2 \text{ ft.}$$

$$y_1' = \frac{-214{,}828}{-7{,}976} = +27.0 \text{ ft.} \qquad y_2' = \frac{+98{,}421}{+6{,}424} = +15.3 \text{ ft.}$$

As in previous examples, the ordinate at one abutment alone is needed; but the others are useful as a check on the accuracy of the drawing.

134. **Equilibrium Polygon; Bending Moments.** — We may now proceed to draw the stress diagram. Lay off 1–2, 2–3, 3–4 and 4–5, parallel successively to the external forces at G, F, E,

and D, and equal to the calculated amounts by any desirable scale; make $5\text{–}6 = H_1'$, and $6\text{–}0 = P_1'$, so that 5–0 shall represent the reaction at A in the proper direction as expressed by the signs obtained above, P_1' being a compression, and H_1' a tension; finally, lay off $0\text{–}7 = P_2'$, and $7\text{–}1 = H_2'$, giving 0–1 for the reaction at B. The closing of 0–1 on the point 1 proves that the diagram has been drawn with care. Having drawn $BQ = + y_2'$, or $BR = + x_2'$, draw through Q or R a line parallel to 0–1, as far as O, where it meets the normal force at G. Then draw O L parallel to 0–2, to cut the force N_3 at L. Follow with L K and K I, parallel to 0–3 and 0–4, closing with a line through I, parallel to 0–5, which, if the polygon has been accurately drawn, will make $AW = y_1'$, as recently computed, or $AU = -x_1$.

As neither H nor P is constant for *oblique* forces on an arch, the bending moment at any point will equal the product of the force acting along a side of the polygon just drawn multiplied by the perpendicular from the point to the side: thus the bending moment at E is $ES \times (0\text{–}3)$, or $ET \times (0\text{–}4)$. If the external forces had been considered as applied at a greater number of points, or as distributed along the principal rafter itself, we should have obtained a polygon which approached nearer to a regular curve, and such a curve has been sketched through the vertices of the polygon just drawn.

135. **Equilibrium Polygons for the Vertical and Horizontal Components.** — Since most of the needful data have already been obtained, we have thought it expedient to draw the equilibrium polygons for the vertical and horizontal components separately, so that they may be compared with each other and with the polygon for normal forces. If a horizontal and a vertical line are drawn from 1 and 5, the components H and V can be at once projected upon them. Upon laying off H_1, and plotting P, we shall locate the pole 0″; and 0″–2″, 0″–3″, &c., will be parallel to the lines of the polygon for horizontal forces. In the same way, P_1 and H for vertical forces will determine 0′. The value of y_2 will be found, upon dividing the M_2 which

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comes from V by H, to be 14.3 feet, giving the starting-point just below Q. Upon drawing the polygon so that the angles are made at the verticals through the loaded points, we obtain the broken line which finally runs below A. This ordinate y_1 may be verified. If M_2 from the H's is divided by P, we have $x_2 = 28.5$ feet, an ordinate a little longer than B R. The polygon, if now drawn, will be the broken line which passes near E′, and extends to a considerable distance, 77.2 feet, to the left of A. All the sides of this polygon except the first are in tension.

136. **Shear and Direct Stress.** — To complete this example, the normal shear at the middle of each division is found, and at the same time the direct stress. The small letters *l*, *m*, *n*, &c., mark the middle of each division. Draw 0–*l* in the stress diagram, parallel to the tangent at *l* in the rib, and 5–*l* perpendicular to it; then will 5–*l* be the normal shear at *l*, and *l*–0 the direct thrust. To satisfy ourselves in regard to the sign of this shear, we note that 5–0 is the thrust in the side U I of the equilibrium polygon, and will therefore be the resultant force on the left of any section between A and D; the forces 5–*l* and *l*–0, in the directions named, will be its components, also on the left of the section *l*: hence we have *positive* shear and a direct thrust. In the same way at *m*, since 4–0 is the thrust in I K, 4–*m* will be the positive shear, and *m*–0 the direct thrust. Between *m* and *n* the shear changes sign; for at *n* we find 3–*n* and *n*–0, the former being drawn *down*, instead of *up*. Passing on, we see that the shear again changes between *r* and *s*, because 1–*r* and 1–*s* run in opposite directions. As noted before, this change of sign occurs at points of maximum bending moment.

137. **Vertical Shear Diagram.** — We may draw a vertical shear diagram, if desired, and from that obtain the normal components; but it is not so conveniently constructed in the case of several forces which are always applied together as for a case of a single load. If *a b* represents the span, P_1' or 6–0 is laid off at *a w*, upwards as usual; then the subtraction of V_1 at D, or 4′–5, brings us to the line *d*; thence a step is made to *e*, to *f*, and finally to *g*, closing at *b* with 0–7, the reaction at B. The horizontal line below *a b* cuts off P, or 0″–3″, so that the vertical components shown in the line 5–1′

might be considered as laid off from this lower line, and the constant quantity P, due to the horizontal components, then subtracted. As the thrust at B is 0–1, a line drawn through 0, parallel to the tangent at B, will cut off from a vertical line drawn from 1 as much vertical force as is required, in addition to 0–7, to give a resultant in the direction of the rib at B. The amount so determined is laid off at $q' r'$. Since it has been shown that all inclined lines are drawn towards the middle of the span c, and are uninterrupted until an external force is encountered, we draw through c the line $r' c s$.

In a similar way, a line 0–10 from 0, parallel to the tangent at A, will cut the vertical through 5 at a distance 5-10, equal to $w u$; a line from 0, parallel to the tangent at D, will cut off the distance from a vertical through 4, which is plotted from d to k; one parallel to the tangent at E will cut off 3–8, which is plotted at $e o$; and the tangent at F gives 0–9, so that 2–9 is laid off at $f p$. If inclined lines are drawn through the points thus found, running towards the point c, the diagram will be completed. Normal components of the ordinates between the two sets of lines just constructed, measured above l, m, n, &c., will agree with the values of the last section, — positive when above the inclined lines, negative when below.

CHAPTER IX.

STONE ARCHES.

138. **Location of Equilibrium Curve determines Thickness of Voussoirs.** — Stone arches may be treated as belonging to the class of ribs with fixed ends, as the voussoirs have sufficient breadth at the skew-backs to make a firm bearing. We can, then, for a given rise, span, and distribution of steady and travelling load, draw the equilibrium curve, and thence determine the required thickness of the arch-ring. To repeat what was mentioned incidentally earlier: if no reliance is placed upon the tenacity of the cement, and if the intensity of pressure at a joint between any two voussoirs or arch-stones is considered to vary uniformly from the outside to the inside edge, the extreme case of deviation of the resultant pressure from the middle of the joint consistent with safety will occur when the pressure is zero at one edge. As the varying intensity of pressure will be represented by the ordinates to an inclined line which passes through the point where the pressure is zero, the total pressure will be equal to the area of a triangle, and the resultant will pass through the centre of gravity of the triangle, or at a distance of one-third the breadth of the ring from that edge where the pressure is most intense. Since the equilibrium curve is the locus of the resultant force at each joint, the condition that the pressure shall never be less than zero at any point, or that there shall be no tension, is equivalent to requiring that the equilibrium curve shall never pass beyond the middle third of the