From this expression, by removing $\frac{H}{EI}$, we obtain the area moment of $A'B'C'$.

The quantities representing v and M will now be introduced in the equation of § 179: hence we get

$$\frac{H}{A\beta}\log\frac{1+\sin\beta}{1-\sin\beta} = \frac{wr^3}{18I}(12\beta\sin^3\beta + 7\sin^2\beta\cos\beta - 9\beta\sin\beta - 4\cos\beta + 4).$$

Find the value of M for the special arch, and value of β, and also the value of v. Let $v \div M = Br^2$; then

$$M = \frac{Hr}{2Br^2AE\beta}\log\frac{1+\sin\beta}{1-\sin\beta}.$$

If the arch is a semicircle,

$$M\,(\max) = \tfrac{1}{2}wr^2(\pi - 2);\quad i = -\frac{wr^3}{4EI}\cdot\frac{\pi}{2};\quad v = \frac{wr^4}{36EI}(\tfrac{3}{2}\pi + 4).$$

181. **Circular Rib Fixed at Ends.** — From the method of treating the parabolic rib with fixed ends, as compared with the parabolic rib with hinged ends, we would suggest that the deflection and the bending moments at crown and springing of the circular arch with fixed ends, due to the compression of the rib from H, may be obtained from a drawing like Fig. 58, when 2–0 is made equal to the H of this case, by plotting the closing line of Fig. 27 on the arch of Fig. 58, at the height above A of $r\left(\frac{\sin\beta}{\beta} - \cos\beta\right)$ (see § 105), projecting the points of contraflexure vertically on A′ B′, drawing the horizontal closing line of this equilibrium polygon, and then finding M and v for the beam fixed at the ends.

For circular arches of moderate rise, the treatment for parabolic arches will probably suffice.

CHAPTER XII.

BRACED ARCH WITH HORIZONTAL MEMBER; OTHER SPECIAL FORMS; CONCLUSION.

182. **The Usual Analysis not Applicable.** — The difficulty in the way of a successful application of the usual formula $\Sigma\, EF \,.\, DE = 0$ for the change of span of the braced arch with horizontal member, of Fig. 60, or, as it is sometimes called, the rib with spandrel bracing, arises from the fact that the *moment of inertia* of successive cross-sections cannot be left out of the equation as a constant. In fact, it varies rapidly; and its amount at any section is unknown until the sizes of the respective pieces are determined. It was shown, in § 72, that **I** must be placed in the denominator of the above formula: and, if not constant, it must come within the sign of summation.

This arch is pivoted at the springings, but continuous at the crown. If it were hinged at the crown by the omission of a piece in either the lower or the upper chord, the thrusts at the abutments could at once be determined by the principles of Chap. II.; and a diagram by the method of Part I., "Roofs," would at once give the stresses in all the pieces for any given load. For the treatment of the case represented in Fig. 60, the following practicable method is offered. It was published in "The Engineer," Feb. 10, 1873, and will also be found in the ninth edition of "The Cyclopædia Britannica," art. "Bridges," where it is attributed to Professor Clerk-Maxwell.

183. **Change of Span from Stress in a Piece.** — From previous statements, we know that the modulus of elasticity **E** is the measure of the extensibility or compressibility of the kind of material to which it refers, so long as the stress does not surpass the elastic limit, and is equal to the quotient of the intensity of the stress on a cross-section divided by the extension or compression of a *unit's* length of the piece in which the stress is exerted. Thus, if l is the length of a piece in inches, A its cross-section in square inches, T the thrust or tension in pounds to which it is exposed, and Δl the change of length produced,

$$\mathbf{E} = \frac{T}{A} \cdot \frac{l}{\Delta l}; \quad \text{or} \quad \Delta l = \frac{T}{\mathbf{E} A} l. \quad (1.)$$

If the piece A of the frame of Fig. 61 is changed in length, and every other piece is unchanged, while the portion of the frame to the right is held firmly in place, the span L of the frame will undergo an alteration ΔL. In this case the motion takes place about the joint opposite to A, and we may write

$$\Delta L : \Delta l = ac : ab, \quad (2.)$$

or the distance described by the point b for a small displacement around the axis a will be to the horizontal movement of d as the arm $a\,b$ to the arm of d, or $a\,c$. A similar proportion will be true, if one of the lower chord pieces is supposed to alter in length. In case any diagonal is changed in length, as, for instance, $f\,g$, the four-sided figure $e\,f\,i\,g$ must alter to $e\,f\,i'\,g'$ of the sketch below, the point i turning about f as a centre, and the point g about e: hence, for a small displacement, the centre of motion is at the point of meeting, o, of $i\,f$ and $g\,e$ prolonged, which, for this arch, will lie in the upper chord; and the perpendicular p, dropped on the line of the piece, will take the place of $a\,b$ above.

184. **Stress in a Piece from H and P.** — Let t be the stress produced in a member by a horizontal force H acting between the springing points. Then the principle of equality of moments as necessary for equilibrium about the point around

which motion would otherwise begin, and which is no other than the point noticed at the close of the last section, will determine the relation of the forces. A general rule for finding the axis about which rotation will begin is, Make a section which shall cut three pieces only; prolong the lines of two of the pieces until they meet: the moment of the stress in the third piece about that point of meeting will equal the moment of H about the same point. Hence we have, for the piece A

$$t \cdot ab = H \cdot ac, \text{ or } t = \frac{ac}{ab} H.$$

Similarly, let t' be the stress produced in A by a vertical force P applied at one springing, while the other end of the frame is held rigidly so that it cannot turn. As the arm of P will be $d\,c$, we may write

$$t' \cdot ab = P \cdot dc, \text{ or } t' = \frac{dc}{ab} P.$$

The distances $d\,c$ and $a\,c$, being respectively horizontal and vertical, may be denoted in general for any piece by x and y. In order to make the symbol $a\,b$ of the last section and of this one general, so as to apply to a diagonal as well as a chord piece, let us write for $a\,b$ the perpendicular p drawn from the axis of rotation upon the action-line of the piece.

Any thrust at the springing having horizontal and vertical components H and P will produce a stress T in the piece, equal to $t + t'$, or

$$T = \frac{ac \cdot H + dc \cdot P}{ab} = \frac{Hy + Px}{p}. \quad (1.)$$

It is evident that heed must be paid to the kind of stress produced by H and P; thus, in any piece of the top member, H will produce tension and elongation, while P will produce compression and shortening: the reverse will be true of the lower member; how the diagonals are affected will be seen when we come to our application. Appropriate signs, therefore, must be given to the arithmetical values of the stress and alteration of length; thus compression and shortening may be called positive; tension and lengthening, negative.

185. **Formula for H.** — From equations (1.) and (2.), § 183, upon writing y and p, as indicated above, for $a\,c$ and $a\,b$, we get the change of span for any stress, T, in a particular piece,

$$\Delta L = \Delta l \frac{y}{p} = \frac{T\,y}{p} \cdot \frac{l}{E\,A},$$

or, upon inserting the value of T from equation (1.), last section,

$$\Delta L = \frac{H\,y^2 + P\,x\,y}{p^2} \cdot \frac{l}{E\,A}.$$

This same quantity can be calculated for the extensibility due to each member of the frame; and the result will not be altered by the slight yielding of all the others, unless this yielding produces sensible deformation, making appreciable changes in $\frac{x}{p}$ and $\frac{y}{p}$: hence the sum of all the changes of span, or the total change of span, will be

$$H\,\Sigma\frac{y^2}{p^2} \cdot \frac{l}{E\,A} + \Sigma\,P\frac{x\,y}{p^2} \cdot \frac{l}{E\,A}.$$

If the abutments do not yield, this expression is zero. If the span changes, by a yielding of the abutments, so that e is the elongation of span for one ton of H, then the above expression for change of span equals e H. P is the vertical component of the reaction at one abutment, found as for any frame loaded as this arch may be: hence H may be found. If the abutments do not yield, we then obtain

$$H = \frac{\Sigma\,P\frac{x\,y}{p^2} \cdot \frac{l}{E\,A}}{\Sigma\frac{y^2}{p^2} \cdot \frac{l}{E\,A}}. \quad (1.)$$

186. **Application of Method.** — Let a single weight, W, be applied at any one of the top joints of the braced arch, Fig. 60. Inclined reactions will be produced at each abutment, whose components will be H and P_1 at the left, H and P_2 at the right. The calculations for the resulting stresses in the pieces are then best made as follows: Construct tables of the values $x \div p$ and $y \div p$ for each member of the frame; the method of sections through the opposite joints, or of moments, will answer best for the top and bottom members, and a diagram such as has been drawn for a roof, for the diagonals; assume a cross-section for each member for an assumed probable value of the abutment thrust; make tables of $\frac{x\,y}{p^2} \cdot \frac{l}{E\,A}$ and $\frac{y^2}{p^2} \cdot \frac{l}{E\,A}$, or, what is equivalent when all the frame is of one material, so that E is constant, make tables of $\frac{x\,y\,l}{p^2\,A}$ and $\frac{y^2\,l}{p^2\,A}$. The summations indicated in (1.), § 185, can then be made. In summing $P \cdot \frac{x\,y\,l}{p^2\,A}$, the value P_1 must be used for all pieces to the left of the loaded joint, and P_2 for all pieces to the right of the load. Equation (1.), above, will now give the value of H for this single load.

The process of finding the numerator of (1.) must be repeated for each joint which is loaded. The abutment reactions having thus been found, the stress in each piece will be computed by (1.) § 184, or will be scaled from a diagram drawn as in Part I., "Roofs." If, upon finding the maximum stresses in the pieces, resulting from the steady load and such rolling loads as will have the worst effect, the assumed sections are not strong enough for these stresses, fresh cross-sections must be assumed, and the whole calculation repeated. The change in cross-sections will cause some change in the values of H; but this tentative process need seldom be repeated but once.

187. **Example; Stresses from H and P.** — These processes will probably be rendered more clear by an example. Let the arched frame of Fig. 60 be 120 feet in span, 12 feet rise to the curved member, and 17 feet rise to the straight member, making the depth at mid-span 5 feet. Let the upper member be divided into panels of 10 feet each, and the parabolic or circu-

lar arc into portions of 10.263 feet each.[1] The radius of the curved member will be 156 feet. Let it be desired to design this arched structure to bear a steady load of ten tons per joint of the top member and a travelling load of the same intensity.

If a horizontal line L O is drawn to represent a certain value of H, we may construct Fig. 62 by the method used in Part I., "Roofs," and by scale determine the magnitude of the stress in each piece due to this H, as the *only force*, applied as a thrust at each abutment; all of the stresses being measured as *fractions of* H, and the kind of stress noted. One-half of the diagram is sufficient, as it will be symmetrical. The magnitude of any stress in a top or bottom piece can be readily proved by the method of moments. We may now fill the columns of a table with these ratios which represent $y \div p$, being not only the ratios of the stresses to H, but of the change of span to change of length. Bow's notation is used, and the stresses in one half of the frame will correspond with those in the other half. The sign + denotes compression, the sign — denotes tension.

VALUES OF $\frac{y}{p}$.

B O — 0.272	A L + 1.203	O A — 0.444	A B + 0.450
D O — 0.639	C L + 1.520	B C — 0.478	C D + 0.480
F O — 1.117	E L + 1.927	D E — 0.500	E F + 0.502
I O — 1.678	G L + 2.427	F G — 0.484	G I + 0.488
K O — 2.185	J L + 2.942	I J — 0.384	J K + 0.386
N O — 2.400	M L + 3.293	K M — 0.153	M N + 0.154

In the same way a diagram constructed upon a vertical line which represents P_1, Fig. 63, will give the stresses in the several pieces caused by this vertical force only, applied in an upward direction at the left abutment, while the right end is held rigidly in place by fixing the end brace in position. This figure will not be symmetrical, and therefore all the pieces must be entered in the table. P_2 at the right abutment, in place of P_1 at the left, will reverse the table, B′ O taking the place of B O, &c. The ratio of these stresses to P will give $x \div p$.

[1] If the arc is parabolic, the length of a piece will be 10.268 feet. The difference is not material for our example.

VALUES OF $\frac{x}{p}$.

B O + 0.718	A L — 0.354	O A + 1.178	A B — 1.189
D O + 1.872	C L — 1.341	B C + 1.505	C D — 1.780
F O + 3.662	E L — 2.833	D E + 1.872	E F — 1.879
I O + 6.226	G L — 4.996	F G + 2.214	G I — 2.232
K O + 9.319	J L — 7.787	I J + 2.341	J K — 2.353
N O + 12.000	M L — 10.655	K M + 1.907	M N — 1.920
K′ O + 13.163	M′ L — 12.592	N M′ + 0.833	M′ K′ — 0.827
I′ O + 12.675	J′ L — 12.978	K′ J′ — 0.371	J′ I′ + 0.309
F′ O + 11.283	G′ L — 12.134	I′ G′ — 1.212	G′ F′ + 1.202
D′ O + 9.698	E′ L — 10.767	F′ E′ — 1.657	E′ D′ + 1.664
B′ O + 8.260	C′ L — 9.387	D′ C′ — 1.876	C′ B′ + 1.880
	A′ L — 8.139	B′ A′ — 1.367	A′ O fixed.

188. **Computation of Tables.** — We may now write a table for $\frac{y^2 l}{p^2}$, and another for $\frac{x y l}{p^2}$, for each piece of the frame. The first table, involving squares, will be positive throughout. The lengths of the horizontal and rib pieces will be multiplied by the footing of their respective columns to save labor; but the lengths of the diagonals are carried in as indicated.

VALUES OF $\frac{y^2 l}{p^2}$.

B O	0.074	A L	1.447	O A	0.197 × 17.72	= 3.491	A B	0.202 × 14.08	= 2.844
D O	0.408	C L	2.310	B C	0.228 × 14.33	3.267	C D	0.230 × 11.17	2.569
F O	1.248	E L	3.713	D E	0.250 × 11.58	2.895	E F	0.252 × 9.15	2.306
I O	2.816	G L	5.890	F G	0.234 × 9.67	2.263	G I	0.238 × 7.75	1.844
K O	4.774	J L	8.655	I J	0.147 × 8.25	1.213	J K	0.149 × 7.17	1.068
N O	5.760	M L	10.844	K M	0.023 × 7.50	0.172	M N	0.024 × 7.07	0.170
	15.080 × 10		32.859			13.301			10.801
	9.320 × 10		2			2			2
	244.000		65.718 × 10.263 = 674.46			26.602			21.602

Summing these columns, and doubling for the whole arch, we obtain $244.00 + 674.46 + 26.60 + 21.60 = 966.66 = \Sigma . \frac{y^2 l}{p^2}$. If, in the first trial, all the sections are supposed equal, A may be omitted from (1.), § 185, and 966.66 becomes the denominator of that expression.

We next compute the following table, and multiply by the length of each piece as we advance. It will be convenient to add other columns, marked Σ, containing successive summations of the factors for each set of pieces, as these numbers will be used in turn. The *summations* are all *negative*, as will be readily seen, and hence the sign — is omitted.

VALUES OF $\frac{x\,y\,l}{p^2}$.

		Σ			Σ			Σ			Σ
B O	— 1.95	1.95	A L	— 4.37	4.37	O A	— 9.27	9.27	A B	— 7.53	7.53
D O	— 11.96	13.91	C L	— 20.92	25.29	B C	—10.30	19.57	C D	— 9.54	17.07
F O	— 40.90	54.81	E L	— 56.05	81.34	D E	—10.84	30.41	E F	— 8.63	25.70
I O	—104.47	159.28	G L	—124.44	205.78	F G	—10.37	40.78	G I	— 8.44	34.14
K O	—203.62	362.90	J L	—235.11	440.89	I J	— 7.42	48.20	J K	— 6.51	40.65
N O	—288.00	650.90	M L	—360.10	800.99	K M	— 2.19	50.39	M N	— 2.09	42.74
K′O	—287.61	938.51	M′L	—425.55	1226.54	N M′	+ 0.95	49.44	M′K′	+ 0.10	42.64
I′ O	—212.69	1151.20	J′ L	—391.85	1618.39	K′ J′	— 1.17	50.61	J′ I′	— 0.85	43.49
F′O	—126.03	1277.23	G′ L	—302.23	1920.62	I′ G′	— 5.68	56.29	G′ F′	— 4.55	48.04
D′O	— 61.97	1339.20	E′ L	—212.94	2133.56	F′ E′	— 9.59	65.88	E′ D′	— 7.64	55.68
B′O	— 22.47	1361.67	C′ L	—146.43	2280.00	D′ C′	—12.85	78.73	C′ B′	—10.07	65.75
			A′ L	—100.48	2380.48	B′ A′	—10.76	89.49	A′ O	fixed.	——

189. **Values of H.** — The calculations for H can now be proceeded with, and they are given below. An explanation of one computation will suffice for all. If a weight W is placed on the third upper joint from the left, the vertical component of the left abutment reaction, P_1, is $\frac{19}{24}$ W. Then, for the two pieces of the upper chord to the left we have $\Sigma P_1 \frac{x\,y}{p^2} l = 13.91\, P_1$; for the two pieces of the rib to the left, we get $25.29\, P_1$, and, for the five web-members to the left, $30.41 + 17.07 = 47.48\, P_1$. On the right of the weight, the nine remaining pieces of the upper chord give $\Sigma P_2 \frac{x\,y}{p^2} l = 1277.23\, P_2$, which will be found opposite F′ O, as the vertical force is now applied at the right end; for the ten pieces of the rib we find $2133.56\, P_2$, and for the rest of the web to E F we find opposite E′ F′ and F′ G′, for the reason

just stated, $65.88 + 48.04 = 113.92\, P_2$. As the piece E L, below the weight, is acted upon by P_1 on one side, and P_2 on the other, it makes no difference whether it is considered to lie to the left or the right of the loaded point. Adding up the respective numbers, multiplying one by $\frac{19}{24}$, and the other by $\frac{5}{24}$, adding, and dividing by $\Sigma \frac{y^2}{p^2} l = 966.66$, we get $H = 0.831$ W for a load on the third joint only. The divisor $966.66 \times 24 = 23{,}200$, is used.

VALUES OF H.

W on 1st Joint.		W on 2d Joint.		W on 3d Joint.	
0	1361.67	1.95	1339.20	13.91	1277.23
0	2380.48	4.37	2280.00	25.29	2133.56
9.27	89.49	19.57	78.73	30.41	65.88
9.27	65.75	7.53	55.68	17.07	48.04
23	3897.39	33.42	3753.61	86.68	3524.71
213.21		21	3	19	5
3897.39		701.82	11260.83	1646.92	17623.55
41.1060÷232=.177 W		11260.83		17623.55	
		119.6265÷232=.516 W.		192.7047÷232=831 W.	

W on 4th Joint.		W on 5th Joint.		W on 6th Joint.	
54.81	1151.20	159.28	938.51	362.90	650.90
81.34	1920.62	205.78	1618.39	440.89	1226.54
40.78	56.29	48.20	50.61	50.39	49.44
25.70	43.49	34.14	42.64	40.65	42.74
202.63	3171.60	447.40	2650.15	894.83	1969.62
17	7	15	9	13	11
3444.71	22201.20	6711.00	23851.35	11632.79	21665.82
22201.20		23851.35		21665.82	
256.4591÷232=1.105 W.		305.6235÷232=1.317 W.		333.0861÷232=1.436W.	

Having completed the computations for six joints, we add the H's, and multiply by two, obtaining 10.764 W as the value of H for an entire load of W on each upper joint.