tension of the lower rib. If the ribs are of equal stiffness, any load may be considered as divided equally between the two systems: if the ribs, while having the same curvature, are not alike in cross-section, the load will probably be distributed in the ratio of their moments of inertia. As the erect arch always tends to move away from its equilibrium curve, and the inverted arch to approach the equilibrium curve, the tangents at the abutment ends will move in the same direction, and therefore the structure should be treated as hinged at the ends, unless each flange is firmly bolted to the skew-back. If the structure is carried on columns or a pier, it appears to us that the ends cannot be rigid, and we judge that the two ribs will begin to turn about the middle of the depth without the introduction of a pivot or hinge.

The effect of temperature is annulled. Also the shortening of the erect arch under the direct compression being opposite to the extension of the inverted arch under the direct tension, the span will tend to remain unaltered; but the ribs themselves will be changed in form, one rib flattening as the other becomes more convex. If, in making such a design, the section of the arch is found to differ much from the section of the inverted rib, it will be well to calculate the relative deflections of the two ribs at the middle. The amount of load each will carry varies inversely as the deflection under equal loads, since they must deflect equally; and hence, if the arch is first designed of such shape, for the purpose of resisting compression, that it is stiffer or has less deflection than the chain, when each has one-half the load, the cross-section of the arch must be increased, and that of the chain may be diminished. This type of structure must not be confounded with a lenticular girder: the absence of bracing between the ribs makes them independent.

CHAPTER XI.

BENDING MOMENTS FROM CHANGE OF FORM.[1]

172. **Displacement from Bending Moments.** — It follows, from the fact that the arched rib moves away from the equilibrium polygon or curve, that the bending moments and chord stresses will have a slight tendency to increase. When the rib changes in shape, however, the equilibrium polygon must also move enough to still satisfy for the new form the equations of condition by which it was first established, and this movement will in some measure counteract the former. Besides, the equilibrium curve for steady load generally runs so close to the axis of the rib, that the change of shape from bending moments is very slight; and, even when the influence of rolling load is added, the increments of the bending moment ordinates are too small to be of material consequence.

The vertical displacement at any point E, Fig. 56, produced by any load, will be found, for the parabolic rib, by taking *area moments*, as explained in Part II., "Bridges," Chap. VI., or for the circular rib by summing the ordinates as usual along the rib. As was done in the treatment of beams, it will here be necessary to find the point D where the tangent to the rib in its new form is horizontal, which point will not be at the crown,

[1] Many of the deductions in this chapter are only intended as guides in practical construction, to indicate where, and to show approximately how much, additional stress may be anticipated from change of form. Exact results are not attempted.

except for symmetrical loads. D is then to be assumed momentarily as a fixed point, and the deflection or area moment of A and E obtained with reference to it: the subtraction of the latter from the former gives the displacement of E relatively to the abutment A; that is, from the area moment between D and A subtract the area moment between D and E; and the remainder, when multiplied by $H \div \mathbf{E}\,\mathbf{I}$, will be the vertical displacement of E. As just stated, these displacements may be neglected.

173. **Displacement and Bending Moments from Compression.**—The thrust which exists at each section of the rib must, by its compression of the particles, cause a shortening of the rib, and, as the shorter rib must fit the same abutments, it is necessarily lowered at the crown. The resulting bending moments may be of consequence. So far as the rib retains sensibly its old form, parabolic or the segment of a circle, the equilibrium polygon is lowered proportionally to the sinking of the rib, as indicated in Fig. 57, in order to still satisfy the equations of condition; but, as the deflection v at the crown is very small compared with k, the alteration of the bending moment ordinates is very trifling. On the other hand, this lowering of the apex of the equilibrium polygon at once increases the value of H, offsetting the change first pointed out. This will be seen, also, from the values of M, § 44, into which k does not enter. The bending moments from the external load are therefore practically unaltered by the change of form.

To produce this change of form, however, or to bring the arch down to its new position, requires a change of inclination, and consequently a bending moment, at most points of the rib. The strains thus induced should be examined. Strictly accurate theoretical investigations for the different ribs cannot easily be made; but formulæ may be deduced which will serve all practical purposes.

174. **Parabolic Rib hinged at Ends.**—The parabolic rib which we have treated varies in cross-section, from the crown

to the springing, according to the secant of the inclination to the horizon, § 37; and, as the magnitude of the direct thrust for a complete uniform load varies in the same way, the intensity of direct compression per unit of cross-section arising from H will be constant, and every unit of length of arc will be shortened by that thrust the same amount, so that the arch will be altered as if exposed to a change of temperature. We will assume that the new form of the rib is still a parabola with a rise k' in place of k, but with the original span $2\,c$.

By definition, Part II., "Bridges," § 85, the modulus of elasticity $\mathbf{E}$ equals the intensity of stress divided by the shortening of a unit's length. Let the constant intensity of thrust equal the thrust at the crown H, divided by the cross-section at the crown A; let the compression of a unit's length equal the difference, s–s', between the lengths of arc before and after compression divided by the original length s. Then

$$s - s' = \frac{H\,s}{A\,\mathbf{E}}.$$

An approximate formula for the length of a parabolic arc is, in our usual notation, $s = 2\,c + \frac{4}{3}\frac{k^2}{c}$. The value of s' will be obtained by writing k' for k; then

$$s - s' = \frac{4}{3\,c}(k^2 - k'^2) = \frac{H\,s}{A\,\mathbf{E}} = \frac{2\,H}{3\,A\,\mathbf{E}} \cdot \frac{3\,c^2 + 2\,k^2}{c}.$$

As v, the deflection at the crown and the difference between k and k', is very small, we may write, without sensible error, $k - k' = v$, and $k + k' = 2\,k$; whence $k^2 - k'^2 = 2\,k\,v$, and we have

$$\frac{8}{3\,c}k\,v = \frac{2\,H}{3\,A\,\mathbf{E}} \cdot \frac{3\,c^2 + 2\,k^2}{c}, \text{ or } v = \frac{H}{4\,A\,\mathbf{E}} \cdot \frac{3\,c^2 + 2\,k^2}{k}.$$

It was proved, in § 36, that this rib deflected vertically like a horizontal beam of uniform section: hence to bring the arch down to its new position will create bending moments at all points such as would accompany the same deflection in a straight beam, supported at the ends, uniformly loaded, and of a cross-section equal to that of the rib at the crown. In Part II., "Bridges," § 95, we found, for a beam supported and loaded as above with w per foot,

$$v = \frac{5}{384} \cdot \frac{w l^4}{\mathbf{E}\mathbf{I}} = \frac{5 w c^4}{24 \mathbf{E}\mathbf{I}} = \frac{5 \mathrm{M}_0 c^2}{12 \mathbf{E}\mathbf{I}},$$

if M_0 is the bending moment at the middle. Equating these two values of v, we obtain

$$\frac{5 \mathrm{M}_0 c^2}{12 \mathbf{E}\mathbf{I}} = \frac{\mathrm{H}}{4 \mathrm{A} \mathbf{E}} \cdot \frac{3 c^2 + 2 k^2}{k},$$

or

$$\mathrm{M}_0 = \frac{3 \mathbf{I} \mathrm{H} (3 c^2 + 2 k^2)}{5 \mathrm{A} c^2 k},$$

the additional positive bending moment at the crown of the arch, caused by its compression under the thrust H.

The bending moments at other points may then be taken to compare with those of the beam, that is, as the ordinates to the parabola, being $\frac{3}{4}\mathrm{M}_0$ at the quarter-span.

175. **Remarks; Example.** — It will be noticed that $\mathbf{E}$ has disappeared from the expression for M_0: hence the bending moment will be the same, whether the material be iron, steel, or wood. As $\mathbf{I}$ contains A, and may be written $n \mathrm{A} h^2$, Part II., "Bridges," § 86, n being a numerical factor, it is seen that the bending moment from deflection of the rib due to compression increases with the square of the depth of the rib, and, as $\mathrm{M} \div h$ equals the flange stress, this stress will increase directly as the depth. To diminish the effect of change of form alone, employ a shallow rib.

If $\mathrm{H} = 20$ tons, $c = 100$ feet or $l = 200$ feet, $k = 20$ feet, and $h = 2\frac{1}{2}$ feet, for a rib with two plate flanges and thin or open web, $\mathbf{I} = 2\{\frac{1}{2}\mathrm{A} \cdot (\frac{1}{2}h)^2\} = \frac{1}{4}\mathrm{A} h^2$, and

$$\mathrm{M}_0 = \frac{3 \times 25 \times 20 \times 30{,}800}{5 \times 16 \times 10{,}000 \times 20} = 2.9 \text{ ft. tons at crown,}$$

giving 1.16 tons compression on upper flange, and an equal tension on lower flange.

176. **Displacement from Change of Temperature.** — The deflection produced by a fall of temperature in the parabolic rib hinged at the ends will be found by taking the area moment of the half parabolic segment, Fig. 16, from the crown to the springing about one abutment, and multiplying by $\mathrm{H} \div \mathbf{E}\mathbf{I}$. Hence, as in Part II., "Bridges," § 95,

$$v_t = \frac{\mathrm{H}_t}{\mathbf{E}\mathbf{I}} \cdot \tfrac{2}{3} c k \cdot \tfrac{5}{8} c = \tfrac{5}{12} \cdot \frac{\mathrm{H}_t}{\mathbf{E}\mathbf{I}} \cdot c^2 k,$$

the deflection at the crown when the temperature falls, and the rise of the crown when the temperature rises. One may prefer to consider the rib in its new position as the proper curve from which to obtain the area moment. If it is assumed to still be a parabola with the rise k', we have

$$v = \tfrac{5}{12} \frac{\mathrm{H}}{\mathbf{E}\mathbf{I}} c^2 k', \text{ and } k' = k \pm v.$$

Substitute this value of k', and v becomes

$$v = \frac{5 \mathrm{H} c^2 k}{12 \mathbf{E}\mathbf{I} \mp 5 \mathrm{H} c^2}.$$

This deflection is the result of the bending moments arising from H_t, and is not to be regarded in the light of the preceding section. The moments were computed in § 74. These moments will be slightly altered by the movement, as it shortens or lengthens the ordinates; but H_t will be changed in the opposite direction, reducing the actual modification of the moments. Since

$$\mathrm{H}_t = \frac{15}{8} \cdot \frac{t e \mathbf{E}\mathbf{I}}{k^2}, \quad v_t = \frac{25}{32} \cdot \frac{t e c^2}{k},$$

a quantity independent of the cross-section of the rib, and, so far as the material is concerned, affected by the co-efficient of expansion only.

The bending moments due to the direct thrust, whether arising from a load or change of temperature, have been considered, as well as the resulting deflection. When the temperature rises, H_t is thrust, and in itself tends to shorten the rib, and thus reduce the above amount of rise due to expansion. The ratio of the two deflections will be

$$\frac{v}{v_t} = \frac{H_t}{4\,A\,E} \cdot \frac{3\,c^2 + 2\,k^2}{k} \div \frac{5}{12}\,\frac{H_t}{E\,I}\,c^2\,k = \tfrac{3}{5}\,n\,h^2\left(\frac{3}{k^2} + \frac{2}{c^2}\right).$$

In the example previously cited this ratio becomes

$$\frac{v}{v_t} = 0.6 \times \frac{25}{16}\left(\frac{3}{400} + \frac{2}{10{,}000}\right) = .0072,$$

a reduction of three-fourths of one per cent. When the temperature falls, H_t is a tension, and, in lengthening the rib, slightly reduces the deflection.

The deflection for a co-efficient of expansion of .000007 and a range of temperature of 30° will be, in our example of § 175,

$$v_t = \frac{25 \times 30 \times .000007 \times 10{,}000}{32 \times 20} = .082 \text{ ft.} = 1 \text{ inch.}$$

[The expansion or contraction of a straight bar may be conveniently stated as ¼ inch in one hundred feet for 30° F.] The theoretical movement of the rib at the crown for a range of 30° above and below the temperature at which it was constructed will therefore be two inches. The actual movement is generally less than theory would indicate, owing to gradual transition from one extreme to another, protection of some portions of the structure from extremes of temperature, as by shielding from the direct rays of the sun, &c., and, finally, imperfect freedom of motion.

177. **Initial Camber for Arch.** — It may be expedient to make the rib a little longer than the distance between the springings to compensate for the amount of compression which will arise from the steady load, or else to wedge up the springing points until the crown of the rib, when not under strain, shall be a distance v above its normal position: the rib will then, when in place and under its steady load, come down to the curve for which it is designed, and will be free from that portion of initial bending moment due to change of form from steady load. This will be true, because, in forcing the rib up,

we have introduced bending moments of the opposite kind to an equal amount. An additional allowance may be made for an ordinary travelling load. If the rib is to be made longer to offset the compression, find v, § 174, or H from steady load, and make the parabolic rib of a span $2\,c + u$ and a rise k, so that, when sprung into place on a span $2\,c$, it would rise to a height $k + v$, if it were not compressed at the same time.

Noticing, from § 174, that this compression acts like a fall of temperature in shortening the rib, we have, from § 74,

$$H_t = \frac{15}{8} \cdot \frac{E\,I}{c\,k^2} \cdot t\,e\,c = \frac{15}{8} \cdot \frac{E\,I}{c\,k^2} \cdot \frac{u}{2},$$

since u must equal $2\,t\,e\,c$. But $H_t = \frac{12}{5}\,\frac{E\,I}{c^2 k}\,v$, by § 176, and, equating these two values, we get

$$\frac{15}{16} \cdot \frac{E\,I}{c\,k^2} \cdot u = \frac{12}{5}\,\frac{E\,I}{c^2\,k}\,v,$$

or

$$u = \frac{64}{25} \cdot \frac{k}{c} \cdot v = \frac{16}{25} \cdot \frac{H}{A\,E} \cdot \frac{3\,c^2 + 2\,k^2}{c}.$$

If, in our preceding example, A is eight square inches, and E is 24,000,000, u becomes half an inch.

178. **Parabolic Rib with Fixed Ends.** — In this case the deflection will naturally correspond with that of a beam of uniform section, uniformly loaded, and fixed at the ends, as will be seen by comparing the equilibrium curve of Fig. 17, where H from temperature alone acts, with that of such a beam. In Part II., "Bridges," § 99, and Fig. 47, we found that

$$v = \frac{w\,l^4}{384\,E\,I} = \frac{w\,c^4}{24\,E\,I} = \frac{M_0\,c^2}{4\,E\,I},$$

if M_0 is the bending moment at the middle. Equating this value of v with the one found in § 174, we obtain

$$M_0 = \frac{I\,H\,(3\,c^2 + 2\,k^2)}{A\,c^2\,k}.$$

The bending moment at the springings will be double this amount, and of the opposite sign.

The deflection produced by a change of temperature will be found by taking the area moment of the semi-segment of the parabola already obtained in § 176, and subtracting the area moment of the rectangle whose height is $\frac{2}{3}k$ and base c.

$$v_t = \frac{H_t}{EI}\left(\tfrac{5}{12}c^2 k - \tfrac{2}{3}ck \,.\, \tfrac{1}{2}c\right) = \tfrac{1}{12}\frac{H_t}{EI}c^2 k.$$

Applying the data of the previous example of § 175, we have

$$M_0 = \frac{25 \times 20 \times 30{,}800}{16 \times 10{,}000 \times 20} = 4.8 \text{ ft. tons at crown,}$$

giving 1.92 tons, compression on upper flange and an equal tension on lower flange at crown, and 3.85 tons, tension on upper flange with an equal compression on lower flange, at either springing.

To find such additional length of span for the parabolic rib fixed at the ends, that, when compressed under steady load, it may have no bending moments due to change of form, we pursue again the method of § 177. From § 76,

$$H_t = \frac{45}{4} \cdot \frac{EI}{ck^2} \cdot tec = \frac{45}{4} \cdot \frac{EI}{ck^2} \cdot \frac{u}{2}.$$

As above,

$$H_t = \frac{12\,EI}{c^2 k} v;$$

therefore

$$u = \frac{32}{15} \cdot \frac{k}{c} \cdot v = \frac{8}{15} \cdot \frac{H}{AE} \cdot \frac{3c^2 + 2k^2}{c},$$

a quantity five-sixths of that for the rib with hinged ends.

179. **Circular Rib hinged at Ends.** — It is more difficult to obtain the amount of deflection from change of form produced by the compression at each section of a circular rib, even approximately. As the equilibrium polygon for steady load will not deviate much from the axis of the rib, the thrust T may be assumed to vary as secant θ, the inclination of the rib

at successive points to the horizon: hence the shortening of a small portion, ds, of arc under the thrust will be

$$d(s - s') = \frac{T\,ds}{AE} = \frac{H\,ds}{AE}\text{secant}\,\theta = \frac{Hr}{AE} \cdot \frac{d\theta}{\cos\theta};$$

as the section is constant,

$$s - s' = \frac{Hr}{AE}\int_{-\beta}^{+\beta}\frac{d\theta}{\cos\theta} = \log\frac{1 + \sin\beta}{1 - \sin\beta} \cdot \frac{Hr}{AE}. \quad (1.)$$

(The symbol *log* denotes the hyperbolic logarithm; to obtain it, multiply the common logarithm by 2.30158.)

As, with a small deflection, the rib will vary but slightly from its original form, let it be assumed to be an arc of a circle after compression. We have then $s - s' = 2r\beta - 2r'\beta'$, where r' is the new radius, and β' the new angle subtended by the half-arch. Now

$$r = \frac{c^2 + k^2}{2k}, \quad r' = \frac{c^2 + (k - v)^2}{2(k - v)}, \text{ and } \sin\beta' = \frac{c}{r'}.$$

By assuming a value for v, r' and β' can be obtained, and the value of $2(r\beta - r'\beta')$ calculated: if it agrees with the value $s - s'$ of equation (1.), the assumed v is sufficiently near the truth; if not, the process of approximation may be repeated. We may adopt, as a value which will answer very well in many cases, $v = \frac{s - s'}{\beta}$. Then

$$v = \frac{Hr}{AE\beta}\log\frac{1 + \sin\beta}{1 - \sin\beta}.$$

This logarithmic expression may be written as a series,

$$v = \frac{Hr}{AE\beta}(\sin\beta + \tfrac{1}{3}\sin^3\beta + \tfrac{1}{5}\sin^5\beta, \text{ \&c.}).$$

It was shown in § 36 that the vertical deflections of two beams of the same cross-section, and carrying the same gross load uniformly distributed, — one inclined at an angle i, and the other the horizontal projection of the former, — were in the proportion of $1 : \cos i$. If, then, the load on the horizontal beam is increased in intensity in the ratio $\sec i : 1$, the vertical deflections of the two beams will be the same. We desire to find the amount and distribution of load on a straight beam of the same span as the circular arch, Fig. 58, and the same cross-section, which shall produce the same deflection at the middle. By what has just been stated, the load on any horizontal foot of a straight beam must be to the intensity on an inclined beam as $w \sec \theta$ to w. A small portion of the arch $ds = \sec \theta \, dx$; hence it follows, that, if the arch is carrying w per horizontal foot over the whole span, a horizontal beam, as above, loaded with the varying intensity $w \sec \theta = w \frac{ds}{dx}$ per foot, will have the same deflection. This load will be the projection of a load of uniform intensity measured along the rib, or the load on the beam is $w s$, or $2 w r \beta$, in our usual notation.

In any particular case we may easily solve the problem graphically. Lay off 1–2, Fig. 58, equal w . A B; divide A B into a number of equal parts, and 1–2 into the same number, with half-loads at 1 and 2 as usual. Make 2–0 equal to H for this load, and, with 0 as a pole, draw the equilibrium polygon A′ B′, which, for an arch of moderate rise, will be a close approximation to a catenary. C′ B′ . (0–2) will be the desired bending moment M_0, for a deflection found by taking the area moment of A′ B′ C′ about A′, multiplying by 0–2, and dividing by **EI**. Use these values as we did those of § 174. In constructing, increase the length of the rib by (1.) if thought desirable. The values of the following section may be taken if preferred.

180. **Analytical Discussion.** — The exact values may be deduced by the usual process for finding the deflection of a beam. If x is the distance of any point of the beam from one abutment (Fig. 59), β, the angle subtended at the centre by the half-arch, θ, the angle from the crown to any point whose projection is x, and w, the load per foot on the arch, and also at the middle of the beam, then $x = r(\sin \beta - \sin \theta)$, $dx = -r \cos \theta \, d\theta$, the load at any point $= w \sec \theta$ per foot, and load on $dx = w \sec \theta \, dx$

$= - w r \sec \theta \cos \theta \, d\theta = - w r \, d\theta$. The load on one-half of the span is shown in the figure.

$$\text{Load on half-span} = \int_0^c w \sec \theta \, dx = w r \int_0^\beta d\theta = w r \beta.$$

This expression is the reaction P_1 at the abutment. If x' is the distance from the abutment to any section at which we desire the bending moment, and the corresponding angle is θ', we have the bending moment

$$M = P_1 x' - \int_0^{x'} (x' - x) \, w \sec \theta \, dx$$

$$= w r^2 \beta (\sin \beta - \sin \theta') - w r^2 \int_\beta^{\theta'} (\sin \theta' - \sin \theta) \, d\theta$$

$$= w r^2 (\beta \sin \beta + \cos \beta - \theta' \sin \theta' - \cos \theta'),$$

which becomes at the middle

$$M\,(\max) = w r^2 (\beta \sin \beta + \cos \beta - 1) = w r (c \beta - k).$$

Writing the usual expressions for inclination and deflection, and dropping the accents, we have

$$i_x = \int_x^c \frac{M}{\mathbf{EI}} dx = - \frac{w r^3}{\mathbf{EI}} \int_0^\theta (\beta \sin \beta + \cos \beta - \theta \sin \theta - \cos \theta) \cos \theta \, d\theta$$

$$= - \frac{w r^3}{\mathbf{EI}} (\beta \sin \beta \sin \theta + \cos \beta \sin \theta - \tfrac{3}{4} \sin \theta \cos \theta - \tfrac{3}{4} \theta + \tfrac{1}{2} \theta \cos^2 \theta).*$$

The slope at the abutment, when $\theta = \beta$, is $- \frac{w r^3}{4 \mathbf{EI}} (\beta \sin^2 \beta - \beta \cos^2 \beta + \sin \beta \cos \beta)$, which, if we remove $\frac{H}{\mathbf{EI}}$, is the area of the half equilibrium polygon A′ B′ C′ of Fig. 58. The deflection of the centre is

$$v = \int_0^c i \, dx = \frac{w r^4}{\mathbf{EI}} \int_0^\beta (\beta \sin \beta \sin \theta + \cos \beta \sin \theta - \tfrac{3}{4} \sin \theta \cos \theta - \tfrac{3}{4} \theta + \tfrac{1}{2} \theta \cos^2 \theta) \cos \theta \, d\theta$$

$$= \frac{w r^4}{\mathbf{EI}} (\tfrac{1}{3} \beta \sin^3 \beta + \tfrac{7}{36} \sin^2 \beta \cos \beta - \tfrac{1}{4} \beta \sin \beta - \tfrac{1}{9} \cos \beta + \tfrac{1}{9}).*$$

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* These expressions are reduced. To aid any who desire to prove them, we give the following integrals: $\int \theta \cos \theta \, d\theta = \theta \sin \theta + \cos \theta$; $\int \theta \sin \theta \cos \theta \, d\theta = - \tfrac{1}{2} \theta \cos^2 \theta + \tfrac{1}{4} \cos \theta \sin \theta + \tfrac{1}{4} \theta$; $\int \cos^2 \theta \, d\theta = \tfrac{1}{2} \sin \theta \cos \theta + \tfrac{1}{2} \theta$; $\int \theta \cos^3 \theta \, d\theta = \theta \cos^2 \theta \sin \theta + \tfrac{7}{9} \cos^3 \theta + \tfrac{2}{3} \theta \sin^3 \theta + \tfrac{2}{3} \sin^2 \theta \cos \theta$.

From this expression, by removing $\frac{H}{EI}$, we obtain the area moment of A′ B′ C′.

The quantities representing v and M will now be introduced in the equation of § 179: hence we get

$$\frac{H}{A\beta}\log\frac{1+\sin\beta}{1-\sin\beta}=\frac{wr^3}{18I}(12\beta\sin^3\beta+7\sin^2\beta\cos\beta-9\beta\sin\beta-4\cos\beta+4).$$

Find the value of M for the special arch, and value of β, and also the value of v. Let $v \div M = Br^2$; then

$$M=\frac{Hr}{2Br^2AE\beta}\log\frac{1+\sin\beta}{1-\sin\beta}.$$

If the arch is a semicircle,

$$M\,(\max)=\tfrac{1}{2}wr^2(\pi-2);\ i=-\frac{wr^3}{4EI}\cdot\frac{\pi}{2};\ v=\frac{wr^4}{36EI}(\tfrac{3}{2}\pi+4).$$

181. **Circular Rib Fixed at Ends.** — From the method of treating the parabolic rib with fixed ends, as compared with the parabolic rib with hinged ends, we would suggest that the deflection and the bending moments at crown and springing of the circular arch with fixed ends, due to the compression of the rib from H, may be obtained from a drawing like Fig. 58, when 2–0 is made equal to the H of this case, by plotting the closing line of Fig. 27 on the arch of Fig. 58, at the height above A of $r\left(\frac{\sin\beta}{\beta}-\cos\beta\right)$ (see § 105), projecting the points of contraflexure vertically on A′ B′, drawing the horizontal closing line of this equilibrium polygon, and then finding M and v for the beam fixed at the ends.

For circular arches of moderate rise, the treatment for parabolic arches will probably suffice.

CHAPTER XII.

BRACED ARCH WITH HORIZONTAL MEMBER; OTHER SPECIAL FORMS; CONCLUSION.

182. **The Usual Analysis not Applicable.** — The difficulty in the way of a successful application of the usual formula Σ E F . D E = 0 for the change of span of the braced arch with horizontal member, of Fig. 60, or, as it is sometimes called, the rib with spandrel bracing, arises from the fact that the *moment of inertia* of successive cross-sections cannot be left out of the equation as a constant. In fact, it varies rapidly; and its amount at any section is unknown until the sizes of the respective pieces are determined. It was shown, in § 72, that **I** must be placed in the denominator of the above formula: and, if not constant, it must come within the sign of summation.

This arch is pivoted at the springings, but continuous at the crown. If it were hinged at the crown by the omission of a piece in either the lower or the upper chord, the thrusts at the abutments could at once be determined by the principles of Chap. II.; and a diagram by the method of Part I., "Roofs," would at once give the stresses in all the pieces for any given load. For the treatment of the case represented in Fig. 60, the following practicable method is offered. It was published in "The Engineer," Feb. 10, 1873, and will also be found in the ninth edition of "The Cyclopædia Britannica," art. "Bridges," where it is attributed to Professor Clerk-Maxwell.