might be considered as laid off from this lower line, and the constant quantity P, due to the horizontal components, then subtracted. As the thrust at B is 0–1, a line drawn through 0, parallel to the tangent at B, will cut off from a vertical line drawn from 1 as much vertical force as is required, in addition to 0–7, to give a resultant in the direction of the rib at B. The amount so determined is laid off at $q' r'$. Since it has been shown that all inclined lines are drawn towards the middle of the span c, and are uninterrupted until an external force is encountered, we draw through c the line $r' c s$.

In a similar way, a line 0–10 from 0, parallel to the tangent at A, will cut the vertical through 5 at a distance 5-10, equal to $w u$; a line from 0, parallel to the tangent at D, will cut off the distance from a vertical through 4, which is plotted from d to k; one parallel to the tangent at E will cut off 3–8, which is plotted at $e o$; and the tangent at F gives 0–9, so that 2–9 is laid off at $f p$. If inclined lines are drawn through the points thus found, running towards the point c, the diagram will be completed. Normal components of the ordinates between the two sets of lines just constructed, measured above l, m, n, &c., will agree with the values of the last section, — positive when above the inclined lines, negative when below.

CHAPTER IX.

STONE ARCHES.

138. **Location of Equilibrium Curve determines Thickness of Voussoirs.** — Stone arches may be treated as belonging to the class of ribs with fixed ends, as the voussoirs have sufficient breadth at the skew-backs to make a firm bearing. We can, then, for a given rise, span, and distribution of steady and travelling load, draw the equilibrium curve, and thence determine the required thickness of the arch-ring. To repeat what was mentioned incidentally earlier: if no reliance is placed upon the tenacity of the cement, and if the intensity of pressure at a joint between any two voussoirs or arch-stones is considered to vary uniformly from the outside to the inside edge, the extreme case of deviation of the resultant pressure from the middle of the joint consistent with safety will occur when the pressure is zero at one edge. As the varying intensity of pressure will be represented by the ordinates to an inclined line which passes through the point where the pressure is zero, the total pressure will be equal to the area of a triangle, and the resultant will pass through the centre of gravity of the triangle, or at a distance of one-third the breadth of the ring from that edge where the pressure is most intense. Since the equilibrium curve is the locus of the resultant force at each joint, the condition that the pressure shall never be less than zero at any point, or that there shall be no tension, is equivalent to requiring that the equilibrium curve shall never pass beyond the middle third of the

arch-ring, however the distribution of the load may be varied: hence, when the equilibrium curves are drawn, the thickness of the voussoirs is readily determined. The tensile strength of the cement after it has become firm, and any deviation from the assumption that the force between two stones must be distributed over the whole joint, increase the safety of the structure, and thus give what is akin to the factor of safety in other cases.

139. **Intensity of Pressure.** — When the stability of the arch-ring is thus assured, it is an easy matter to find the greatest intensity of pressure, and hence to see whether the material proposed for the arch will have strength enough. When the equilibrium curve passes through the centre of the joint, the pressure on the square inch will be found by dividing the thrust at that joint by the area of the bearing surface. If the curve touches the extreme limit, the edge of the middle third, the most intense pressure, at the edge of the joint nearest to the curve, will be twice the mean pressure; for the height of the triangle whose ordinates represent the varying intensities is twice its mean ordinate. In some rare cases, where the span is large, and the stone is of a weak quality, we may have to increase the depth of the arch-ring in order to provide sufficient strength.

140. **Circular Arch; Load for Equilibrium.** — Although the curve of the arch-ring may be any one of a number of forms, the circular arch is the more common type, and we have therefore thought it best to take such an arch as an example of this method: the steps will apply to any form. The Gothic arch will be classed with the example of § 194. If the load is entirely, or almost entirely, steady, as in the aqueduct or canal bridge, it will be advisable, on the score of economy, to find that distribution of the load which shall cause the equilibrium curve to coincide with the centre line of the arch-ring. Then, by arranging the filling and the empty spaces above the arch-ring so as to conform to that distribution, the voussoirs can be made of moderate depth.

Thus, if B C, Fig. 45, be one-half of an arch which it is desired to load in this way, divide it, by vertical lines, into quite a large number of parts, equal horizontally. If the divisions are small, the areas of these portions between the soffit of the arch and the upper line may be considered trapezoids, and the middle ordinate of each division will be proportional to its volume for unity of thickness, and to its weight, if homogeneous. It is then evident, that, if there is to be no bending moment at any point, the equilibrium curve must coincide, either with the tangents to the centre line of the ring at these loaded points, or with the chords drawn between these points, according as the first loaded point is taken at half a division's distance from the abutment, or at the abutment itself. See Part II., "Bridges," § 58. Let this weight be concentrated, in imagination, on each middle ordinate.

Upon drawing, from any point 0, radiating lines parallel to the tangents, or perpendicular to the radii, at the successive points of division, and cutting them all by a vertical line 1–12 at any convenient distance, loads in each division, supposed to be concentrated at the intersection of the above tangents,* and proportional to the several portions of the vertical line intercepted by the inclined lines, will be the ones required for equilibrium; and the distributed loads spread over all of each division, or, in other words, a continuous load over the whole arch, will thus be found. If 1–2 is placed at such a distance from 0 that it will represent, by a convenient scale, the mean depth, as well as the weight of the load, in the first division on the right of C, 2–3, 3–4, &c., will represent the required depth of loading in the succeeding divisions. As the angle made by 0–2 with the horizontal line is the same as that subtended at the centre by the first division near C, there is no difficulty in finding, by calculation, the exact length of 0–1, when 1–2 is given, in case the angle at 0 is too acute to give any accurate result graphically. In our figure the depth of the load at the

* The tangents will not intersect exactly in the middle of each division.

crown was assumed to be five feet, and the intercepted portions of the vertical line were then plotted from the points where verticals at the middle of each division would cut the centre line of the arch. The curved line drawn through the upper ends of these ordinates will then show the desired amount of homogeneous load to be spread over the arch to produce equilibrium.

141. **Limiting Angle for Arch-Ring without Backing.** — It is now worthy of notice, that, while the required depth of loading increases but slowly for some distance after we leave the crown, when we reach the haunches, the ordinates rapidly lengthen, and the curve through their upper ends will finally become vertical, if the arch springs vertically from the abutment. This point was also referred to in § 89. It is apparent, therefore, that it is not practicable to so load with vertical forces a circular arch, beyond a certain distance from the crown, that the line of thrust shall coincide with the centre line of the arch-ring. As the roadway must not deviate greatly from a horizontal line, we see, that, for an arch extending 60° each way from the crown, the amount of material as heavy as masonry required over the springing will fill all of the available space, and, when the spandrel filling is lighter, the limiting angle will probably be in the neighborhood of 45°. In ordinary cases of loading, the equilibrium curve will deviate so much from the centre line in this portion of the rib as to require very deep voussoirs to retain the curve within the middle third when the attempt is made to extend the unassisted arch-ring much farther. It is customary, therefore, to carry the masonry backing, in horizontal courses, up to the neighborhood of the point where the arch-ring is inclined at an angle of 45°: below this point any attempt of the arch-ring to move outwards under the thrust of the upper portion is immediately resisted by the backing, and the arch will be designed as if the springing points were at the joints level with the top of this masonry backing. The portion below really forms a part of the abutment.

142. **Example; Data.** — In accordance with the above statements, and as an example of the application of preceding principles, we propose to design a circular segmental arch of stone, for a railroad bridge, which shall subtend 100°, with a radius, for the centre line of the voussoirs, of 100 feet, making the span, from centre to centre of skew-backs, about 153 feet, and the rise about 36 feet. The rolling load will be 3,000 pounds per running foot of track, and the width of the bridge over which this load is distributed will be ten feet. The backing will be carried up to the point where the rib is inclined at 45°, and the remainder of the spandrel will be filled with such material, or will have such an amount and distribution of empty spaces, that it shall weigh, on the average, one-half as much per cubic foot as does the masonry of the arch-ring. The equilibrium curve for steady load will now first be found; then such possible combinations of rolling load will be discussed as will increase the deviation of the steady load curve at those points where it already deviates most from the centre line of the arch-ring; and, finally, the necessary depth of the voussoirs will be determined by the rule suggested in § 138. The depth of the voussoirs at the crown is assumed, in our present ignorance of the final dimensions, at five feet; two feet of filling, earth or some other material, is added at that point, and the horizontal line drawn seven feet above the soffit at the crown will be the upper boundary of the spandrel filling. If, then, the arch-ring is taken at a uniform thickness of five feet, as shown at A C, on the left half of Fig. 45, the depth of a homogeneous load equal to stone will be found by shortening each ordinate above the arch ring one-half. Thus was obtained the curve D E. By dividing the area between this curve and the soffit into small portions by vertical lines, we may find the weight to be concentrated on the several assumed loaded points of the arch-ring.

143. **Calculations for Steady Load.** — From the equations of § 92, after making $\beta = 45°$, and giving to α the successive values, 5°, 10°, 15° . . . 40°, we have worked out the quantities y_1, y_0, and y_2, for a weight at such distances from the crown, and

these quantities are given in the first portion of the following table, it being understood that the weights are here placed on the left of the crown to correspond with our figure: —

	a.	y_1.	y_0.	y_2.	H.	P_1.	P_2.
C	0°	.0449 r	.3587 r	.0449 r	1.126 W	.5 W	.5 W
F	5	.0252	.3585	.0607	1.095	.596	.411
G	10	.0001	.3578	.0735	1.007	.683	.325
I	15	—.0341	.3569	.0842	0.866	.760	.244
K	20	—.0817	.3555	.0930	0.690	.830	.172
L	25	—.1536	.3537	.1012	0.498	.890	.111
N	30	—.2730	.3515	.1078	0.311	.939	.063
O	35	—.5137	.3487	.1142	0.150	.972	.027
P	40	—1.2407	.3470	.1183	0.040	.993	.007

These values of y_1, y_0, and y_2, have been plotted on the arch of Fig. 44, and the several stress diagrams have been drawn on a vertical line which represents W. From this figure the amounts of H and of the vertical components of the abutment reactions for a load W at successive points can be scaled off, and thus we obtain the last three columns of the above table. H, P_1, and P_2, can also be easily calculated by the formulæ of § 63, if we make $c = r \sin \beta$, and $b = r \sin \alpha$.

Having divided the centre line C A of the arch-ring of Fig. 45 at points C, F, G, &c., distant five degrees from one another, the weight to be concentrated at each of these loaded points is next computed, for an arch one foot thick, perpendicular to the plane of the paper, by scaling the area between the dotted ordinates, marked on the horizontal line, and placed midway between the points of division, and multiplying this area by the weight of a cubic foot of masonry, here assumed at 150 pounds. The weights at the several points, to the nearest hundred pounds, will then be

$$C = 7{,}500, \quad F = 7{,}600, \quad G = 8{,}400, \quad I = 9{,}600, \quad K = 11{,}100,$$
$$L = 12{,}800, \quad N = 14{,}600, \quad O = 16{,}600, \quad P = 19{,}300 \text{ lbs.};$$

making the weight of the half-arch (when we take one-half of the load at C, and add 9,800 pounds for the load at A), = 113,450 pounds.

Calculate H for steady load by multiplying each co-efficient of H in the table above by its W in pounds just ascertained, and adding all the results for both halves of the arch. The work in detail is below. As the two halves of the arch are alike, we add up the column for H, add in again all but the amount for the load at the crown, and have H′ for the entire arch. Each vertical reaction will equal the weight of the half arch.

To find the ordinate $y_1' = y_2'$, for the combined weights, multiply each H by its y_1, add the products, and divide by H′. As, for each weight on one half of the arch, there will be a corresponding and equal weight on the other half, it will shorten the process to add y_1 and y_2 together for each point on one-half of the rib, *except the centre one* at C.

			W.	H.	$y_1 + y_2$.	M_1.	
C.	0°	1.126 ×	7,500 =	8,445 lbs.	.045 r +	380.0 r lbs.	
F.	5	1.095	7,600	8,322	.086	715.7	
G.	10	1.007	8,400	8,459	.074	626.0	
I.	15	0.866	9,600	8,314	.050	415.7	
K.	20	0.690	11,100	7,659	.011	84.2	
L.	25	0.498	12,800	6,374	—.053		—337.8 r lbs.
N.	30	0.311	14,600	4,541	—.165		749.2
O.	35	0.150	16,600	2,490	—.400		996.0
P.	40	0.040	19,300	772	—1.123		867.0
				55,376 lbs.		+2,221.6	—2,950.0
				46,931		—2,950.0	
				H′ = 102,307 lbs.		) —728.4 × 100 (—.712 ft. = y_1'.	

144. Equilibrium Curve for Steady Load. — Plot the weights of the above table on a vertical line from 1′ to 10′, lay off H′ from the middle of 1′–2′ to 0′, and, starting at 0.71 feet below A, draw an equilibrium polygon with its sides successively parallel to the lines which would radiate from 0′. This polygon will run quite close to the centre line, crossing it twice between A and C, and passing 0.4 feet below it at the crown. In any actual example the whole polygon should be drawn, as its accuracy will be proved by its striking the ordinate from B at the proper distance. If this arch were never to be subjected to any other than a steady load, or should the travelling load always be light, voussoirs of moderate depth would contain this polygon within their middle third. The true equilibrium curve will pass through the angles of the polygon just drawn.

145. **Calculations for Rolling Load.** — But, as we stated that a line of railroad was to be carried over this arch, let us suppose that the rolling load of one ton and a half per foot of track, or 3,000 pounds, is distributed over the ten feet of width of the arch; the moving load will then amount to 300 pounds per foot of span on the rib of our figure. The sleepers, the filling over the rib, and the bond of the arch-stones, will distribute any concentrated load over a considerable area.

At the crown of the arch the curve already drawn falls somewhat below the centre line. Upon inspecting Fig. 44 we see that six of the polygons there drawn pass below the crown of the rib. If, therefore, we place upon the stone arch a rolling load which covers six points of division from each abutment, that is, from Q to R on one side, and a corresponding distance on the other half arch, this distribution of load, if a practicable one under the usual method of running trains, will cause the greatest deviation of the equilibrium curve at the crown C.

To draw the polygon for this rolling load alone: first multiply each horizontal distance belonging to I, K, L, &c., by 300 pounds, to obtain the concentrated load on each point; then multiply by the proper co-efficients of H already obtained; sum the products, and double the results for both halves of the arch; multiply each H by its y_1 and y_2; divide the algebraic

sums of these products by H″. The operations are carried out below.

		W.			H.	$y_1 + y_2$.		
I.	8.4	2,520	× .866	=	2,182	.050	+ 109.1 r lbs.	
K.	8.2	2,460	.690		1,697	.011	18.7	
L.	7.9	2,370	.498		1,180	— .053		— 62.6 r lbs.
N.	7.5	2,250	.311		700.	— 165		115.4
O.	7.1	2,130	.150		320.	— 400		127.8
P.	6.7	2,010	.040		80	1.123		90.0
		13,740			6,159		+ 127.8	— 395.8

$$H'' = 12,318\,)\,-268.0 \times 100\,(\,-2.2 \text{ ft.} = y_1''.$$

Lay off the loads for one-half of the rib on a vertical line from 4″ to 10″; make 4″– 0″ = H″; and, laying off $y_1'' = -2.2$ feet, at A, draw the polygon which passes horizontally below C at a distance, by scale, of 2.3 feet.

146. **Increase of Bending Moment at Crown; Required Depth of Keystone.** — We can now find how much this added load increases the negative bending moment at the crown of the rib, or how much it causes the equilibrium curve to move inwards. If we multiply H′ and H″ by the ordinates to their respective curves at the crown, which ordinates are 0.4 feet and 2.3 feet, as lately stated, and add the products, we shall obtain the existing moment at the crown, and, upon dividing by H′ + H″, we get the ordinate from the centre line at C to the curve for the combined loads. It is worthy of note how little effect the rolling load produces, owing to the great thrust of the masonry itself.

102,307	×	0.4	=	40,922.8 ft. lbs.
12,318	×	2.3	=	28,331.4
114,625				)69,254.2
Ordinate at C			=	0.60 ft.

In order that this deviation of 0.6 feet from the middle of the joint shall not bring the equilibrium curve outside of the