sive. A depth equal to or greater than $\frac{1}{12}$th of the span is certain to give sufficient stiffness, and the usual method is to assume the depth and then to observe whether experiment gives a deflection agreeing with that found by calculation. The calculation is made by finding the radius of curvature of the beam at a series of sections, and then determining the curve assumed by the whole beam either by integration or by an approximate graphic method. When the curve is known the deflection or versed sine is found either from the equations of the curve, or by actual measurement on the diagram to be presently described. Let R be the radius of curvature of the neutral surface of a beam at a given section, under a load producing a maximum stress p_1 at the outer elements of the section at a distance y_1 from the neutral surface. Consider a short length x of the beam fig. 20. Before deflection the length of the

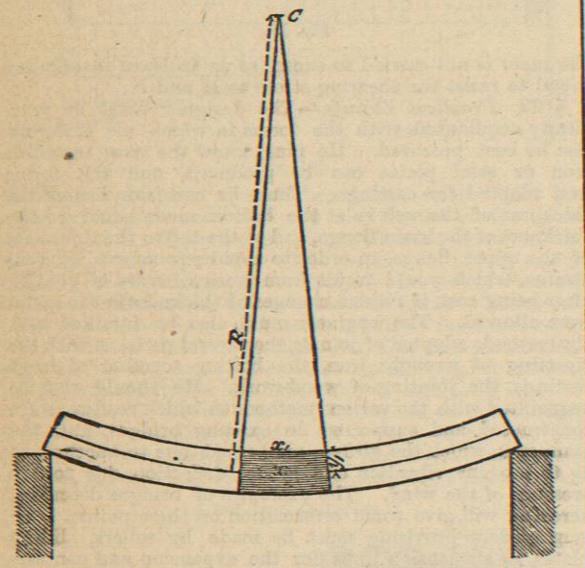

Fig. 20.

outer element is equal to x, but after deflection (if we consider the upper member) the length of the outer element will be shortened to x_1, while the length of the element in the neutral surface remains equal to x. By similar triangles we have $x : x_1 = \mathrm{R} : \mathrm{R} - y_1$, or $\mathrm{R} = \frac{y_1 x}{x - x_1}$, but $x - x_1$ is the extent to which the most compressed element is shortened, and by the definition of the modulus of elasticity we have $\mathrm{E} = p_1 \frac{x}{x - x_1}$, hence

$$1. \quad . \quad . \quad . \quad . \quad . \quad . \quad \mathrm{R} = \frac{\mathrm{E}y_1}{p_1},$$

from which the radius of curvature at any section can be obtained in terms of E, y_1, and p_1, all known quantities with a given cross section, material, and load. Another form of the same expression, which is sometimes more convenient, is obtained by remembering that $\mathrm{M}_1 = \frac{p_1 \mathrm{I}}{y_1}$, hence

$$2. \quad . \quad . \quad . \quad . \quad . \quad . \quad \mathrm{R} = \frac{\mathrm{EI}}{\mathrm{M}_1},$$

where M is the bending moment which will produce p_1. If p_1 is made equal to f_1 the maximum safe stress on the material, equation 1 or 2 gives the minimum safe radius of curvature.

We will first consider the special case in which the beam under consideration is of *equal depth and uniform strength* at all cross sections, and which we will call a beam of class 1; these conditions can only be fulfilled by any one beam for a given constant distribution of load; the beam of uniform strength for a load at the centre, for instance, will clearly not be the beam of uniform strength for a uniformly distributed load. In beams of class 1 for a given load, both y_1 and p_1 are constant, and therefore R, by equation 1, will also be constant throughout the whole length of the beam, or, in other words, the beam will bend into a circular arc. The approximate expression for the versed sine of an arc having a chord L and radius R will therefore give the deflection, and we have

$$3. \quad . \quad . \quad . \quad . \quad . \quad . \quad v = \frac{\mathrm{L}^2}{8\mathrm{R}},$$

and employing for R the value given by equation 1, we have

$$4. \quad . \quad . \quad . \quad . \quad . \quad . \quad v = \frac{\mathrm{L}^2 p_1}{8\mathrm{E}y_1}$$

and if $y_1 = \frac{1}{2}d$, we have

$$5. \quad . \quad . \quad . \quad . \quad . \quad . \quad v = \frac{\mathrm{L}^2 p_1}{4\mathrm{E}d}.$$

If for p_1 we substitute f_1, the maximum safe stress for the given materials, equation 4 or 5 will give the maximum safe deflection, which may be called v_1; we observe, then, that the safe deflection for a beam of this class will be proportional to the square of the span, and inversely proportional to the depth of the beam.

In beams of class 1, the deflections which different loads produce will be simply proportional to the values of p_1 produced by those loads; thus, for a given distribution of load, the deflection will be simply proportional to the load; if we change the distribution of the load, keeping the total load constant, the same rule will give the solution; for instance, since a load uniformly distributed produces only half the stress p_1 which would be produced by the same load at the centre of a beam of the same span and cross section, we see that the uniformly distributed load will produce only half the deflection that would be produced by the same load at the centre of a beam of the same span and same *section at the centre* (both beams belonging to class 1). It would not be correct to say that a load uniformly distributed would produce half the deflection produced by the same load at the centre of the *same beam*, because the same beam cannot be uniformly strong throughout its length for two different distributions of load.

We may compare the deflections produced by the same load on various beams of similar cross section as follows:—By equation 4, § 14, we see that for a given moment M, p_1 is proportional to $\frac{d}{\mathrm{I}}$; moreover, for equal loads, the moment M at any cross section similarly placed will be proportional to L; hence we may write

$$p_1 \propto \frac{\mathrm{L}d}{\mathrm{I}} \propto \frac{\mathrm{L}}{bd^2}.$$

Substituting this expression for p_1 in equation 5, we have

$$6. \quad . \quad . \quad . \quad . \quad . \quad . \quad v \propto \frac{\mathrm{L}^3}{bd^3},$$

an equation which expresses the fact that, for beams of class 1, the deflection produced by a given total load similarly applied will be proportional to the cube of the length, and inversely proportional to the breadth and to the cube of the depth. (In I girders the expression breadth must be understood as proportional to S, the section of the flange.)

Passing from beams of class 1 to beams in general, the

fundamental difference to be observed will be that the curve assumed by the neutral surface will not be that of a circular arc, so that equation 1 is no longer true. Where, however, the law according to which R varies can be stated algebraically, integration will give the value of v.

Combining equations 2 and 3 for class 1, we have $v = \frac{1}{8}\frac{\mathrm{M}_1\mathrm{L}^2}{\mathrm{EI}}$, and as $\mathrm{M}_1 \propto \mathrm{LW}$, we have $v \propto \frac{\mathrm{WL}^3}{\mathrm{EI}}$. Now this proportional equation can be shown to hold good for beams of uniform cross section and uniform depth, which may be called beams of class 2, also for beams of uniform cross section and uniform breadth, which may be called beams of class 3, hence for the three classes of beams we may write

$$7. \quad . \quad . \quad . \quad . \quad . \quad . \quad v = n\frac{\mathrm{WL}^3}{\mathrm{IE}},$$

where n will have different values in the three classes and for each distribution of load.

Similarly it can be proved, that where beams are so designed that the loads produce the same maximum value p_1 of the stress on the outer elements, we have the deflection proportional to $\frac{p_1\mathrm{L}^2}{\mathrm{E}y_1}$,—equation 4 being one case of the general law: we may therefore write

$$8. \quad . \quad . \quad . \quad . \quad . \quad . \quad v = n_1\frac{p_1\mathrm{L}^2}{\mathrm{E}y_1},$$

where n_1 is a constant differing for each class of beam and each distribution of load.

Table X. gives the values of n and n_1 for the three classes of beam, and for two distributions of load. The value of p_1 in the case of a beam of uniform cross section applies to the stress at the centre; the stresses elsewhere in that beam will be less.

TABLE X.—*Values of* n *and* n_1.

Description of Beam.	W at Centre.		W uniformly distributed.	
	n	n_1	n	n_1
Uniform strength and depth..	$\frac{1}{32}$	$\frac{1}{8}$	$\frac{1}{64}$	$\frac{1}{8}$
Uniform strength and breadth	$\frac{1}{24}$	$\frac{1}{6}$	0·0178	0·1427
Uniform cross section..........	$\frac{1}{48}$	$\frac{1}{12}$	$\frac{5}{384}$	$\frac{5}{48}$

In any actual bridge girder the deflection should lie between the value calculated for the beam of uniform strength and that calculated for the beam of uniform cross section.

§ 25. *Graphic Method of finding Deflection.*—Divide the span into any convenient number n of equal parts of length l, so that $nl = \mathrm{L}$; compute the radii of curvature R_1, R_2, R_3 for the several sections. Let measurements along the beam be represented according to any convenient scale, so that calling L_1 and l_1 the lengths to be drawn on paper, we have $\mathrm{L} = a\mathrm{L}_1$; now let r_1, r_2, r_3 be a series of radii such that $r_1 = \frac{\mathrm{R}}{ab}$, $r_2 = \frac{\mathrm{R}_2}{ab}$, &c., where b is any convenient constant chosen of such magnitude as will allow arcs with the radii r_1, r_2, &c., to be drawn with the means at the draughtsman's disposal. Draw a curve as shown in fig. 21 with arcs of the length l_1, l_2, l_3, &c., and with the radii r_1, r_2, &c. (note, for a length $\frac{1}{2}l_1$ at each end the radius will be infinite, and the curve must end with a straight line tangent to the last arc), then let v be the measured deflection of this curve from the straight line, and V the actual deflection of the bridge; we have $\mathrm{V} = \frac{av}{b}$, approximately. This method distorts the curve, so that vertical ordinates of the curve are drawn to a scale b times greater than that of the horizontal ordinates. Thus if the horizontal scale be one-tenth of an inch to the foot, $a = 120$, and a beam 100 feet in length would be drawn equal to 10 inches; then if the true radius at the centre were 10,000 feet, this radius, if the curve

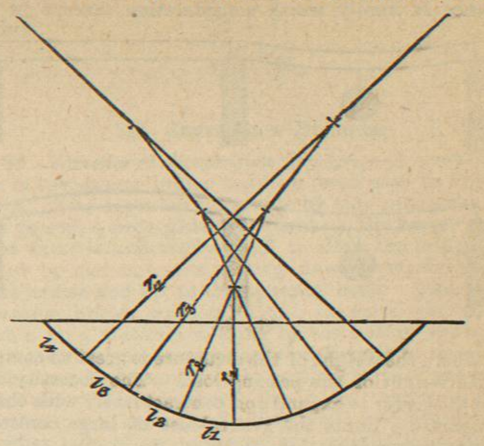

Fig. 21.

were undistorted, would be on paper 1000 inches, but making $b = 50$ we can draw the curve with a radius of 20 inches. If we now measure the versed sine of an arc drawn with a length 10 inches and a radius 20 inches, we shall approximately find it equal to 0·64 inches, hence $\mathrm{V} = \frac{120 \times \cdot 64}{50} = 1{\cdot}54$ inches. The vertical distortion of the curve must not be so great that there is any very sensible difference between the length of the arc and its chord. This can be regulated by altering the value of b. In fig. 21 distortion is carried much too far; this figure is merely used as an illustration, and is not to be taken as an example.

§ 26. When a girder has more than two supports it is called a *continuous girder*. The distribution of the stresses in a continuous girder differs very materially from that in a simple girder, as will be at once apparent by the inspection of fig. 22, which shows the way in which a continuous

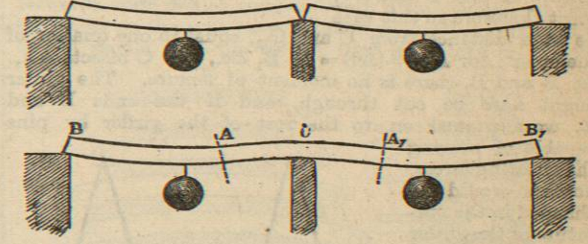

Fig. 22.

girder of two spans and two simple girders bend when employed to carry equal weights across equal openings. The continuous girder when both spans are loaded is bent upwards at C over the centre pier; in other words, the bending moment at this and neighbouring points is negative. The direction of the flexure changes at certain sections, as at A and A_1, *i.e.*, the bending moment is positive on one side of these sections, negative on the other side; and at the section where the direction of flexure changes the bending moment is *nil*. Again, when only one of two simple girders is loaded, the girder over the second span is not bent in either direction, but with the continuous girder there may be a negative bending moment produced throughout the whole unloaded span as shown in fig. 23. Continuous girders require less material for the same depth, span, and permanent load than simple girders; but the difference is hardly worth consideration, except in large

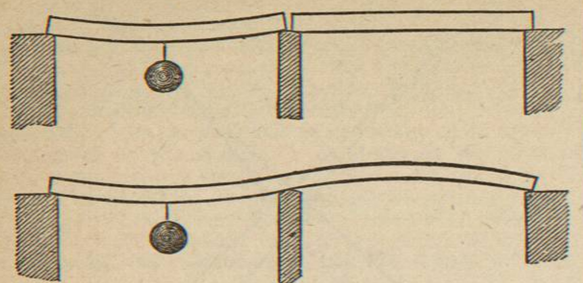

Fig. 23.

spans where the weight of the structure is great as compared with the weight of the passing load. The necessity of allowing the girder to expand and contract freely with changes of temperature limits the general use of large continuous girders to two spans.

Consider a continuous girder having an indefinite number of supports equally spaced; let the girder be of constant depth, uniformly loaded, and so proportioned as to be of uniform strength at all sections. The method of designing such a girder has not yet been shown, but it may be admitted that the design is possible. Then, as we have already seen (§ 24), the curvature assumed by the girder will everywhere be constant, *i.e.*, the curves will be circular arcs of constant radius. The points of inversion of flexure A and B (fig. 24) must therefore in this case be at a distance from C and C_{11}, equal to one quarter of the span; for $AA_1 = AB = A_1 B_1$ &c., and C bisects AA_1. At B and B_1 there is no moment of flexure. The girder might here be cut through, and if the ends B and B_1 were pinned on to the rest of the girder by pins

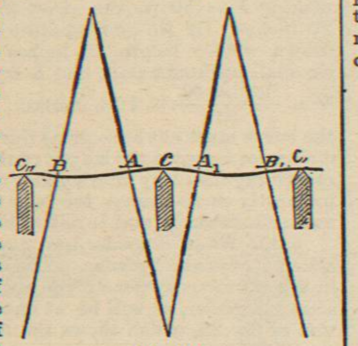

Fig. 24.

capable of bearing the shearing stress, nothing would be changed in the curvature of the girder nor in the distribution of stress. Quite similarly, we may suppose the ends B and B_1 of the portion $BACA_1B_1$ to rest directly on supports or piers introduced for the purpose, the rest of the imaginary girder being wholly removed. We shall then have (fig. 25) the curve assumed by a continuous girder of two spans of constant depth and uniform strength. The points AA_1 of inversion of flexure will be at a distance CA from the middle pier equal to one-third of the span. The top and bottom members at A and A_1 might vanish (if only one distribution of load were to be carried), for at this point there is only a shearing stress and no bending moment. The girder is, as it were, made up of three girders: BA and A_1B_1 hang on to AA_1, which is supported in the middle; half the weight of AB is borne by the pier B; half the weight of A_1B_1 is borne by the pier B_1; the rest of the weight between B and B_1 is borne by the middle pier. Thus let L be the length of one span in feet, w the load per foot run, P the load borne by each of the end piers, and P_c the load borne by the centre pier, then we have—

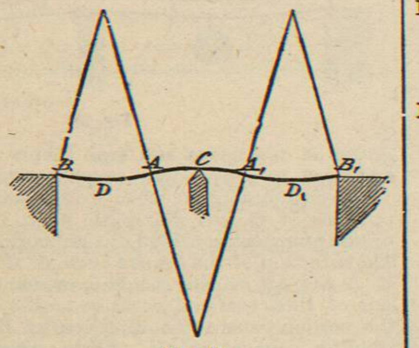

Fig. 25.

1. . . . . $P = \frac{1}{3}wL$,
2. . . . . $P_c = \frac{4}{3}wL$.

The curve of bending moments can now be calculated for each girder precisely as for a simple girder; the moment at any section is equal to the sum of the moments of all the external forces on one side of the section; the beam between A and B will be subject to bending moments equal to those produced by a uniform load on a simple girder if the span BA be similarly loaded. Between A and A_1 the moments will be negative, *i.e.*, the left-handed moment produced by the downward action of all the weights to the left will exceed the right-handed moment produced by the upward reaction of the pier at B (or of the two piers B and C if the section considered lies to the right of C). The full black line in fig. 25*a* shows a curve of bending moments for this case.

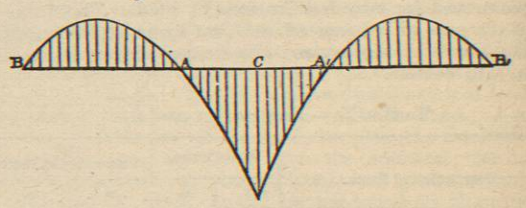

Fig. 25*a*.

The maximum moment is negative and occurs over the centre pier; let it be called M_c. The maximum positive moment occurs at a distance from B equal to one-third of the span; let this moment be M_d. Then we have—

3. . . . . . . $M_c = -\frac{1}{6}wL^2$,
4. . . . . . . $M_d = \frac{1}{18}wL^2$.

The shearing stresses F_b, F_d, F_a, F_c, at the points B D, A, and C, are

5. . . . . . . $F_b = F_a = \frac{1}{3}wL$,
6. . . . . . . $F_d = 0$,
7. . . . . . . $F_c = \frac{2}{3}wL$.

Fig. 26 gives a diagram of the shearing stresses for two

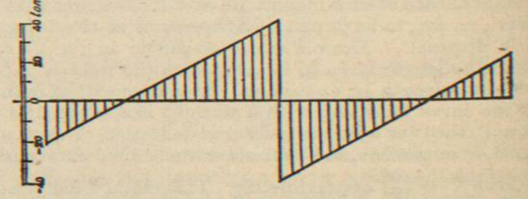

Fig. 26.

spans of 60 feet, with a uniform load of 1 ton per foot run.

When the beam is of uniform depth and uniform cross section, the curves in which it deflects are no longer circular; the defect of strength over the centre pier has

the effect of increasing the curvature over it, and shortening the distance AA_1; analysis shows that in this case the point of inversion of flexure will be at a distance from the centre pier equal to ·2683L; then the length of the part of the beam subject to a positive bending moment will be ·7317L, instead of ·66L as when the beam was of uniform strength; the load on each of the end piers will be ·36585wL; the load on the centre pier 1·2685wL; and we have—

8. . . . . . $M_c = \cdot 1341wL^2$,
9. . . . . . $M_d = \cdot 0669wL^2$,
10. . . . . . $F_b = F_a = \cdot 36585wL$,
11. . . . . . $F_d = 0$,
12. . . . . . $F_c = \cdot 6341wL$

In any actual bridge uniformly loaded the values of the moments and shearing stresses will be intermediate between those given for a beam of uniform strength and those for a beam of uniform cross section. It must be observed that the above theory assumes that the girder is unstrained when being built as a whole; if, as is often the case, the separate spans are separately built and lifted into position, and then joined on the piers, special provision must be made to bring the desired bending moment over the piers into existence; it is obvious that merely joining the two independent beams in the upper part of fig. 22 will not make these into a continuous girder, such as is shown in the lower figure,—to do this, besides joining the upper and lower flanges, we must pull the top flanges together over the piers and put the lower flanges under compression. This may be done practically by tilting one end, or both ends, before the junction at the centre is made, and afterwards allowing the ends to sink until the curve assumed by the girder shows that the required distribution of stress has been attained. The complete analysis of the problem of continuous girders of any number of spans equal or unequal with any number of loads has been given by Mr Heppel (*Proc. R. S.*, 1870–71).

§ 27. *Allowance for Weight of Beam. Limiting Span.*—When the weight of the beam is a considerable and uncertain part of the whole load, it can be allowed for as follows. Design a beam of the desired depth and span, fit to carry a total load equal to the external or passing load W_1; calculate the weight of this beam and call it B_1; the beam so designed will really be fit to carry an external load $W_1 - B_1$. Let b_1 be the area of any cross section of this beam; let b be the area of cross section required at the same point for the beam of weight B actually necessary to carry a total load W. Then since the strength of the properly proportioned girder of constant depth and span is simply proportional to the quantity of metal employed, and therefore to the area of cross section, we have the proportion $b : b_1 = W_1 : W_1 - B_1$, or

1 . . . . $b = \frac{W_1 b_1}{W_1 - B_1}$.

The weight B is given by the expression—

2. . . . . $B = \frac{B_1 W_1}{W_1 - B_1}$.

The whole load W is given by the expressio

3. . . . . $W = \frac{W_1^2}{W_1 - B_1}$.

For any given design of beam there is a limiting length which cannot be exceeded (the beams of different spans being assumed to be *similar* in the geometrical sense). Let L_1 be the limiting length of a beam of a given design which for the span L weighs B, and carries a gross load W, then the ratio of B to W can be shown to increase in direct proportion to the length of the span until this ratio reaches unity. Hence—

$$\frac{B}{W} \cdot \frac{B_1}{W_1} = f_1 : L_1,$$

or when $\frac{B_1}{W} = 1$,

4. . . . . . . $L_1 = \frac{WL}{B}$.

III. Suspension Bridges.

§ 28. *Varieties of Suspension Bridges.*—A very simple form of suspension bridge has long been used in Peru and Thibet. Two ropes are hung side by side across the gorge to be passed, a rude platform is laid on the ropes, and the dip of these is sufficiently small to allow the bridge to be crossed by men or beasts passing down from the one side to the centre and up to the opposite bank. The modern suspension bridge consists of two or more *chains*, from which a level *platform* is hung by suspension *rods*. The chains may in some cases be secured directly to the sides of the chasm to be crossed, but the configuration of the ground seldom allows this to be done. The chains, therefore, usually pass over *piers*, as in fig. 2, Plate XIX., and are led down on either side to an *anchorage* at a considerable distance from the piers. The chains between the piers and the anchorage are generally used to support part of the platform. The chains where they pass over the piers rest on *saddles*, which are made of two different types. One construction, shown in fig. 27, allows the chain to slip backwards and

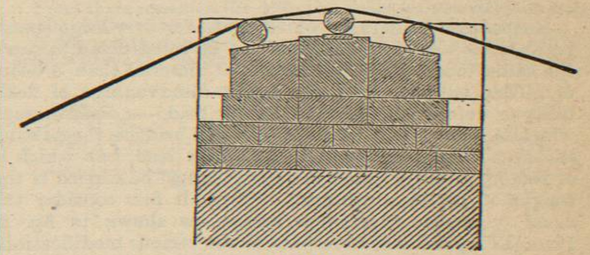

Fig. 27.

forwards over it with comparatively little friction, so that the stress on the rope may be taken as equal on both sides of the saddle. In the second type, as shown in fig. 28,

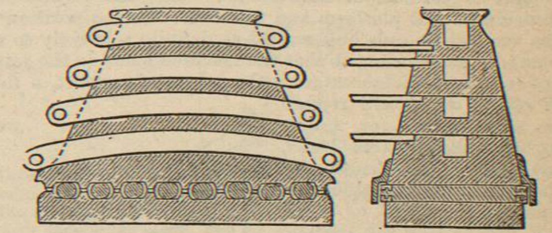

Fig. 28.

the chain is secured to the saddle, which, however, is free to move horizontally on the top of the pier. With the first form of saddle the resultant pressure on the pier will not be vertical unless the chain leaves the pier with an equal inclination on each side, and even when the bridge is designed with an equal slope of chain on both sides of the pier, a change in the distribution of weight due to any passing load will cause some departure from the equal slope of the chains, and therefore from the truly vertical pressure on the piers. This departure is easily allowed for in the design of the pier. The friction on the saddle renders the