but the ratio $\overline{sc}:\overline{sg}$ is constant, and may be designated by the letter k; so that we may write $k\,(\Sigma y.\overline{sg})=\Sigma y^2$, from which equation we find the value of k—

$$6. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad k=\frac{\Sigma y^2}{\Sigma y.\overline{sg}}.$$

If the cross section is not constant we have for k the more complex expression—

$$7. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad k=\frac{\Sigma \frac{y^2}{I}}{\Sigma \frac{y.\overline{sg}}{I}};$$

but $\overline{sc}=k.\overline{sg}$, so that the required ordinate sc is at once obtained in terms of k and the known ordinate $\overline{sg}$.

When the actual linear arch $oc_1c_2c_3$, &c., has been thus obtained, it is easy to calculate the horizontal thrust; for let $\overline{s_5c_5}$ be the maximum ordinate, the direction of the thrust will at this point of the curve be horizontal, and therefore calling W the weight on one side of this ordinate, and x the distance of its centre of gravity from the springing, we have—

$$8. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad Wx=h.\overline{s_5c_5},$$

from which h can be found.

h and v being known give the position and direction of the resultant thrust at the springing of the rib. The magnitude of the thrust at any other point is easily computed graphically or by moments

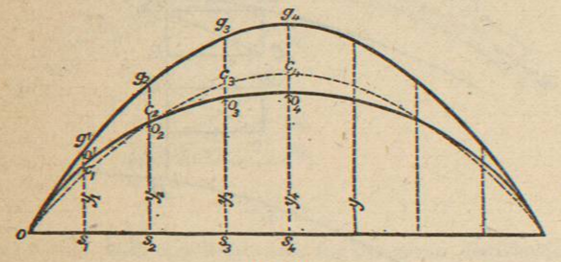


Fig. 61.

round the springing; then the resultant thrust at a given point being known, the intensity of the stress on any part of a section at that point is to be computed by first resolving the thrust into two components, one normal to the section and one in the plane of the section; the latter gives rise to a shearing stress (analogous to the force which causes one stone to slip on another in the masonry arch), while the component normal to the section will (if not axial) give rise to a uniformly varying stress, the magnitude of which at each distance from the axis can be computed by the formulæ given in § 8.

The value of h is determinate, if the direction of the rib be supposed fixed at the springing, but this cannot be ensured in large structures, and the theory need not therefore be developed. It simply requires $\Sigma\,\overline{co}=0$.

When the rib (as is generally the case in existing bridges) abuts against a flat springing the exact value of h is indeterminate. When the rib is hinged the friction at the bearing renders the thrust indeterminate within limits depending on the possible bending moment at the springing due to the friction.

§ 47. *Process of Designing a Rib.*—In future designs of ribbed arches it is to be hoped that the practice will be adopted of allowing the rib freedom to turn at the springing. This can be done by ending the rib in a bearing, curved as in fig. 62; the resultant thrust will then be approximately axial, and the stress on every part of the rib can be determined with as much accuracy as on the several parts of a girder. When the span is large a cast-iron metal arch can be made lighter than a wrought iron girder for the same load, but the imperfection of the theory of the stresses on the ribs has hitherto led to great waste of metal in their construction. In what follows it is assumed that the resultant at the springing passes through the geometrical centre of the cross section of the rib.

If the rib were to carry a load distributed only in one way we ought clearly to make the form of the axis of the rib coincide with a linear arch for that load. There would then be no bending moment on any part of the rib. As in practice we must provide for all the possible combinations of passing load, we need take little pains in designing the curvature of the rib—a flat arc of a circle with a rise of say $\frac{1}{4}$th will answer well. The semicircular or elliptical forms are not good, for no linear arch with any practical distribution of load can even approximately coincide with a form in which the rib springs vertically from the abutments.

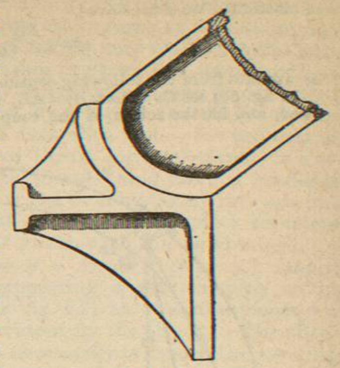


Fig. 62.

The general character of the cross section should be similar to that for a girder, inasmuch as the rib will have to resist bending moments as well as direct compression. The depth need not, however, be nearly so great as the depth of a girder. Let a cross section be chosen in which the area is assumed as approximately say 5 per cent. more than that which would be sufficient to sustain the thrust resulting from a linear arch suitable for the maximum load and coinciding approximately with the axis of the rib. (If the load be nearly uniform per foot run of platform we may for this first approximation take $h=\frac{wl^2}{8d}$, where d is the rise of the linear arch above the springings.) With the rib thus designed determine by the method given in § 46 the actual linear arches resulting from the following arrangements of passing load (combined with the permanent load):—(1), Bridge half covered from one end; (2), three-quarters covered from one end; (3), wholly covered; (4), covered by the passing load over the middle half of the bridge, the haunches being unloaded. Draw these linear arches on the rib by the method given in § 38, and choosing at each one of some eight or ten selected sections the two curves which most nearly approach the top and bottom flanges respectively, compute the maximum intensity of stress on the top and bottom flanges at each section from the two thrusts corresponding to these two linear arches; where the stress is excessive add metal; remove it where the maximum stress is less than the safe stress for the material. If no great change is made in the design this process will be sufficient, but if the cross section is seriously modified by the alteration we must make a second approximation by recalculating the linear arches for the new form of rib, and thus proceed by trial and error until the stresses corresponding to the actual linear arches are met by sufficient metal at all points. The rib need not be of uniform depth throughout, and may be increased in depth at the places where the stress due to bending moment has been found excessive.

In large spans the effect of a change of temperature must be taken into account. This can be done by finding the linear arch given by the expression—

$$1. \quad . \quad . \quad . \quad . \quad \Sigma\frac{M\Delta l}{I}y=\Delta s,$$

where Δs is the alteration in span which would result from the expansion or contraction of the span if free to expand or contract with the change of temperature.

In a series of arches abutting against comparatively slender piers, account must be taken of the thrust transmitted from the neighbouring arch. This thrust will only be due to the passing load, and a part may be considered as taken by the pier; the remainder which the pier cannot counterbalance must be compounded with the reaction due to the linear arch calculated for the permanent load of the unloaded span. With the reaction thus computed the linear arch resulting in the unloaded span must be constructed, and the stresses examined on the top and bottom flanges of the rib.

The theory now given for a stiff rib used as an arch is equally applicable to a stiff rib hung as a suspension bridge.

§ 48. *Wooden Arches.*—Arches have occasionally been built of wood with ribs elaborately constructed of bent timber, scarfed and bolted together; the strength of such a rib could be calculated in the way indicated for metal ribs, but the mode of construction is not to be recommended. When wood is employed it should be used in simple straight balks built into a framed arch.

§ 49. *Practical Details of Metal Arches.*—The common form of metal arch is a cast-iron rib of ⌶ section and of small depth. This rib is intended to be sufficient, unaided, to bear the whole weight of the superstructure. The spandrils, made of some kind of lattice work (or occasionally a mere arcade), bear the roadway, and to some extent stiffen the rib beneath. The rib may with advantage be made much deeper than has been the practice, and may consist of tubes framed as in the St Louis Bridge, fig. 5, Plate XVIII., so as to form a single stiff rib. Where, to gain headway, a rib of small depth at the crown is desirable, the rib might with advantage be deepened at the haunches. Wrought iron is a very suitable material for small arches, where the permanent load is insufficient to prevent tension from occurring in some parts of the rib. Cast-iron and cast-steel are better materials for large spans; for moderate spans a good form of metal arch will be shown under the head of "Frames" (fig. 77), being that in which a lower member is braced to the upper member carrying the roadway so as to form a true frame; for very large spans a single deep rib, or a frame with parallel members arranged as an arch, may be adopted. This design has the advantage over that shown in fig. 77 of avoiding very long bracing at the abutments.

V. FRAMES.

§ 50. *Preliminary.*—A frame is a rigid structure composed of straight struts and ties. The struts and ties are called the members or pieces of the frame. The frame as a whole may be subject to a bending moment, but each bar, pillar, rod, or cord in the structure is thereby simply extended or compressed so that the total stress on a given member is the same at all its cross sections, while the intensity of stress is uniform for all the parts of any one cross section. This result must follow in any frame, the members of which are so connected that the joints offer little or no resistance to change in the relative angular position of the members. Thus if the members are pinned together, the joint consisting of a single circular pin, the centre of which lies in the axis of the piece, it is clear that the direction of the only stress which can be transmitted from pin to pin will coincide with this axis. The axis becomes, therefore, a line of resistance, and in reasoning of the stresses on frames we may treat the frame as consisting of simple straight lines from joint to joint. When the members of a frame consist of iron rods as ties, combined with struts formed by angle iron or T iron of the usual sizes, or by pieces of timber of the ordinary dimensions, it is found by experiment that the stresses on the several members do not differ sensibly whether these members are pinned together with a single pin or rigidly jointed by several bolts or rivets. Frames are much used as girders, and they also give useful designs for suspension and arched bridges. A frame used to support a weight is often called a *truss*; the stresses on the various members of a truss can be computed for any given load with greater accuracy than the intensity of stress on the various parts of a continuous structure such as a tubular girder, or the rib of an arch. Many assumptions are made in treating of the flexure of a continuous structure which are not strictly true; no assumption is made in determining the stresses on a frame, except that the joints are flexible, and that the frame shall be so stiff as not sensibly to alter in form under the load. Both assumptions are consistent with the facts in the case of any bridge truss.

§ 51. *Classes of Frames used as Trusses.*—Frames used as bridge trusses should never be designed so that the elongation or compression of one member can elongate or compress any other member. An example will serve to make the meaning of this limitation clearer. Let a frame consist of the five members AB, BD, DC, CA, CB (fig. 63), jointed at the points A, B, C, and D, and all capable of resisting tension and compression. This frame will be *rigid, i.e.*, it cannot be distorted without causing an alteration in the length of one or more of the members; but if from a change of temperature or any other cause one or all of the members change their length, this will not produce a stress on any member, but will merely cause a change in the form of the frame. Such a frame as this cannot be *self-strained*. A workman, for instance, cannot produce a stress on one member by making some other member of a wrong length. Any error of this kind will merely affect the form of the frame; if, however, another member be introduced between A and D, then if BC be shortened AD will be strained so as to extend it, and the four other members will be compressed; if CB is lengthened AD will thereby be compressed, and the four other members extended; if the workman does not make CB and AD of exactly the right length they and all the members will be permanently strained. These stresses will be unknown quantities, which the designer cannot take into account, and such a combination ought therefore never to be adopted. A frame of this second type is said to have one *redundant member*.

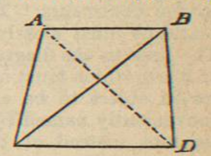


Fig. 63.

If the members AD and CB were flexible cords there would be no redundant members; for the tightening of one diagonal would throw no sensible stress on the other diagonal, since it is supposed incapable of resisting a thrust. Both diagonals, if flexible, are required to prevent the quadrilateral from getting out of shape. Members capable of bearing only one kind of strain might receive the name of *semi-members*.

§ 52. *External Forces on Frame.*—Frames used as bridge trusses are in equilibrium under the external forces applied to them. These forces are—(1) the loads, (2) the reactions at the points of support. The loads are to be referred to the joints as follows:—(1) find the resultant of the load carried by any two joints; (2) resolve that load into two vertical components acting through the two joints; (3) compound the several components acting at each joint into one resultant. This process gives a frame with external forces equivalent to the actual loads, but acting only at the joints. The frames are always supported at a joint, and the reactions of the supports are therefore also forces acting at joints. The load between any two joints is directly supported by the member of the frame joining them; the stresses due to the direct action of this partial load must, where great accuracy is wanted, be added to the stresses computed on the assumption that the loads have been applied directly to the joints. Generally the stresses due to the direct action of the load between two joints may be neglected except where a member of the frame is employed to carry the roadway.

Fig. 64 shows a common form of bridge truss known as

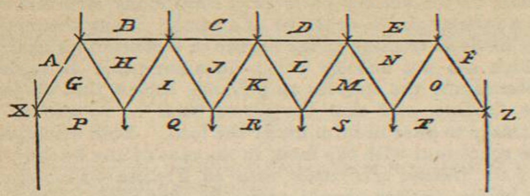


Fig. 64.

a *Warren girder*, with lines indicating external forces applied to the joints; half the load carried between the two lower joints next the piers on either side is directly carried by the abutments. The sum of the two upward vertical reactions must clearly be equal to the sum of the loads. The lines in the diagram represent the directions of a series of forces which must all be in equilibrium; these lines may, for an object to be explained in the next paragraph, be conveniently named by the letters in the spaces which they separate instead of by the method usually employed in geometry. Thus we shall call the first inclined line on the left hand the line AG, the line representing the first force on the top left hand joint AB, the first horizontal member at the top left hand the line BH, &c.; similarly each point requires at least three letters to denote it; the top first left hand joint may be called ABHG, being the point where these four spaces meet. In this method of lettering, every enclosed space must be designated by a letter; all external forces must be represented by lines *outside* the frame, and each space between any two forces must receive a distinctive letter; this method of lettering was first proposed by Mr R. H. Bow (*Economics of Construction*), and is convenient in applying the theory of reciprocal figures to the computation of stresses on frames.

When the weight of the truss is small it is usual to refer the weights of the parts of the truss itself to the same joints as carry the roadway, and to treat all other joints as unloaded.

The reactions at the points of supports of a framed arch or suspension bridge are inclined, as in fig. 65; the manner

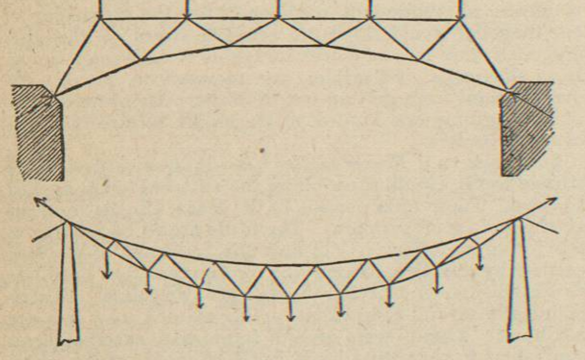


Fig. 65.

of computing the stresses on these frames when the direction of these reactions is known will be first explained, and subsequently the manner of finding this direction will be given.

§ 53. *Reciprocal Figures.*—Prof. Clerk Maxwell has given (*Phil. Mag.* 1864), the following definition of reciprocal figures:—"Two plane figures are reciprocal when they consist of an equal number of lines so that corresponding lines in the two figures are parallel, and corresponding lines which converge to a point in one figure form a closed polygon in the other."

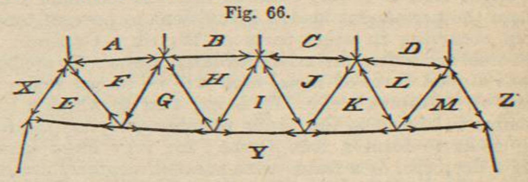
Let a frame (without redundant members), and the external forces which keep it in equilibrium, be represented by a diagram constituting one of these two plane figures, then the lines in the other plane figure or the reciprocal will represent in direction and magnitude the forces between the joints of the frame, and, consequently, the stress on each member as will now be explained.

Reciprocal figures are easily drawn by following definite rules, and afford therefore a simple method of computing the stresses on members of a frame.

The external forces on a frame or bridge in equilibrium under those forces may, by a well-known proposition in statics, be represented by a closed polygon, each side of which is parallel to one force, and represents the force in magnitude as well as in direction. The sides of the polygon may be arranged in any order, provided care is taken so to draw them that in passing round the polygon in one direction this direction may for each side correspond to the direction of the force which it represents.

This polygon of forces may, by a slight extension of the above definition, be called the *reciprocal figure* of the external forces, if the sides are arranged in the same order as that of the joints on which they act, so that if the joints and forces be numbered 1, 2, 3, 4, &c., passing round the outside of the frame in one direction, and returning at last to joint 1, then in the polygon the side representing the force 2 will be next the side representing the force 1, and will be followed by the side representing the force 3, and so forth. This polygon falls under the definition of a reciprocal figure given by Clerk Maxwell, if we consider the frame as a point in equilibrium under the external forces.

Fig. 66.



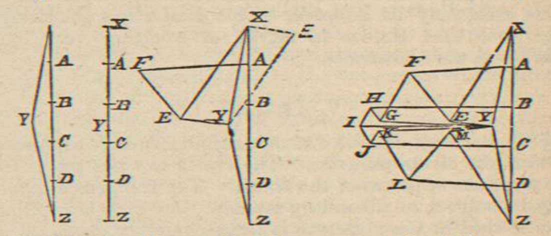


Fig. 66*a*. Fig. 66*b*. Fig. 66*c*. Fig. 66*d*.

Fig. 66 shows a frame supported at the two end joints, and loaded at each top joint. The loads and the supporting forces are indicated by arrows. Fig. 66*a* shows the reciprocal figure or polygon for the external forces on the assumption that the reactions are slightly inclined. The lines in fig. 66*a*, lettered in the usual manner, correspond to the forces indicated by arrows in fig. 66, and lettered according to Mr Bow's method. When all the forces are vertical, as will be the case in girders, the polygon of external forces will be reduced to two straight lines, fig. 66*b*, superimposed and divided so that the length AX represents the load AX, the length AB the load AB, the length YX the reaction YX, and so forth. The line XZ consists of a series of lengths, as XA, AB......DZ, representing the loads taken in their order. In subsequent diagrams the two reaction lines will, for the sake of clearness, be drawn as if slightly inclined to the vertical (as practised by Mr Bow).

If there are no redundant members in the frame, there will be only two members abutting at the point of support, for these two members will be sufficient to balance the reaction, whatever its direction may be; we can therefore draw two triangles, each having as one side the reaction YX, and having the two other sides parallel to these two members; each of these triangles will represent a polygon of forces in equilibrium at the point of support. Of these two triangles, shown in fig. 66*c*, select that in which the letters X and Y are so placed that (naming the apex of the triangle E) the lines XE and YE are the lines parallel to the two members of the

same name in the frame (fig. 66). Then the triangle YXE is the reciprocal figure of the three lines YX, XE, EY in the frame, and represents the three forces in equilibrium at the point YXE of the frame. The direction of YX, being a thrust upwards, shows the direction in which we must go round the triangle YXE to find the direction of the two other forces; doing this we find that the force XE must act down towards the point YXE, and the force EY away from the same point. Putting arrows on the frame diagram to indicate the direction of the forces, we see that the member EY must pull and therefore act as a tie, and that the member XE must push and act as a strut. Passing to the point XEFA we find two known forces, the load XA acting downwards, and a push from the strut XE, which, being in compression, must push at both ends, as indicated by the arrow, fig. 66. The directions and magnitudes of these two forces are already drawn (fig. 66*a*) in a fitting position to represent part of the polygon of forces at XEFA; beginning with the upward thrust EX, continuing down XA, and drawing AF parallel to AF in the frame we complete the polygon by drawing EF parallel to EF in the frame. The point F is determined by the intersection of the two lines, one beginning at A, and the other at E. We then have the polygon of forces EXAF, the reciprocal figure of the lines meeting at that point in the frame, and representing the forces at the point EXAF; the direction of the forces on EX and XA being known determines the direction of the forces due to the elastic reaction of the members AF and EF, showing AF to push as a strut, while EF is a tie. We have been guided in the selection of the particular quadrilateral adopted by the rule of arranging the order of the sides so that the same letters indicate corresponding sides in the diagram of the frame and its reciprocal. Continuing the construction of the diagram in the same way, we arrive at fig. 66*d* as the complete reciprocal figure of the frame and forces upon it, and we see that each line in the reciprocal figure measures the stress on the corresponding member in the frame, and that the polygon of forces acting at any point, as IJKY, in the frame is represented by a polygon of the same name in the reciprocal figure. The direction of the force in each member is easily ascertained by proceeding in the manner above described. A single known force in a polygon determines the direction of all the others, as these must all correspond with arrows pointing the same way round the polygon. Let the arrows be placed on the frame round each joint, and so as to indicate the direction of each force on that joint; then when two arrows point to one another on the same piece, that piece is a tie; when they point from one another the piece is a strut. It is hardly necessary to say that the forces exerted by the two ends of any one member must be equal and opposite. This method is universally applicable where there are no redundant members. The reciprocal figure for any loaded frame is a complete formula for the stress on every member of a frame of that particular class with loads on given joints. Some examples of these figures will be given, and the reader will easily construct others for himself.

§ 54. *Warren Girders—Reciprocal Figure and Method of Computing Stresses by Method of Sections.*

Case 1. The Warren girder loaded at each top joint, figs. 67 and 67*a*. This diagram differs very slightly from

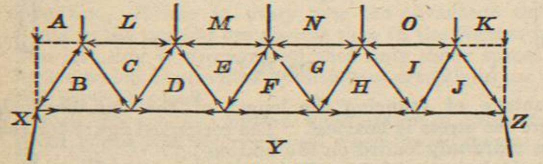


Fig. 67.

that shown in fig. 66. The top and bottom members are in straight lines, and consequently the lines indicating the stresses on the bottom member are superimposed one on the other instead of radiating from Y; the loads XA and KZ are shown as directly borne by the piers. It is clear that if the road is supported by a platform reaching from the end joints to the piers, half of the load on these parts of the platform will be directly supported on the piers. These end loads are shown also in the subsequent diagrams. The truss is generally built of equilateral

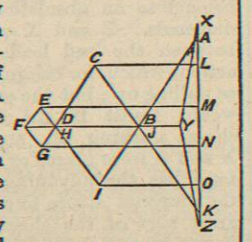


Fig. 67*a*.

triangles, and the inclination of the bracing to the horizon should never be less than 45°.

Case 2. Warren girder loaded on top and bottom, fig. 68. 68*b* shows the polygon of external forces, and 68*c*

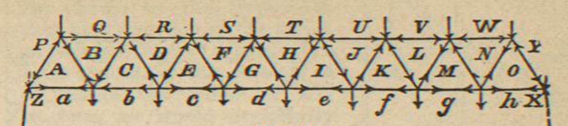


Fig. 68.

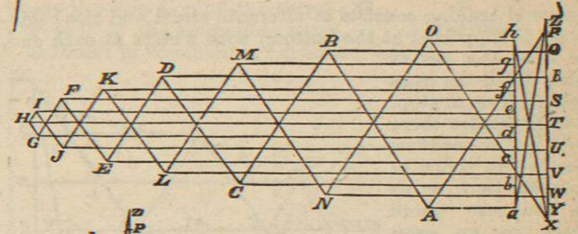


Fig. 68*a*.

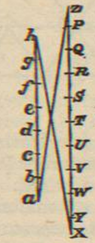


Fig. 68*b*.

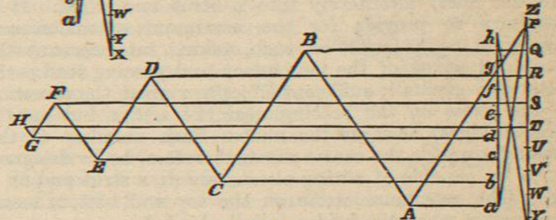


Fig. 68*c*.

shows half the reciprocal figure. These figures have been added to facilitate the comprehension of the complete reciprocal shown by 68*a*.

Case 3. Warren girder with one load, not central, fig. 69.

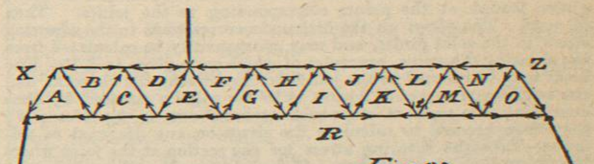


Fig 69

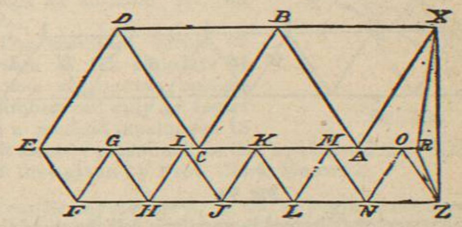


Fig. 69*a*.

69. The polygon RXZ, fig. 69*a*, represents the external forces.