Substituting the modulus of rupture f (Table VIII.), for p_1 in equation 3, § 14, we can from the above values of $\frac{2I}{d}$ calculate the ultimate strength of any given cross section to resist any given bending moment. Thus, if we wish to know what breadth we must give a bar of wrought iron 3 inches deep, supported at points 3 feet apart, so that it may break with 2000 lb at the centre, we find the maximum bending moment from equation 2, § 16; and we find the value of μ from equation 3, § 14, by the help of table IX. Then equating M and μ, we have—

$$1. \quad \ldots \quad \tfrac{1}{4}WL = \tfrac{1}{6} f b d^2 ,$$

from which $b = \frac{6WL}{4d^2 f} = \frac{6 \times 2000 \times 36}{4 \times 9 \times 42000} = \cdot 286$.

Since the strength of the beam is directly proportional to the values of $\frac{2I}{d}$, table IX. shows us how to dispose of the material in the cross section so as to obtain the maximum strength. The expression for a rectangle shows that the strength is proportional to the breadth of a beam, but to the square of its depth; and therefore, as appears in the second column, for the same quantity of material the strength of a rectangular beam increases simply as the depth, so that a deep narrow beam is preferable to a square or to a broad and flat one. The circular cross section is weak compared even with the square (the ratio for the same quantity of material is ·846 to 1). The hollow tube is stronger than the thin rectangular plate of the same depth and cross section, but clearly the material is much best applied when wholly used in the form of two thin flat plates, separated from one another by a web of the maximum depth d which can be practically allowed. Thus the hollow rectangle is the form preferred for large girders, the material intended to resist the bending moment being placed in the top and bottom members of the girder, and kept apart by vertical webs, which add somewhat to the moment μ, but which are chiefly employed to resist shearing stress, as will be presently shown.

The I section, fig. 16, is that employed for small girders: the moment of its elastic forces is exactly the same as that of a hollow rectangle, having the same values for b, d, and d_1, and having a value for b_1 equal to $2b_2$ in fig. 16. It is usual to neglect the strength to resist bending moment given by the vertical web, and to design the girder so that the top and bottom plates are alone sufficient to meet this stress. The model, fig. 10, shows at once that to obtain the full resistance from the material employed to resist tension or compression consistently with a given depth, it must all be placed in two horizontal plates at the extreme top and bottom of the beam; moreover, if $\frac{1}{2}$S is the sectional area of the top or bottom member, and f is the stress it will bear (assumed equal for tension and compression), the maximum moment which the elastic forces can exert is the sum of the moments due to the top and bottom members about the axis or tongue, *i.e.*, $2(\frac{1}{2} f S \times \frac{1}{2} d)$ or $\frac{1}{2} f S d$; or dismissing the idea of an axis through the centre of gravity of the section, we may (to refer again to the illustrative model) take the moment of the elastic forces round the line running through the top row of india-rubber pieces looked upon as a fulcrum, and then the moment of the bottom row will be as before $\frac{1}{2} f_t S d$, and will be the *whole moment* of the elastic forces. Similarly, by taking the bottom row as the fulcrum, we see that the moment due to the top row will be the same. This moment calculated in either way is the *whole* moment of the elastic forces. This latter mode of considering the moment in the simple case of an I or hollow rectangle, enables us to find the value of the moment of elastic forces for those cases in which the ultimate strength may not be the same for tension and compression.

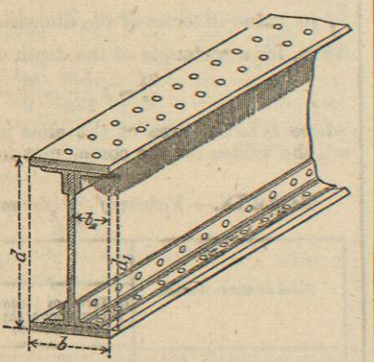

Fig. 16.

Thus, assume that the strength of wrought iron to resist tension, or f_t, is 25 tons, and its strength to resist compression, or f_c, is 20 tons, then calling S_t the area of the bottom member at the section considered, and S_c the area of the top member, we have for μ the two values 25 $S_t d$ and 20 $S_c d$; whichever happens to be the smaller will represent the available value of the moment of the elastic forces. Consequently we must, in order not to waste material, design the beam so that the ratio of the material in the upper member to that in the lower member shall be 5 to 4. That this ratio ought to be adopted is evident from the fact that the strength of the beam will be limited by that of the weaker member. No structure is perfectly designed unless it will when overstrained give way simultaneously in every part. The foregoing theory, the soundness of which is borne out by experiment, tacitly assumes that, although the strength differs, the modulus of elasticity is constant for tension and compression. For example, if the flanges are made with sections bearing to one another the proportion of 4 to 5, the neutral axis (neglecting the web) will, assuming E constant, be at a distance from the top or larger flange equal to $\frac{4}{9}$ths of the depth; then the intensity of stress varying directly as the distance from the neutral axis will for the two flanges be in the desired ratio of 4 to 5. Thus we see not only that the I, or hollow rectangle, has the advantage of being the best form for a girder, but that it allows us easily to arrange the material to the best advantage, even when its strengths to resist tension and compression are dissimilar. Values of the modulus of rupture given above are therefore not to be applied to this design of beam, but the values of f_t and f_c are to be taken from Tables I. and II. In the case of cast-iron the member under tension is made with six times the cross section of the member under compression, the reason being the same as that for making the ratio of the upper and lower members of the wrought iron beam 5 to 4. When a cast-iron beam is thus designed, the moment which any section can exert will for a given depth be proportional to the area S_t of the lower flange. Professor Hodgkinson verified this theory experimentally, and found that the ultimate value of the moment due to elastic forces expressed in lbs and inches for beams thus designed was $\mu = 16500\ S_t d$. The value of the constant agrees closely with the tensile strength of cast-iron. The experiments by Professor Hodgkinson are therefore consistent with the assumption, that although the strength of cast-iron is very different in resisting tension and compression, nevertheless the modulus of

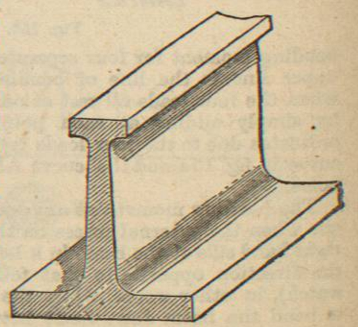

Fig. 17.

elasticity is equal under the two kinds of stress. Fig. 17 shows the cross section usually adopted for cast-iron girders.

§ 18. *Neutral Axis.*—The line ZZ_1, fig. 14, perpendicular to the plane in which a beam is bent, and passing through the centre of gravity of any given cross section, is called the *neutral axis* of the beam at that point. The surface containing all the neutral axes is the *neutral surface.* Practical engineers sometimes apply the term neutral axis to the longitudinal line showing the neutral surface on a side elevation, but in this article, as in Rankine's works, the words will be used as defined above.

When the assumption is made that the modulus of elasticity is the same for any given material whether under tension or compression, notwithstanding any difference in the ultimate strength to resist tension or compression, then it follows, as has been shown, that the neutral surface of a bent beam will separate it into two parts, one of which is compressed while the other is extended. The neutral axis in any cross section then contains the only part of the material which is neither extended nor compressed.

If, however, the average value of E for stresses varying between zero and the maximum intensity of compression to which the beam is subjected be different from the average value between zero and the maximum intensity of tension, then the neutral axis as above defined will not be the *unstrained* axis; the neutral axis is determined as soon as the cross section of the beam is known, being independent of the material used; the unstrained axis may differ in beams of the same cross section but made of different materials; for if the average of E be greater say for compression than for tension, this will raise the unstrained axis above the neutral axis. It is not improbable that the position of the unstrained axis may vary in the same beam with loads similarly distributed, but of different magnitude, and also with different distributions of load. Until experiments shall have accurately determined the relation of E to the intensity of stress we have no means of determining accurately the position of the unstrained axis. Even when E is constant the neutral axis, as above defined, will not always in practice correspond with the unstrained axis; for instance, in a beam which was not only bent across, but also compressed endways, the unstrained surface would no longer contain the neutral axis as above defined. The unstrained surface might be near one edge of the beam, or, indeed, if the general compression were large and the bending small, the whole beam might be under compression, so that no part was unstrained. By restricting the use of the words neutral axis to the above definition, and using the words unstrained axis, or unstrained surface, for the second idea all ambiguity will be avoided.

The actual position of the unstrained axis in any beam of any material subject to a bending moment depends on the relative values of the modulus of elasticity for the material under all stresses positive and negative, great and small; but as the simple hypothesis of a constant modulus is borne out by experiment in the most important materials, it is unnecessary to pursue this subject further.

§ 19. *Shearing Stress.*—The theory hitherto given shows the relation between longitudinal stresses (such as are resisted by the india-rubber in the model) and the load on the girder; but in designing a girder we have also to provide for the shearing stress or transverse force tending at each imaginary cross section to make the more heavily loaded of the two parts into which it divides the bridge slide down past the other. This shearing stress was resisted by the tongues of the model. The total shearing stress at any section is the sum of all the vertical forces acting on the beam on one side of the section. The shearing stress at any section will be called positive if the sum of the external forces on the right hand part of the beam tends to lift that portion up. Diagrams may be conveniently used to show shearing stresses, and, as in the case of bending moments, the shearing stress at any section due to two or more loads is simply the algebraic sum of the shearing stresses due to each load.

Example 1. Load W at centre between supports (fig. 18); weight of beam neglected. The shearing stress is

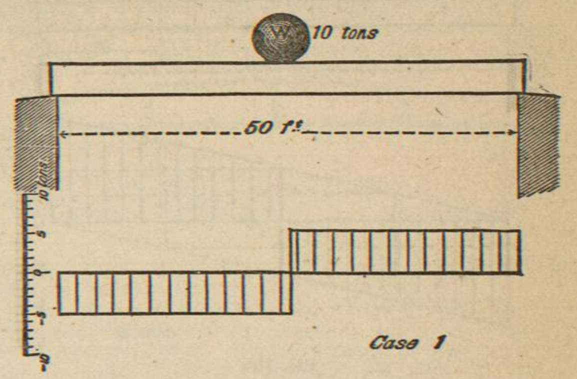

Fig. 18.

equal to $\frac{1}{2}$W all along the beam, being the reaction at one pier; the stress is positive to the right of the load, negative to the left.

Example 2. Uniformly distributed load w per foot run (fig. 18a). The shearing stress is $\frac{1}{2}w$L at the points of

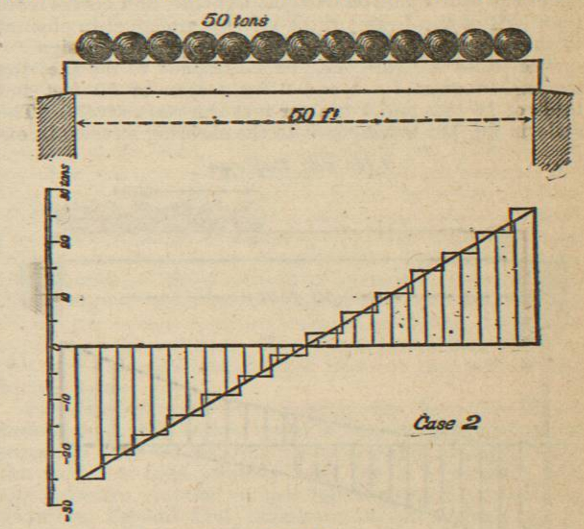

Fig. 18a.

support, and zero at the centre of the span; at any section distant c from the centre of the beam it is wc.

Example 3. A single load rolling from right to left of a beam of span L (fig. 18b). When the load is at the distance x from the right hand support the shearing stress to the right of the load is $W\frac{L-x}{L}$, and to the left it is $-\frac{Wx}{L}$. The maximum stress for each section occurs when the load reaches that section; it is positive for the right half, and negative for the left half of the beam.

Example 4. Uniform advancing load of w per unit of length (fig. 18c). When the load covers a length x measured from the right hand pier, the shearing stress at all points beyond x towards the left is $-\frac{wx^2}{2L}$; when x is greater than $\frac{1}{2}$L

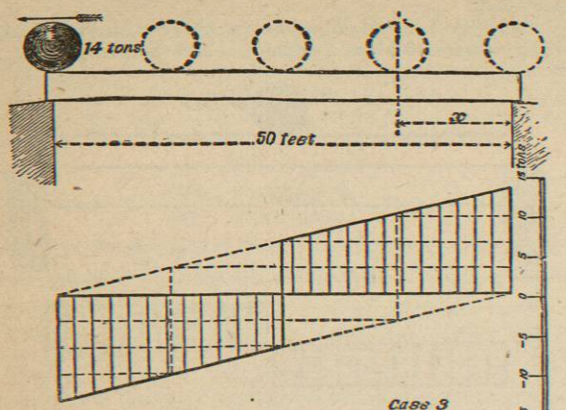

Fig. 18b

the above expression gives the maximum stress which can occur on that section with any distribution of the given uniform load. Thus the maximum shearing stress at the left end occurs when the whole bridge is loaded, and is $-\frac{1}{2}w$L; the maximum stress at the centre occurs when the bridge is half loaded, and is equal to $-\frac{1}{8}w$L. The maximum shearing stresses on the other half of the beam occur when the load comes on from the left side, and covers more than half of the beam; these stresses are equal in amount to the stresses in the left half, but are positive in sign.

The scales in figures 18, 18a, correspond to the shearing stresses in examples 1 and 2 for a span of 50 feet and loads of 10 tons and 1 ton per foot run respectively. The scale in fig. 18b corresponds to the shearing stresses in ex-

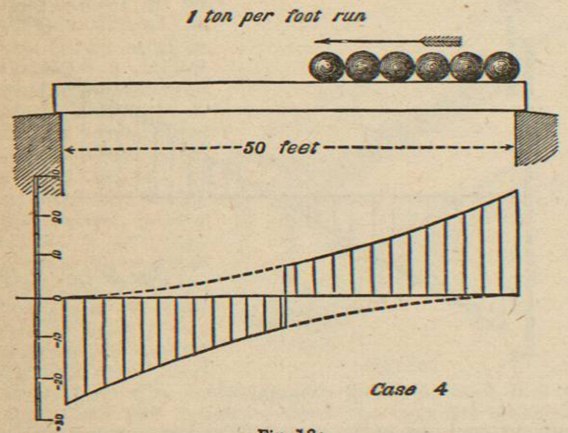

Fig. 18c.

ample 3 with a single passing load of 14 tons. The scale in fig. 18c, example 4, gives the maximum shearing stress which the advancing load of one ton per foot run can produce at each section. As a train leaves a bridge it produces the same shearing stresses as when it comes on to the bridge from the opposite end, the same portions being similarly loaded. The maximum shearing stress due to a passing load of this kind changes its sign at the centre of the span, as appears by diagram 18c.

In I girders with solid vertical webs the shearing stress is practically all borne by the web unassisted by the top or bottom members, it being clear that these would fold down at the sides under a small fraction of the total stress. Sufficient material is therefore employed in the web or upright plates to reduce the intensity of the stress to the desired amount. The shearing stress may, however, when the web is a thin iron plate, cause failure by crumpling the web, or causing it to buckle, instead of by shearing it across. This tendency is prevented by stiffening the web with angle or T irons rivetted to the sides. Mathematical analysis has not yet been very successfully applied to the determination of the amount of stiffening required; experience has given a sufficient number of examples to guide the practical designer. In cast-iron girders the web is generally excess of the strength required to resist shearing.

§ 20. *Factor of Safety.*—In designing a girder the load which it will have to carry is multiplied by a number called the factor of safety, varying from 3 to 6, and the girder is so designed that it shall not yield at any point with less than the load thus multiplied. If, for instance, the girder is practically to carry 1 ton per foot run, it is designed so that at no place shall it *break or yield injuriously* with less than say 5 tons per foot run. The multiplier is called the *factor of safety.* The factor of safety is required to allow for imperfections in the material as compared with picked specimens, for the wear and tear by which the strength of a structure is gradually reduced, for unforeseen loads, for jars and vibrations, for imperfection of theory, and for the sake of obtaining stiffness. This last property might be the subject of calculation, and in some cases must be separately examined. The particular factor employed depends on the judgment of the engineer. A larger factor of safety is required for a passing or moving load than for a permanent load, there being a greater uncertainty as to the stress which may be caused by vibrations or impulses due to what is sometimes called a live load. Moreover, the mere presence of a large permanent load tends by its inertia to diminish the dangerous effect of the impulses or stresses due to the passing load, so that the factor of safety should be chosen with reference to the ratio $\frac{\text{max. passing load}}{\text{max. permanent load}}$ being larger as this ratio increases. Rankine recommends that the factor of safety should for the moving load be double that employed for the permanent load. Sometimes the factor is more conveniently employed as a divisor to deduce the safe stress f_1 from the ultimate strength f of the material, rather than as a multiplier for the load. Thus the same number of square inches will be obtained in the tension member of a wrought iron girder to bear 1 ton per foot run, whether we use in the calculations 25 tons as the value of f, the ultimate strength of the material, and a load w of 5 tons per foot run, or if we use 5 tons as f_1, the safe stress on the material, and a load w of 1 ton per foot run; in short, if we call the factor of safety K, we may in equation 5, § 14, use $KM = \frac{2fI}{d}$, or making $f_1 = \frac{f}{K}$, we may write

$$1. \quad . \quad . \quad . \quad . \quad M = \frac{2f_1 I}{d}.$$

The same remark applies, of course, to equation 6, § 14.

§ 21. *Weight of Girders and Roadway.*—When two girders are employed for each line of way on a railway, the weight of the iron girders per foot run will, with the usual proportions, probably lie between 0·0017L tons and 0·0025L tons, L being the span in feet. It follows from the theory given above, that for similar beams the quantity of material in the whole girder will be proportional to the square of the length, and, therefore, the quantity per foot run will be proportional to the

simple length. The constants given above are derived from practice. The weight of girders for a common road, if placed from 7 to 8 feet apart, will be nearly the same as for railway girders of the same span. The weight of a cast-iron railway girder (two girders per way) will be about 0·005L tons per foot run. The weight of the *roadway* in a railway bridge will probably be from 0·14 to 0·22 tons per girder, or double this for each line. For a turnpike road with metalling the weight will much exceed this, and should in each case be computed.

§ 22. *Design of a Girder.*—(1.) From the span and load to be carried the engineer will determine the material and form to be employed. Cast-iron may in some districts be the cheapest material for girders under 30 feet span. Wrought iron I girders are very generally employed for spans of from 30 feet to 100 feet; beyond that span lattice or framed girders are more usually employed. For extreme spans exceeding, say, 300 feet, a hollow rectangle or tubular bridge may be used, carrying the road on its top or inside the tube. The depth of the cross section is limited by the consideration that the web must be sufficiently stiff not to buckle; but for this consideration the deeper a girder could be made the better. In practice the depth is made from $\frac{1}{8}$th to $\frac{1}{15}$th of the span. The engineer will also determine whether he will keep the depth of the girder constant throughout or diminish the depth at the ends. It is impossible to graduate the material so as to give absolutely uniform strength at all sections, but by diminishing the depth towards the ends, some material may be saved without attenuating the top and bottom members to such an extent as to be inconvenient. When the general character of the design has thus been settled, the engineer will compute the probable weight of the girders and roadway or total permanent load; he will next determine the passing load for which he intends to provide.

(2.) The value of M, the bending moment, must next be computed for a sufficient number of cross sections of the beam, and for various distributions of load. For a small cast-iron girder of uniform cross section a single value of M will be sufficient, computed for the section at the centre when the girder is wholly covered with the greatest uniform load and also supports the greatest single load at the centre of the span. When, as in larger girders, the design is intended to give a structure of approximately equal strength throughout, the maximum value of M should be found for eight or ten sections; this maximum value will be that obtained when the bridge is wholly loaded with its maximum uniform load and has the maximum single load resting just over the section in question.

(3.) The maximum shearing stress must next be calculated for each of the above sections. The designer will bear in mind that the maximum stress occurs at the points of support, and that at the centre it is greatest when the bridge is half covered with the passing load.

(4.) The engineer can now compute the number of square inches S_c and S_t required at each section in the upper and lower members consistently with the factor of safety he chooses to employ; this he obtains from the expressions—

$$1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad S_t = \frac{M}{f_{t1} d},$$

$$2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad S_c = \frac{M}{f_{c1} d}.$$

It is here assumed that the best and strongest form of girder is employed, but if a mere square or circular beam is to be used, the cross section will be obtained by equating the values of M and μ, using a safe modulus of rupture f_1.

(5.) The web will next be designed by giving it such a thickness as will, with the depth already fixed, supply the number of square inches required to reduce the stress per square inch to the *safe* or *proof* shearing stress, say 4 or 5 tons on wrought iron. When the web is a thin wrought iron plate it must be stiffened with ⊥ or angle irons. In a cast-iron girder the web must have at least the number of square inches required by the shearing stress, but the exigencies of the foundry generally require a design resulting in a great excess of strength in this part of the beam, except in beams which are tapered towards the ends, as in fig. 19. With these beams care must be taken that

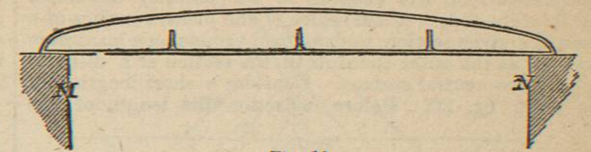

Fig. 19.

the taper is not carried to excess so as to leave insufficient metal to resist the shearing stress at M and N.

§ 23. *Practical Details.*—The designer must be practically acquainted with the forms in which his materials can be best procured. He must know the sizes in which iron or steel plates can be produced, and the forms best adapted for castings. Thus, in cast-iron beams the thickness of the web is at the bottom made equal to the thickness of the lower flange, and at the top to the thickness of the upper flange, in order to avoid permanent internal strains, which would result from unequal rates of cooling after being cast, if sudden changes of thickness in the metal were allowed. The engineer must also be familiar with the methods adopted of joining the several parts, as with the rivetting of wrought iron, the bolting together of large castings, the jointing of wood-work. He should also be acquainted with the various methods in which roadways are constructed and supported on existing bridges, and the manner in which the girders are braced one to another, so as to prevent vibration and lateral deflection due to the pressure of the wind. The examples of bridges described hereafter will give some information on these points. In long girders provision must be made by rollers, sliding plates, or suspension links for the expansion and contraction due to changes of temperature. The range in Great Britain may be taken as about 45° C. If the ends of the girder could be firmly secured at a constant distance apart this change of temperature would produce a stress of about 6 tons per square inch in wrought iron, and 3 tons per square inch in cast-iron. The result in practice would be that any attempted fastening of stone or iron work would be torn loose.

§ 24. *Deflection.*—When a bridge has been erected its deflection at the centre under a known passing load is generally observed with the object of ascertaining whether the work has been properly done, for it is assumed that any defective material or bad jointing would increase the deflection beyond that calculated on the assumption of sound material and perfect workmanship. Sometimes the practical test applied is a rough one, a certain fraction of an inch being allowed per foot of span as a safe deflection. If an inspector of bridges, having authority, chooses to limit the deflection to a constant fraction of the span, the ratio of the depth to the span must be made sufficiently great to give the desired stiffness and maintained constant for all spans; equation 5 below shows that when p_1 is kept constant and d is a given fraction of L, the deflection v will be proportional to the span. For the proof or maximum possible load, Rankine gives as the result of practice a value for the deflection of from $\frac{1}{200}$L to $\frac{1}{800}$L; but one foot deflection in a span of 200 feet would certainly be exces-