masonry above the haunches of the ring; it is carried back between the spandrils to the pier or abutment. If the backing is not carried up to the roadway, as is seldom the case, the rough material employed between the backing and the roadway is called the *filling*. The *parapet* rests on the outer spandrils. The abutments and piers have the same signification as in other bridges. The masonry arch differs from the superstructure of other bridges in the following respect: it depends for its stability on the presence of a permanent load specially arranged, and so considerable in amount that the changes produced in the direction and magnitude of the stresses by the passing load are insignificant. The theories of the masonry arch often neglect the passing load entirely, and simply teach the student how to distribute the permanent load, so that the voussoirs may be in equilibrium. The permanent load consists of the ring, the backing, the filling, the spandrils, and the roadway. Inasmuch as the ring is that part of the structure which by its special strength and arrangement carries the superstructure in the same sense as a beam or chain carries it, the arch in this article will be treated simply as a ring of voussoirs springing from two abutments and loaded with weights, some permanent and some passing. Where the backing strengthens the arch, it becomes virtually part of the ring.

§ 37. *Equilibrium of a Single Voussoir.*—A block, such as a voussoir, ABCD, fig. 42, resting on one of its surfaces, such as the joint AB separating it from the next voussoir, is in equilibrium when the resultant of all the forces acting upon it (including its own weight) falls within the supporting surface, while the direction of this resultant makes an angle ϕ with the normal to the surface less than the angle of repose; (the tangent of the angle of repose is the coefficient of friction). If the resultant, as $R_{,}$, falls without the surface, the block will heel over, pivoting on the edge A. If the resultant, as $R_{,,}$, although falling within the surface of which AB is the trace, is yet much inclined to the normal, the block ABCD will slide up on the joint AB without heeling over.

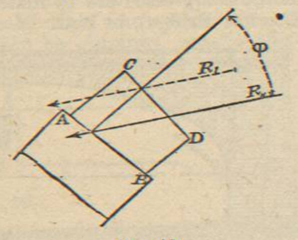

Fig. 42.

The block, if used as the voussoir of a bridge, must not only be in equilibrium under the forces applied to it, but must also be of sufficient strength to resist these forces. The intensity of crushing stress due to the external forces must nowhere exceed the safe crushing strength of the material. This latter condition would in most arches be fulfilled by an extremely thin ring of stones or brick if the resultant passed through the geometrical centre of the joint AB in a direction normal to it. In that case the stress on the joint would be a uniformly distributed stress; if, however, the resultant stress passes near one edge, the intensity of stress at that edge will be much greater than elsewhere, and would indeed be infinite if the resultant passed exactly through the edge at A or B; while, therefore, the condition of equilibrium is satisfied if the resultant passes within either edge of the voussoir at no great inclination, the condition of strength requires that this resultant shall not cut the joint very near the edge, and the common practical rule is that it shall always fall within the middle third of the joint. This rule is based on the condition that the pressure on a joint shall nowhere be negative; in other words, that no tension shall occur at any part of any joint. The principles explained in § 8 show that the minimum stress on any joint $p_1 = p_0 - \frac{Px_0x_1}{I}$, or that the stress will be zero, when $p_0 = \frac{Px_0x_1}{I}$. Let d be the depth of the rectangular joint, and b the breadth; then

$$I = \frac{bd^3}{12}, \text{ and } x_1 = \frac{d}{2}, \text{ and } p_0 = \frac{P}{bd};$$

hence

$$\frac{P}{bd} = \frac{12Px_0d}{2bd^3}, \text{ or } \frac{d}{6} = x_0,$$

an equation expressing the condition that the centre of pressure lies at the edge of the middle third; any greater value of x_0 will give a negative value to p_1. We shall see that the actual resultant is, according to the theory practically in use, indeterminate within certain limits; it is therefore useless to attempt to calculate the exact maximum stress on any one stone. In the rest of this article the *ring* is to be held to mean the *middle third* of the actual masonry, or brick ring, wherever the theory requires that the blocks are to resist practical loads. As bridges are subject to a sensibly equal load on all parts of their breadth between the parapets, it is usual to consider a portion of the ring one foot in width, each other strip being under precisely similar conditions. Similarly the joint may be spoken of for convenience as the line which is its trace, and the edge as the point which is its trace.

The external forces which act on any voussoir are—1st, the vertical force, being the resultant of its own weight and the load which is directly over it; 2d, the thrust from the voussoir above it; and 3d, the reaction from the voussoir on which it rests (fig. 43). It is sometimes difficult to determine exactly what portion of the superincumbent load a voussoir may properly be said to carry, but a sufficient approximation is obtained for practical purposes by assuming that the mass vertically above any voussoir is carried by the voussoir when the back of the voussoir is not much inclined. If the materials had little cohesion, the direction of the force produced by the load would not be vertical, but inclined at an angle depending on the coefficient of friction; in practice, the direction of the force is uncertain and even variable with changes in the condition of the superincumbent filling. If, however, the stability of the arch is calculated with a reasonable margin or coefficient of safety, on the hypothesis that the force produced by the load is vertical, there is every probability that the arch will be stable under any actual stress which may arise in practice.

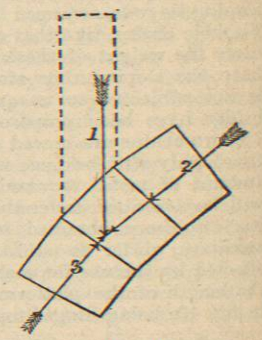

Fig 43.

§ 38. *Equilibrium of any three Voussoirs; Equilibrated Polygon.*—The simplest arch would be an arch of three voussoirs resting on two abutments, and any actual arch consisting of many voussoirs may be considered as composed of successive triplets, the voussoirs on each side of which act as abutments. If, therefore, we can show the conditions of equilibrium for three voussoirs we shall have determined the conditions for the whole ring.

Let three voussoirs be taken from any part of the ring (fig. 44), and let the lines 1, 2, and 3 represent the position of the resultants of the three known loads w_1, w_2, and w_3 (including the weight of the voussoirs) borne by each voussoir.

Let NA represent the position of the reaction t due to the abutting voussoir on one side. Let A be the point where the prolongation of the line NA cuts the line 1; then if the magnitude of the

forces t and w_1 are known, these determine the magnitude and direction of the equilibrating force t_1, which must act at A to balance them. Let the direction AB and the magnitude of the force t_1 be

Fig. 44a.

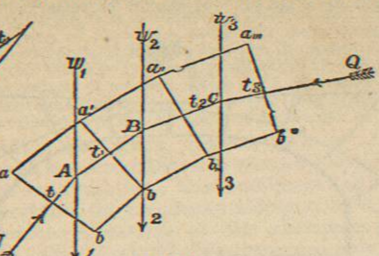

Fig. 44.

found by the ordinary parallelogram of forces, as in fig. 44*a*, and let B be the point of intersection of the direction of this force with the line 2; then the direction and the magnitude of the equilibrating force t_2 can be found as for t_1; similarly the direction of this force gives the point C by its intersection with the line 3, and finally we obtain, by the resolution of forces, the direction, magnitude, and position of the force t_3, by means of which the reaction of the second abutment will keep the system in equilibrium. When the position and magnitude of t are known, the position and magnitude of all the other forces are determinate; the conditions of equilibrium are that the lines NA, AB, BC, and CQ, shall not cut the joints above or below the edges a or b, for in that case the blocks would heel over on the edge beyond which the resultant passed; also, the direction of the lines NA, AB, &c., must be such as not to exceed the angle of repose with the normal to the joints, otherwise one stone will slip on the other. The abutment producing by its reaction the force t_3 must not yield with a less force than t_3, and must not be pushed forward so as to produce a greater force than t_3. The line NABCQ, if inverted, is in form identical with that which a cord would assume, loaded at the points A, B, and C, with the loads 1, 2, and 3, and having the direction of NA determined. This line will, in the rest of this article, be called an *equilibrated polygon*.

When the joints are supposed indefinitely near, or the voussoirs thin sheets, the equilibrated polygon becomes a curve called a *linear arch*.

The reasoning applied to three blocks is clearly applicable to any number, and we may therefore say that any series of loaded voussoirs will be in equilibrium when a reaction of known magnitude and direction is applied at one abutment, provided the equilibrated polygon required by this reaction and the given loads can be drawn so that its sides cut all the joints within the ring (or within the

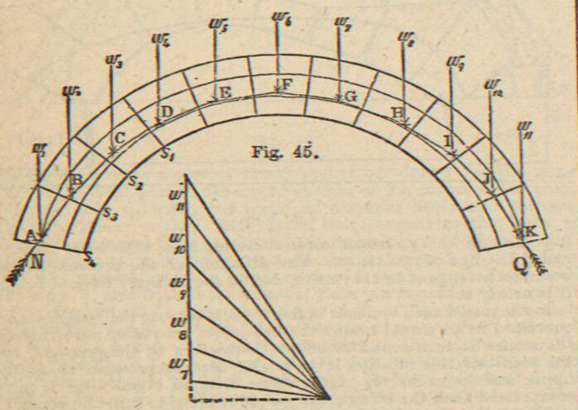

Fig. 45.

Fig. 45*a*.

middle third, where strength is an element of the question) at an angle greater than the complement of the angle of repose for the material used. An equilibrated polygon, ABC... ..Q, for a complete arch is shown in fig. 45 Fig. 45*a* is the diagram giving the slopes KJ, JI, &c., as for the loaded chain fig. 30 and 31.

§ 39. It will be shown in next paragraph that the arch will be in equilibrium if with any value of the horizontal thrust h an equilibrated polygon can be drawn fulfilling the conditions required. In most arches equilibrated polygons fulfilling these conditions can be drawn with values of h varying between two limits differing by a considerable amount. In that case the smallest value of h will be the true value, and give the true stresses; for the abutments being inert will not give back a greater thrust than is just required to balance the structure. This would render the thrust h determinate if the equilibrated polygon might actually approach the true edge of the ring, but as this would require infinitely strong materials, we are still left in uncertainty as to the true value of h, but may feel sure that it will be the smallest value consistent with a safe stress on the material. If we provide abutments capable of reacting with a force h sufficient to keep the equilibrated polygon (where it cuts the joints) within the middle third, our abutment will certainly be amply strong enough, and this is the value of h to be adopted in all practical calculations of the stability of an arch.

In the example with the three voussoirs it is clear that equilibrium would be obtained with very widely different reactions t at the one abutment. And this fact is also true for a bridge of many voussoirs. The vertical component (which may be called v) of this total thrust t is indeed determinate if we suppose the point where t cuts the joint to be known, being the same as the vertical reaction from a beam carrying the same weights and supported at the points where t and v cut the abutments; but the horizontal component or horizontal thrust, which has been called h, cannot be determined by any considerations hitherto mentioned.

§ 40. *Experimental Demonstration that the Equilibrium of a series of Voussoirs is stable if any Equilibrated Polygon can be drawn fulfilling the conditions stated above.*—Let us suppose an arch, fig. 48, to be constructed, the bed-joints of which are not plane but curved, so that each stone touches its neighbour only along a horizontal line, the trace of which in a drawing may be called the *point of contact*. Such an arch will differ from an ordinary arch in this respect, that the centre of pressure at joints will be shown by the points of contact, while the stones will be able by rolling to alter the points of contact if not in equilibrium. In such an arch the voussoirs in the first place may be put together so as to touch at any desired series of points, but the forces called into play when external support is withdrawn will rearrange the voussoirs so as to bring them into equilibrium, if any equilibrated arch consistent with the loads can be drawn so that the lines forming it cut the joints inside the ring, and a model will show the points of contact, or, in other words, the places where these lines cut the joints. (It is assumed that the obliquity of the sides of the polygon to the joints which they cut is insufficient to produce slipping.)

An actual model shows the action very prettily, but the following considerations will easily allow the student to see how it is that the voussoirs always arrange themselves so as to build a true arch.

Suppose, first, that the arch consisted merely of three stones, fig. 46, and that the weight on the centre one was so great that the linear arch, or equilibrated polygon, became sensibly two inclined straight lines like rafters. As soon as the voussoirs are left to themselves, the pressure at the surface a_2 b_2, and the reaction at the surface a_1 b_1, will lie in one straight line, which, meeting a similar straight line from the other abutment, will give one equilibrated polygon, satisfying the required conditions; but if the horizontal force required for this polygon is not supplied by the abutments, the two forces at joints 1 and 2 will, as shown by the small straight arrows, constitute a couple tending to turn the stone A round, so that the point of contact at joint 1 will be lower, and the point of contact in joint 2 will be higher than before. The same action will occur in stone C, and the result will be that the weight may be balanced with a smaller horizontal force. At the same time the rotation of the stones A and C, coupled with the descent of B, tends to push back the abutments N and Q, and therefore to increase their horizontal reaction supposing them to be stable. If the abutments N, Q continue to yield, the stones A

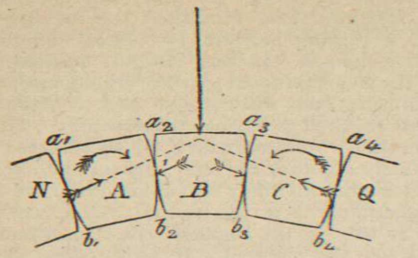

Fig. 46.

and B will continue to turn until the points of contact reach a_2 and a_3, or b_1 and b_4. The horizontal thrust which the abutments require to meet will therefore diminish as the stones turn, and the little structure will only fail to support the weight in case the abutments N and Q are insufficiently strong or stable to supply the minimum thrust consistent with an equilibrated polygon

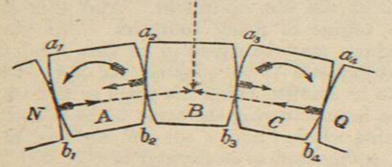

Fig. 47.

cutting the joints inside the ring (or in case the polygon cut the joints at such an angle that the stones slip). If, on the other hand, the abutments were so made as to press in upon A and C with a greater horizontal force than is consistent with two lines of pressure passing through the actual points of contact, then, as in fig. 47, the direction of the couples on the stones A and B would be reversed, and they will roll round so as to bring the points of contact more nearly into the position required to meet an excessive horizontal thrust, and at the same time the changed position of the stones, by allowing N and Q to come forward, will tend to relieve or diminish the original excessive horizontal thrust, where this is due to the elasticity of the stones N and Q, or of the stones supporting these abutments. The structure will not fail unless the points of contact reach a_1 and a_4, or b_2 and b_3, when the structure would fail by the sides being squeezed in, and the stone B being lifted up out of the arch. This could not happen with stones of the proportions shown in fig. 47, as before the limiting position was reached, the points of contact would lie on a straight line corresponding to an infinite horizontal thrust. In conclusion we see that, whether the horizontal force supplied by the reaction of the abutting stones be too small or too great, the three voussoirs tend to move so as to adapt the centre of pressure and the actual horizontal force to one another. The equilibrium produced is stable, that is to say, if by some external force the arrangement of the blocks is slightly disturbed, when the force is removed the blocks return to their original position. In the above demonstration it is assumed that the blocks when first put together touch at some point not far from the centre of the bed,—a condition corresponding to reasonably good fitting in the case of the plane joints of a stone arch before the centring is removed.

If a model be prepared (fig. 48), having a number of voussoirs of wood with their bed-joints slightly curved and roughened, the result of the above theory will be very clearly and beautifully seen. The action explained in the case of three blocks holds good for any three, and therefore for the whole series. If an additional weight is placed at the crown, as in fig. 48, the crown is a little lowered, but the curve passing through the lines of contact rises at the crown and is lowered at the haunch by the rotation of the blocks, until the lines of contact at the joints arrange themselves, so that the resultant pressures forming the imaginary polygon pass through these lines of contact. If the extra load be placed at the haunches the crown rises, but the points or lines of contact between the voussoirs are lowered at the crown and raised at the haunches, as in fig. 48*a*. If one haunch only is weighted, the curve passing through the lines of contact rises at that haunch and is lowered at the other, as in fig. 48*b*; if the model be distorted by the hand it oscillates up and down on each side of the position of equilibrium, as a string similarly loaded would do. Figures 48, 48*a*, and 48*b* are taken from photographs of a model. (It should be remarked that the abut-

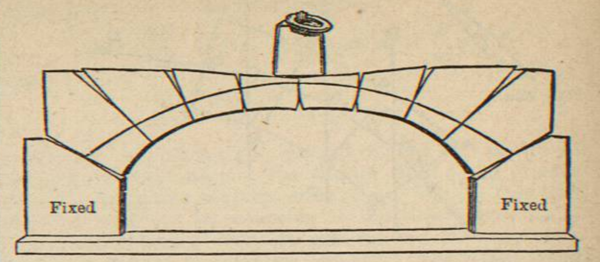

Fig. 48.

ments were screwed to the supporting board; it is obvious that otherwise they would not have been in equilibrium.) The general character of the curve passing through the points of contact may be easily conceived by thinking of a string similarly loaded and

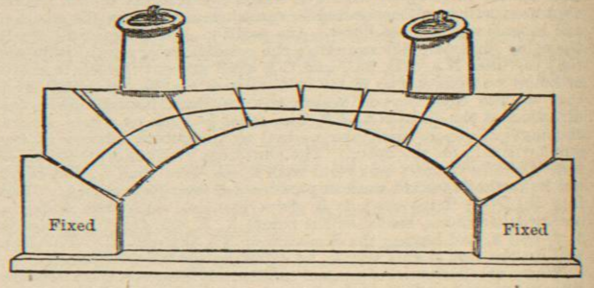

Fig. 48*a*.

inverted. The equilibrated arch will be one of those forms which a string might take when similarly loaded, but when the load is changed, the length of the curve will not be constant in the arch, whereas it must be constant with any given chain. The curve pass-

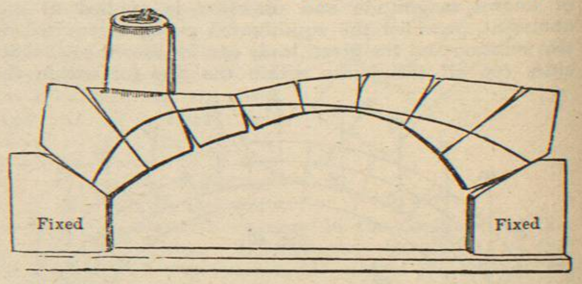

Fig. 48*b*.

ing through the points of *contact* corresponds with what Moseley called the line of resistance. The direction of the pressure is not necessarily tangent to this curve, but in the ordinary form of bridge it is nearly so.

In the model each voussoir is free to roll, because the bed-joints are curved. In an actual bridge the bed-joints are plane, nevertheless, the stones do turn round to adapt themselves to the pressure, but the result of this rotation is to render the compression along the upper and lower halves of the stone unequal. One edge is more compressed than the other; the couple tending to turn the voussoir, and actually allowed to do so in the model, is met by an equal and opposite couple, due to the unequal compression of the stone.

This couple is the necessary result of a pressure which is not axial, *vide* § 8; an equilibrated polygon cutting the joints at various distances from the centre is therefore as correct an indication of the actual forces present in a practical arch with flat joints as in the model with curved joints; but we must remember that where the joints are flat, the pressure will be unequally distributed wherever the line of the equilibrated polygon does not cut the centre of the

joint. Greater or less elastic resistance in the stone corresponds to greater or less curvature in the surface of the joint. A small distortion of the arch will restore equilibrium when the curvature is great, or when the stone has a high modulus of elasticity. The ring with plane bed-joints is in stable equilibrium, and adapts itself to new distributions of load for precisely the same reasons as the model with curved joints, but in the one case the couple called into play to move the voussoir is actually cancelled by the new position which the points of contact assume; in the other case it is balanced by the equal and opposite couple resulting from the resistance to motion due to the hardness of the stone.

The preceding paragraph showed how to determine whether an arch was in equilibrium when a known reaction was applied at one abutment; the experiment and reasoning now given show that the incipient yielding of an arch under loads will produce a reaction at the abutments suited to keep the whole ring in equilibrium, provided only an equilibrated polygon can be drawn, cutting the joints within the ring at suitable angles.

§ 41. *Practical Investigation of the Stability of a given Arch under a given Load—Joint of Rupture.*—This investigation resolves itself into finding that equilibrated polygon or linear arch which can be drawn within the (middle third of the) ring from the crown to the *lowest possible joint* of the ring (or to the springing if this be possible). This lowest possible joint must in any case be treated as the springing of the arch, and if the linear arch goes out of the (middle third of the) ring above the actual springing, as will be the case in all semicircular or elliptical rings, masonry must be provided in the backing capable of taking the actual thrust into the abutment and constituting the real arch, which often differs widely from the form indicated by the ring of stones in the face. The linear arch in a circular or segmental bridge loaded simply by its own weight generally has a smaller radius of curvature than the ring at the crown, and a much larger radius towards the haunches. Consequently, the longest linear arch which can be drawn within the ring will approach the upper surface of the ring at the crown and the soffit towards the haunches.

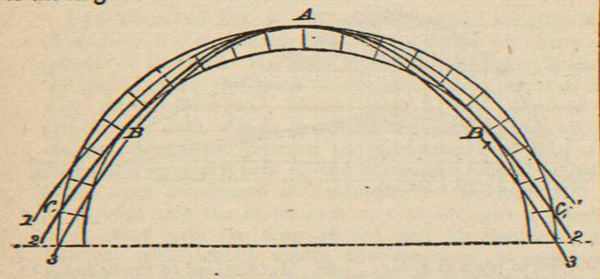

Fig. 49.

Fig. 49 shows a series of linear arches, all drawn for the same load, and all tangent to the upper surface of the crown of the arch, differing only in being the result of different horizontal thrusts. The curve drawn with a thick black line, tangent to the soffit, is clearly the longest linear arch which can be drawn within the ring. Any smaller value of the horizontal thrust h would give a linear arch like urve 3, and any larger value of h would give a linear arch like curv 1, and both these values of h are incompatible with equilibrium for the whole arch down to joint C; if, therefore, the arch fails by the yielding of the abutment, or of the lower portion of the ring, the failure will first be apparent at the joints A and B, where this black line is tangent to the ring, and at joint C, where the linear arch cuts the back of the ring. Smaller values of h will keep the stones in equilibrium above and below joint B, but unless the arch below the joint B, as well as the abutment, can resist the tendency of the arch to spread, or, in other words, supply *at least* the horizontal reaction h required for this linear arch, the joint B will open at the top, the centre joint A will open at the bottom, the joint C will open at the back, and the crown fall in as shown in fig. 49*a*. The joint B, where the longest linear arch is tangent to the soffit, is called the *joint of rupture*. The value of h required to make a linear arch tangent to the back of the ring at the crown pass through the edge of the joint of rupture at the soffit, is larger than the value of h required to give a linear arch passing through the edge of any other joint at the soffit; at the same time, it is the smallest value of h consistently with which the arch can remain in

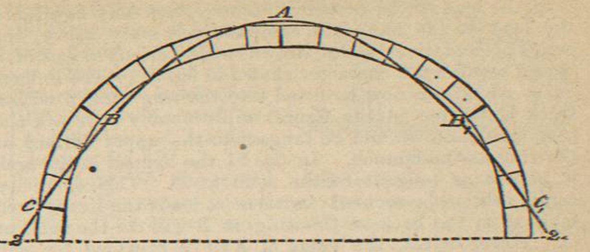

Fig. 49*a*.

equilibrium down to B and from B to C. In circular arches the joint of rupture generally makes an angle of about 30° with the horizontal plane; in elliptical arches the angle is usually about 45°. Its position is easily found as follows:—Let y_1, y_2, y_3, &c. (fig. 50), be the heights of the upper surface of the crown A above any points

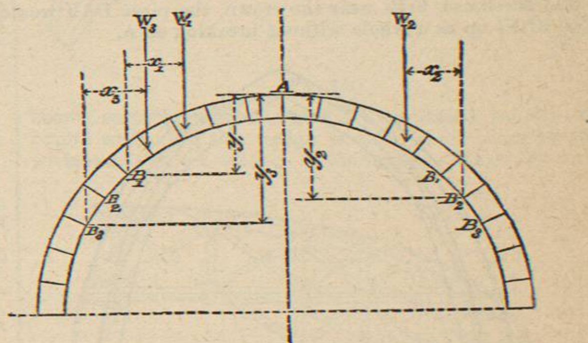

Fig. 50.

B_1, B_2, B_3 at the lower edges of the soffit; let W_1, W_2, W_3 be the weights of the portions of the arch with its load carried by the ring from B_1 to A, from B_2 to A, from B_3 to A, &c. (The load is in the fig. assumed to be symmetrically disposed relatively to the centre of the span.) Let x_1, x_2, x_3 be the horizontal distances of the centres of gravity of w_1, w_2, w_3 from the points B_1, B_2, B_3, &c.; then taking moments round B_1, B_2, B_3 in succession, we have, if the linear arch be assumed to pass through any point B—

$$Wx = hy;$$

taking the successive values of h for a series of joints B, we shall find that one joint gives a maximum value. This value corresponds with that of the linear arch tangent to the soffit (of the middle third) at the joint of rupture; for this arch has the maximum thrust of any passing through the points B_1, B_2, &c., as appears by simple inspection of fig. 49. The joint of rupture can thus be tentatively found, and the value of h, or the thrust which the abutment must resist, is obtained at the same time. If the backing is carried well up above C, a larger value of h than that obtained by this method would be consistent with the stability of the arch, and might actually occur; but we need not provide for this larger value, since the yielding of the abutment under it would diminish the thrust till it fell to the value as above determined. If the abutments could resist this thrust, the bridge would then remain in equilibrium. If the arch is flat there may be no joint of rupture, and in that case the value of h is to be taken as that given by a linear arch passing through the bottom of the (middle third of the) springing and tangent to the crown of the arch, *i.e.*, to the summit of the middle third of the ring.

When the apparent springing lies much below the joint of rupture, we find that the linear arch leaves the ring on the upper surface at a joint (C) lower down, where failure must result by the opening of the joint at the lower surface, unless the pressure is taken by masonry outside the ring. It is for this purpose that the *backing* is required. Obviously the best mode of supplying backing is to thicken