imperfect manner. To make the frame into a true beam, this tendency of the loaded frame to slip through between the others must be counteracted by tongues, T, T_1, T_2, projecting from one frame and working in a groove in the neighbouring frame (*vide* § 19). Each tongue should be made so that it does not abut against the bottom of the groove, and is thus incapable of resisting any horizontal force—it must neither prevent the whole beam nor any part of it from being extended or compressed longitudinally.

The structure will now be found capable of carrying weights as a beam. It deflects or bends as in fig. 11

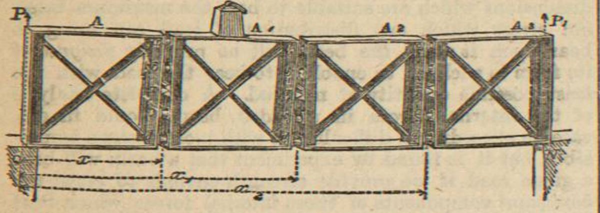

Fig. 11.

under the action of a load on A_1; all the pieces of india-rubber above the centre of the beam are then compressed; all those below the centre are then extended; and at the sections between the frames the horizontal internal forces are wholly met by the elastic reaction due to these horizontal extensions and compressions.

At any one section, say at a distance x_1 from the support Q, the pieces c_1 and d_1 are just as much compressed as the pieces $b_1\,a_1$ are extended; the equal and opposite parallel resultants of these forces consequently constitute a couple, and the moment of this couple must be equal to the moment of the couple tending to bend the beam at this section, or to what is called the *bending moment;* now the *bending moment* in this simple case is due to the upward vertical reaction P_1 at N and the equal downward force with which the frame A_1 bears on the tongue T_1; for it is clear that, neglecting the weight of the frame, if a weight W on frame A_1 is borne by two forces P and P_1 at the two piers, the tongue next N must also bear a vertical force P_1 and the tongue next Q a vertical force P. The stresses borne by the india-rubber pieces are exactly the same as if the frame A_1 were firmly held, and a vertical force P_1 applied to pull up the right hand part of the beam, while the tongue T_1 acted as a hinge; the moment tending to turn the right hand part round in a left-handed direction would be $P_1\,(L-x_1)$. This moment is resisted by the elasticity of the india-rubber, which must exert an equal and opposite moment round the same point. Calling s_1, s_2, s_3 the sectional areas of the pieces of rubber, and y_1, y_2, y_3 their distances from the axis where the section is neither extended nor compressed, and p_1, p_2, p_3 the intensity with which each piece is strained, the moment due to the elasticity of the pieces of rubber tending to turn back the left hand part of the beam in a right-handed direction will be Σpsy. Now, if the modulus of elasticity of the rubber is constant, the stresses p_1, p_2, p_3 will be proportional to their distance from the unstrained axis; thus if b_1 is 18 inches from the axis and c_1 only 9 inches, b_1 being shortened twice as much as c_1, the stress on b_1 will be twice that on c_1, and calling a the stress at unit distance from the axis we have $p_1 = ay_1$ and $p_2 = ay_2$, $p_n = ay_n$, so that we may write $\Sigma psy = a\Sigma sy^2$ as the expression for the moment of the elastic forces. Hence equating the bending moment and the moment of the elastic forces, we have—

$$1. \quad \ldots\ldots \quad P_1(L-x) = a\Sigma sy^2.$$

From this equation when the load is given we may determine a, and hence the intensity of the stress $p = ay$ at any distance y from the axis. If this intensity is less than the safe stress for the material, the beam is, at the section considered, strong enough to bear the load so far as the horizontal extending and compressing forces are concerned.

Thus if the dimensions of the beam be those given above and it be supported so that L may be 6 feet and the distance x_1 2 feet 3 inches, the distance y of the outermost piece of rubber from the unstrained axis 8 inches, the weight 50 lb, and the section of the rubber in each row 2 inches (two cylinders side by side, each with a section of 1 inch) we shall have as the numerical values in the above equation, neglecting the weight of the frame itself (P_1 being nearly = 18·7), 18·7 × 45 in. = a (4 × 8² + 4 × 4²) from which a = 2·64; then the force supported by each inch of either of the rows of rubber a_1 or d_1 will be $p = 8 \times 2{\cdot}64 = 21{\cdot}12$ lb; the stress on each of the inner rows will be half this amount; the same equations give the load which (so far as that particular section is concerned) can safely be placed on the frame A_1 consistently with a given stress per square inch on the rubber. The strength required in the tongue T_1 is still more easily found, the stress tending to shear it off is P_1, and it must have a sufficient cross section to bear that shearing stress. Similar reasoning would allow us to calculate the strength of our beam at either of the two other sections where the india-rubber pieces and tongues are placed. The general relation between the external and internal forces in any beam is similar to that illustrated by the model; at any section the moment due to the elastic forces must balance the moment due to the external forces tending to bend the beam at that section. The problem, therefore, of determining the strength of a beam at any section resolves itself, so far as the horizontal forces are concerned, into finding expressions for these two moments and equating them. The equation thus stated will give the maximum horizontal stress thrown on the material of a given girder by a given load, or it will give the maximum load on a given girder consistent with a safe stress on the material, or if, as is generally the case in bridges, the load and maximum safe stress on the material are given, the equation will allow us to fix the dimensions required for the cross section of the beam so far as the horizontal forces are concerned. The provision to be made for resisting the shearing stress for which the tongues are required in the model will be explained in § 19.

In any solid beam the stresses do not divide themselves into horizontal and vertical components. This division is made by the engineer to simplify his calculations. In the beam the actual stress at any point will be the resultant of the horizontal stress (borne by the india-rubber in the model) and the vertical stress (borne by the tongue in the model). The diagram, fig. 12, shows the direction of

Fig. 12.

the resultant stress at each point of a beam of rectangular cross section. The curves are called the lines of principal stress.

§ 14. *Bending Moment and Moment of Elastic Forces or Moment of Stress.*—The bending moment for a given section of a given beam under a given distribution and magnitude of load is the sum of the moments, taken relatively to the section, of all the external forces acting on one of the two segments into which the section divides the beam. It is a matter of indifference which segment is considered, but the moment on one segment will be positive and that on the other negative. Let x be the distance of the section from the left hand abutment Q of any beam of span L (fig. 13). Let P be the load borne by abutment Q, and P the load borne by

abutment N. Let the load on the beam, including the weight of the beam, be divided into any number of equal or unequal parts. Let those loads which lie to the left of the

Fig. 13.

section be called w_1, w_2, w_3, &c., and let the distances of their centres of gravity from the section be called l_1, l_2, l_3. Let those loads which are to the right of the section be called u_1 u_2, u_3, and let their distances from the section be called r_1, r_2, r_3 &c. Let the bending moment at the section be called M, then by the above definition

$$1 \qquad M = Px - \Sigma wl = \Sigma ur - P_1(L-x).$$

This bending moment is resisted at the section by the elastic forces in the beam called into play by compression at the top and extension at the bottom, or, in other words, called into action by the stresses on the material. Limiting our consideration to those cross sections which have a vertical axis of symmetry, and to those materials which have a constant modulus of elasticity for tension and compression, it is easy to find a general expression for the moment of the elastic forces, which will hereafter be designated by the letter μ. Let the horizontal axis ZZ_1 (fig. 14) be a line along which the material is unstrained,—

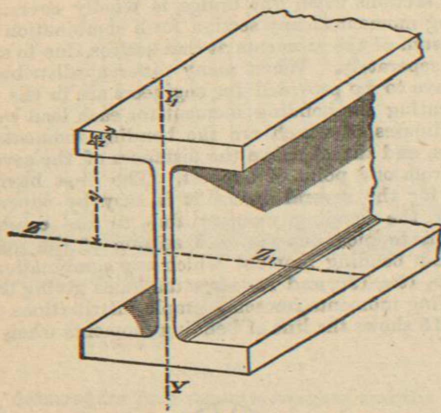

Fig. 14.

all the portions above this line being compressed, and all the portions below this line being extended. Conceive the section divided into little elementary surfaces, the area of each being equal to the product of the little elementary length Δy into the little elementary length Δz, the force exerted by the elasticity of this element will be proportional to its area $\Delta y \Delta z$ and to its distance y from the axis ZZ_1, for the longitudinal extension or compression is directly proportional to this distance; then calling a, as before, the rate at which the stress increases with the increase of y, or in other words the intensity of the stress at the unit distance from ZZ_1, we shall have for the force exerted by each element the expression $ay\Delta y\Delta z$. Now, as the section is not moved as a whole along the beam in either direction, we must have $\Sigma ay\Delta y\Delta x = 0$, that is to say, the sum of the positive must be equal to the sum of the negative forces. Let the constant quantity $a\Delta y\Delta x$ be conceived as the weight w of a thin plate of the area $\Delta y\Delta x$, the moment of this weight relatively to the axis will be wy, but as we have the sum of the moments $\Sigma wy = \Sigma ay\Delta y\Delta x = 0$, the axis ZZ_1 round which these moments are taken must pass through the centre of gravity of the section. This axis is called the *neutral axis* of the beam (*vide* § 18). The expression for the force exerted by each element being $ay\Delta y\Delta x$, the moment of this force, or, in other words, for the moment due to the stress, is given by the expression $ay^2\Delta y\Delta z$; then for the moment μ of all the forces round ZZ_1 we have—

$$2. \quad \ldots\ldots \quad \mu = a\Sigma y^2\Delta y\Delta z = aI,$$

where I is the moment of inertia of the surface of the cross section. This expression μ is Rankine's moment of resistance to flexure; it may also be called the moment of the elastic forces, or, as suggested by Professor Kennedy, the moment of stress.

The greatest intensity of stress p_1 occurs at the greatest distance from ZZ_1, or at the distance $\frac{1}{2}d$ if the depth of the beam be called d, and the axis ZZ_1 be equidistant from top and bottom. In this case we have—

$$p_1 = \frac{ad}{2}, \text{ or } a = \frac{2p_1}{d};$$

hence we may write for sections with two axes of symmetry, and for materials having equal strengths and constant moduli of elasticity under tension and compression—

$$3. \quad \ldots\ldots\ldots \quad \mu = 2\frac{p_1 I}{d}.$$

Now, the general condition of equilibrium between the external forces and the elastic forces at the section is simply $M = \mu$, hence when the beam does not break we have also—

$$4. \quad \ldots\ldots\ldots \quad M = \frac{2p_1 I}{d}.$$

Whenever the amount and distribution of load on a given girder is known the bending moment M is to be calculated from equation 1, and then equation 4 allows us to calculate p_1; or if we know the value of f, the ultimate strength of the material to resist tension and compression, this equation enables us to find what moment M_1 is required to break the beam at that section, and therefore what load distributed in a given way the beam can bear; thus we have—

$$5. \quad \ldots\ldots\ldots \quad M_1 = \frac{2fI}{d}.$$

In a beam of uniform strength the value of $\frac{2fI}{d}$ will at every section be equal to the value of M_1, the moment due to external forces at that section.

When the material is not equally strong to resist tension and compression, let f_t and f_c be the two strengths (per unit of cross section), but let the modulus of elasticity be assumed constant as before. Then, as above, the unstrained axis will be the axis passing through the centre of gravity of the section, and the intensity of stress at any distance y above or below that axis will be ay; let y_c be the distance of the uppermost element of the cross section from the horizontal axis, and let y_t be the distance of the lowermost element. Then the greatest stress will occur at the top if y_c be greater than y_t, but at the bottom if y_t be greater than y_c; since, however, the material is not equally strong to resist tension and compression, it does not follow that it will give way where the stress is greatest, and the beams will yield first at that edge where the ratio $\frac{y}{f}$ is greatest.

We must, therefore, on a beam of this kind ascertain whether $\frac{y_c}{f_c}$ or $\frac{y_t}{f_t}$ is the greater, and replace $\frac{d}{2}$ in equations 3 and 4 by the value of y in the larger of the two expressions. Having selected y and f in this manner, we have—

6. . . . . $M_1 = \frac{fI}{y}$.

§ 15. *Modulus of Rupture.*—If the above hypothesis of a constant modulus of elasticity in a given material under both descriptions of stress, and under stresses of all magnitudes, were accurate, we should require no fresh experiments to determine the values of f_c and f_t; these would be already known from direct experiments on tension or compression. For both wrought and cast iron beams of the maximum strength, such as will be described hereafter, experiment gives results closely in accordance with strengths calculated by equation 6; and for wrought iron beams the hypothesis appears to be approximately true when tested by experiment with any form of beam. But it is usual, in calculating the strengths of beams or bars of simple rectangular or circular cross section, to assume that y is equal to $\frac{1}{2}d$ and to employ equation 5 instead of equation 6. The imperfection of the theory is then to some extent corrected by determining f from direct experiments on solid rectangular bars. The value of M_1, I, and d being known, f is calculated from equation 5, and the number thus determined has received the name of modulus of rupture.

TABLE VIII.—*Modulus of Rupture* = f (*Rankine*).

Name of Material.	Lbs. per sq. inch.
Wrought Iron Plate Beams	42,000
Cast-iron Solid Bars	33,000 to 43,000
Fagersta Steel (Kirkaldy)	110,000 to 191,000
Red Pine	7100 to 9540
Larch	5000 to 10,000
Oak (British and Russian)	10,000 to 13,600
Indian Teak	14,770
Sandstone	1100 to 2360

Experiments on the modulus of rupture have generally been made by hanging weights at the centre of a rectangular bar, supported at both ends, and increasing the weights till the bar breaks. Then let b and d and l be the breadth, depth, and length of the bar in inches, and W the breaking weight. M is a maximum at the centre, and, neglecting the weight of the bar, is equal to $\frac{1}{4}Wl$; substituting the value of I for a rectangle in equation 3, we get

$$\mu = \tfrac{1}{6} f b d^2,$$

and equating M and μ we have—

1. . . . . . . $f = \frac{3Wl}{2bd^2}$,

or, if the span is measured in feet and called L_1, while b and d are measured in inches, we have—

2. . . . . . . $f = 18\,\frac{WL_1}{bd^2}$.

Hence the modulus of rupture is stated by Rankine to be "eighteen times the load required to break a bar of 1 inch square, supported at two points 1 foot apart, and loaded in the middle between the points of support."

The use of a modulus of rupture determined by experiment on a special form of beam is not based on any satisfactory principle. The employment of this modulus is an imperfect means of correcting a defective theory. A different value is found for f when bars of different sizes or cross sections are tested. Even the same bar broken in different ways will give a sensibly different value for f. The use of this modulus is, however, convenient when great accuracy is not required.

§ 16. *Expressions for the Bending Moment caused by Special Distributions of Load.*—From equation 1, § 14 it is easy to deduce the following values for the bending moments in beams subject to various simple distributions of load:—

Case 1. Let a single load W be placed at the centre of a beam; let M_x be the moment at any section taken at the distance x from the nearest pier; let M_c be the moment at the centre of the span, we have—

1. . . . . . . $M_x = \frac{1}{2}Wx$,

2. . . . . . . $M_c = \frac{1}{4}Wl$.

M_c is the maximum moment at any section of the beam.

Case 2. Let a single load W be placed at the distance x_1 from the nearest pier; let M_{x_1} be the moment at the section where the load is applied—

3. . . . . . . $M_{x_1} = W\frac{lx_1 - x_1^2}{l}$.

M_{x_1} is the greatest moment which the load at that place produces at any part of the beam, and increases as the load rolls from the end to the centre in proportion to the product $(l - x_1)\, x_1$ of the parts into which the load divides the beam.

Case 3. Let the load be uniformly distributed along the beam so that each unit of length bears an equal load $w = \frac{W}{l}$; when the unit of length adopted is the foot, the quantity w is called the *load per foot run.* Let M_c and M_x be as before the moments at the centre of span, and at any distance x from the nearest pier, we have—

4. . . . . . . $M_c = \frac{1}{8}wl^2$,

5. . . . . . . $M_x = \frac{1}{2}w\,x(l - x)$.

When a uniform passing load, such as is approximately represented by a train of locomotives, of length at least equal to the span of the bridge, comes on to a bridge at one end and passes over to the other, gradually covering the whole span, the bending moment reaches a maximum for all sections when the bridge is wholly covered. The bending moment at any section for a combination of loads is the sum of the moments at that section due to each load taken separately. When many different distributions of load have to be provided for engineers are in the habit of representing the bending moment for each load by a line, the ordinates of which are the bending moments at the sections, and the abscissæ the distances of the several sections from one point of support. The lines having been drawn for the several loads, it is easy by superposition to find the bending moment due to the combination, and thus to pick out for each section of the girder the maximum bending moment which any combination gives. Figs. 15, 15a, 15b, and 15c show diagrams giving the curve of bending moments for some simple distributions.

Fig. 15 shows the line of bending moments when a single

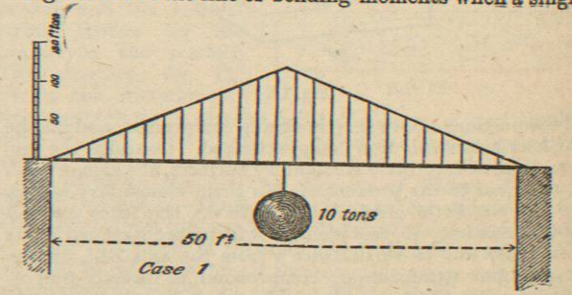

Fig. 15.

load of 10 tons hangs at the centre of a span of 50 feet. The vertical ordinates measured on the vertical scale give the bending moments at each section. Fig. 15a shows the curve of bending moments for a uniformly distributed load

of 1 ton per foot run on a span of 50 feet. The upper curve in fig. 15b shows the curve of bending moments

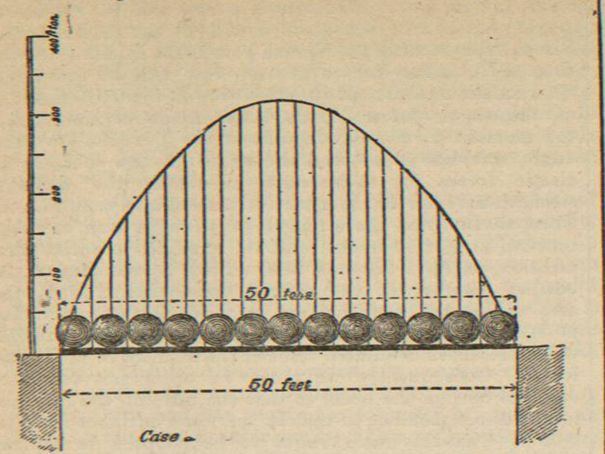

Fig. 15a.

when the loads of case 1 and 2 both occur at once. This curve is obtained by adding the ordinates in case 1 to those in case 2. Fig. 15c shows the four separate lines of

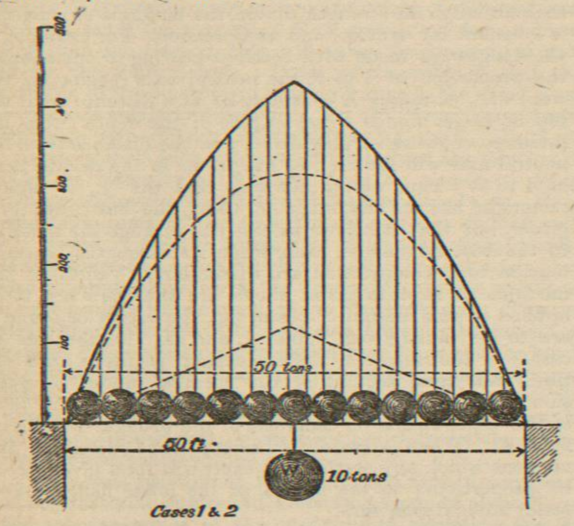

Fig. 15b.

bending moment for four separate weights, and the broken upper line is the line of bending moments for the case when the four loads all rest at once on the beam; it is got by simply adding at each point of the span the four ordinates due to the four loads considered separately. The curve in fig. 15a and the curve ABCD in fig. 15c are parabolas.

The bending moment at any section is reckoned as *positive* when the external forces on the beam tend to turn the right hand side of the beam in a left-handed direction (or in the direction opposite to that followed by the hands of a watch), in other words, when the bending moment tends to bend the beam downwards between the supports. We shall hereafter see that in beams with more than two supports the bending moment is at places *negative*, tending to bend the beam up; the curve of bending moments is drawn below the datum line where the moments are negative. Fig. 25a gives an example of a curve of bending moments of this class.

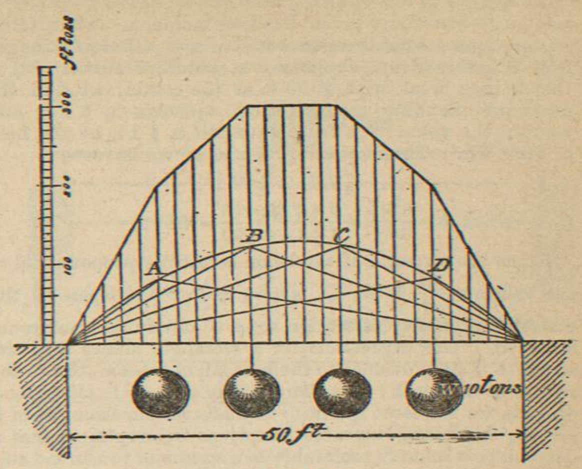

Fig. 15c.

§ 17. *Moment of Elastic Forces.*—On the hypothesis above stated, viz., that for any one material the modulus of elasticity may be taken as constant for all stresses, and assuming that our investigation is to be confined to those cases in which the cross section of the beam has a vertical axis of symmetry, and in which the centre of gravity lies at a point equidistant from the top and bottom of the beam, the general equation—

$$\mu = aI = 2\,\frac{p_1 I}{d}$$

(3, § 14), allows simple expressions of the value of μ to by obtained for all practical cases by substituting for the ratio $\frac{2I}{d}$ its value in terms of the dimensions of the cross section. Thus, for a rectangle of the depth d and breadth b,

$$\frac{2I}{d} = 2\,\frac{bd^3}{12d} = \frac{bd^2}{6} = \frac{Sd}{6},$$

where S is the area of the cross section. The following are the values for the commonest forms of cross sections.

TABLE IX.—*Values of* $\frac{2I}{d}$ *for various Cross Sections.*

Form of Cross Section.	$\frac{2I}{d}$ — In Terms of Linear Dimensions of the Cross Section.	$\frac{2I}{d}$ — In Terms of the Area of the Cross Section (S) and the Depth.
Square, side b	$\frac{b^3}{6}$	$\frac{Sb}{6}$
Rectangle, breadth b depth d	$\frac{bd^2}{6}$	$\frac{Sd}{6}$
Circle, radius r	$\frac{\pi r^3}{4}$	$\frac{Sd}{8}$
Hollow circle, external radius r, internal r_1	$\frac{\pi(r^4 - r_1^4)}{4r}$	$\frac{Sd}{4}$ when r_1 differs little from r.
Hollow rectangle, internal depth d_1, breadth b_1	$\frac{bd^3 - b_1 d_1^3}{6d}$	$\frac{Sd}{2}$ when moment of web is neglected, and S is taken as area of cross section of flanges only, and d_1 differs little from d.