upon a particle within its mass, whose co-ordinates are f, 0, h, are

$$X=\frac{4}{3}\pi\rho f\left\{\frac{1}{r^3}\int_0^{c_1}\rho d.c^3(1+2e)-\frac{\lambda}{r^3}\int_0^{c_1}\rho d(ec^5)-\frac{2}{5}\int_{c_1}^{c_0}\rho de\right\}$$

$$Z=\frac{4}{3}\pi\rho h\left\{\frac{1}{r^3}\int_0^{c_1}\rho d.c^3(1+2e)-\frac{\lambda'}{r^3}\int_0^{c_1}\rho d(ec^5)+\frac{4}{5}\int_{c_1}^{c_0}\rho de\right\}$$

We take into account the rotation of the earth by subtracting the centrifugal force $f\omega^2=F$ from X. Now, the surface of constant density upon which the point f, 0, h is situated gives $(1-2e)fdf+hdh=0$; and the condition of equilibrium is that $(X-F)df+Zdh=0$. Therefore,

$$(X-F)h=Zf(1-2e),$$

which, neglecting small quantities of the order e^2 and putting $\omega^2t^2=4\pi^2$, gives

$$\frac{2e}{r^3}\int_0^{c_1}\rho d.c^3(1+2e)-\frac{6}{5r^5}\int_0^{c_1}\rho d(ec^5)-\frac{6}{5}\int_{c_1}^{c_0}\rho de=\frac{3\pi}{t^2}$$

Here we must put now c for c_1, c for r, and $1+2e$ under the first integral sign may be replaced by unity. Two integrations lead us to the following very important differential equation:—

$$\frac{d^2e}{dc^2}+\frac{2\rho c^2}{\int\rho c^2dc}\cdot\frac{de}{dc}+\left(\frac{2\rho c}{\int\rho c^2dc}-\frac{6}{c_2}\right)e=0.$$

When ρ is expressed in terms of c, this equation can be integrated. We infer then that a rotating spheroid of very small ellipticity, composed of fluid homogeneous strata such as we have specified, will be in equilibrium; and when the law of the density is expressed, the law of the corresponding ellipticities will follow. If we put M for the mass of the spheroid, then

$$M=\frac{4\pi}{3}\int_0^c\rho d.c^3(1+2e);\quad\text{and}\quad m=\frac{c^3}{M}\cdot\frac{4\pi^2}{t^2},$$

and putting $c=c_0$ in the equation expressing the condition of equilibrium, we find

$$M(2e-m)=\frac{4}{3}\pi\cdot\frac{6}{5c^2}\int_0^c\rho d(ec^5),$$

Making these substitutions in the expressions for the forces at the surface, and putting $r=1+\epsilon-e\frac{h^2}{c^2}$, we get

$$G\cos\phi=\frac{M}{ac}\left\{1-e-\frac{3}{2}m+\left(\frac{5}{2}m-2e\right)\frac{h^2}{c^2}\right\}\frac{f}{c}$$

$$G\sin\phi=\frac{M}{ac}\left\{1+e-\frac{3}{2}m+\left(\frac{5}{2}m-2e\right)\frac{h^2}{c^2}\right\}\frac{h}{c}.$$

Here G is gravity in the latitude ϕ, and a the radius of the equator. Since $\sec\phi=\frac{c}{f}\left(1+e+e\frac{h^2}{c^2}\right)$,

$$G=\frac{M}{ac}\left\{1-\frac{3}{2}m+\left(\frac{5}{2}m-e\right)\sin^2\phi\right\},$$

which expression contains the theorems we have referred to as discovered by Clairaut.

The theory of the figure of the earth as a rotating ellipsoid has proved an attractive subject to many of the greatest mathematicans, Laplace especially, who has devoted a large portion of his *Mécanique Céleste* to it. In English the principal existing works on the subject are Sir George Airy's *Mathematical Tracts*, where the subject is treated in the lucid style so characteristic of its author, but without the use of Laplace's coefficients, Archdeacon Pratt's *Attractions and Figure of the Earth*, and O'Brien's *Mathematical Tracts*; in the last two Laplace's coefficients are used. In the *Cambridge Transactions*, vol. viii., is a valuable essay by Professor Stokes, in which he proves, without making any assumption whatever as to the ellipticity of internal strata, or as to the past or the present fluidity of the earth that if the external form of the sea—imagined to percolate the land by canals—be a spheroid with small ellipticity, then the law of gravity will be that found above.[1]

An important theorem by Jacobi must not be overlooked. He proved that for a homogeneous fluid in rotation a spheroid is not the only form of equilibrium; an ellipsoid rotating round its least axis may with certain proportions of the axes and a certain time of revolution be a form of equilibrium.[2]

Local Attraction.

In speaking of the figure of the earth, we mean the surface of the sea imagined to percolate the continents by canals. That this surface should turn out, after precise measurements, to be exactly an ellipsoid of revolution is *a priori* improbable. Although it may be highly probable that originally the earth was a fluid mass, yet in the cooling whereby the present crust has resulted, the actual solid surface has been left in form the most irregular. It is clear that these irregularities of the visible surface must be accompanied by irregularities in the mathematical figure of the earth, and when we consider the general surface of our globe, its irregular distribution of mountain masses, continents, with oceans and islands, we are prepared to admit that the earth may not be precisely any surface of revolution. Nevertheless, there must exist some spheroid which agrees very closely with the mathematical figure of the earth, and has the same axis of rotation. We must conceive this figure as exhibiting slight departures from the spheroid, the two surfaces cutting one another in various lines; thus a point of the surface is defined by its latitude, longitude, and its height above the spheroid of reference. Call this height for a moment n; then of the actual magnitude of this quantity we can generally have no information, it only obtrudes itself on our notice by its variations. In the vicinity of mountains it may change sign in the space of a few miles; n being regarded as a function of the latitude and longitude, if its differential coefficient with respect to the former be zero at a certain point, the normals to the two surfaces then will lie in the prime vertical; if the differential coefficient of n with respect to the longitude be zero, the two normals will lie in the meridian; if both coefficients are zero, the normals will coincide. The comparisons of terrestrial measurements with the corresponding astronomical observations have ever been accompanied with discrepancies. Suppose A and B to be two trigonometrical stations, and that at A there is a disturbing force drawing the vertical through an angle δ, then it is evident that the apparent zenith of A will be really that of some other place A′, whose distance from A is $r\delta$, when r is the earth's radius; and similarly if there be a disturbance at B of the amount δ', the apparent zenith of B will be really that of some other place B′, whose distance from B is $r\delta'$. Hence we have the discrepancy that, while the geodetical measurements deal with the points A and B, the astronomical observations belong to the points A′, B′. Should δ, δ' be equal and parallel, the displacements AA′, BB′ will be equal and parallel, and no discrepancy will appear. The non-recognition of this circumstance often led to much perplexity in the early history of geodesy. Suppose that, through the unknown variations of n, the probable error of an observed latitude (that is, the angle between the normal to the mathematical surface of the earth at the given point and that of the corresponding point on the spheroid of reference) be ϵ, then if we compare two arcs of a degree

[1] See also a paper by Professor Stokes, in the *Cambridge and Dublin Mathematical Journal*, vol. iv. 1849.

[2] See a paper in the *Proceedings of the Royal Society*, No. 123 1870, by I. Todhunter, M.A., F.R.S.

each in mean latitudes, and near each other, say about five degrees of latitude apart, the probable error of the resulting value of the ellipticity will be approximately $\pm\frac{1}{500}\epsilon$, ϵ being expressed in seconds, so that if ϵ be so great as 2″ the probable error of the resulting ellipticity will be greater than the ellipticity itself. It is not only interesting, but necessary at times, to calculate the attraction of a mountain, and the consequent disturbance of the astronomical zenith, at any point within its influence. The deflection of the plumb-line, caused by a local attraction whose amount is $A\delta$, is measured by the ratio of $A\delta$ to the force of gravity at the station. Expressed in seconds, the deflection Δ is

$$\Delta=12''.447.\frac{\delta}{\rho}A,$$

where ρ is the mean density of the earth, δ that of the attracting mass,—the linear unit in expressing A being a mile. Suppose, for instance, a table-land whose form is a rectangle of twelve miles by eight miles, having a height of 500 feet and density half that of the earth; let the observer be two miles distant from the middle point of the longer side. The deflection then is 1″·472; but at one mile it increases to 2″·20. At sixteen astronomical stations in the English Survey the disturbance of latitude due to the form of the ground has been computed, and the following will give an idea of the results. At six stations the deflection is under 2″, at six others it is between 2″ and 4″, and at four stations it exceeds 4″. There is one very exceptional station on the north coast of Banffshire, near the village of Portsoy, at which the deflection amounts to 10″, so that if that village were placed on a map in a position to correspond with its astronomical latitude, it would be 1000 feet out of position! There is the sea to the north and an undulating country to the south, which, however, to a spectator at the station does not suggest any great disturbance of gravity. A somewhat rough estimate of the local attraction from external causes gives a maximum limit of 5″, therefore we have 5″ unaccounted for, or rather which must arise from unequal density in the underlying strata in the surrounding country. In order to throw light on this remarkable phenomenon, the latitudes of a number of stations between Nairn on the west, Fraserburgh on the east, and the Grampians on the south, were observed, and the local deflections determined. It is somewhat singular that the deflections diminish in all directions, not *very* regularly certainly, and most slowly in a south-west direction, finally disappearing, and leaving the maximum at the original station at Portsoy.

The method employed by Dr Hutton for computing the attraction of masses of ground is so simple and effectual that it can hardly be improved on. Let a horizontal plane pass through the given station; let r, θ be the polar co-ordinates of any point in this plane, and r, θ, z, the co-ordinates of a particle of the attracting mass; and let it be required to find the attraction of a portion of the mass contained between the horizontal planes $z=0$, $z=h$, the cylindrical surfaces $r=r_1$, $r=r_2$, and the vertical planes $\theta=\theta_1$, $\theta=\theta_2$. The component of the attraction at the station or origin along the line $\theta=0$ is

$$\delta\int_{r_1}^{r_2}\int_{\theta_1}^{\theta_2}\int_0^h\frac{r_2\cos\theta drd\theta dz}{(r^2+z^2)^{\frac{3}{2}}}$$

$$=\delta h(\sin\theta_2-\sin\theta_1)\log\frac{r_2+(r_2^2+h^2)^{\frac{1}{2}}}{r_1+(r_1^2+h^2)^{\frac{1}{2}}}.$$

By taking r_2-r_1 sufficiently small, and supposing h also small, as it usually is, compared with r_1+r_2, the attraction is

$$=\delta(r_2-r_1)(\sin\theta_2-\sin\theta_1)\frac{h}{r},$$

where $r=\frac{1}{2}(r_1+r_2)$. This form suggests the following procedure. Draw on the contoured map a series of equidistant circles, concentric with the station, intersected by radial lines so disposed that the sines of their azimuths are in arithmetical progression. Then, having estimated from the map the mean heights of the various compartments, the calculation is obvious.

In mountainous countries, as near the Alps and in the Caucasus, deflections have been observed to the amount of as much as 29″. On the other hand, deflections have been observed in flat countries, such as that noted by Professor Schweitzer, who has shown that, at certain stations in the vicinity of Moscow, within a distance of 16 miles the plumb-line varies 16″ in such a manner as to indicate a vast deficiency of matter in the underlying strata. But these are exceptional cases.[1] Since the attraction of a mountain mass is expressed as a numerical multiple of $\delta:\rho$, the ratio of the density of the mountain to that of the earth, if we have any independent means of ascertaining the amount of the deflection, we have at once the ratio $\rho:\delta$, and thus we obtain the mean density of the earth, as, for instance, at Schiehallion, and more recently at Arthur's Seat. A compact mass of great density at a small distance under the surface of the earth will produce an elevation of the mathematical surface which is expressed by the formula

$$y=a\mu\left\{\frac{1}{(1+k^2-2k\cos\theta)^{\frac{1}{2}}}-1\right\},$$

where a is the radius of the (spherical) earth, $a(1-k)$ the distance of the disturbing mass below the surface, μ the ratio of the disturbing mass to the mass of the earth, and $a\theta$ the distance of any point on the surface from that point, say Q, which is vertically over the disturbing mass. The maximum value of y is at Q, where it is

$$y=a\mu\frac{k}{1-k}$$

The deflection at the distance $a\theta$ is

$$\Delta=\frac{\mu k\sin\theta}{(1+k^2-2k\cos\theta)^{\frac{3}{2}}},$$

or since θ is small, putting $h+k=1$,

$$\Delta=\frac{\mu\theta}{(h^2+\theta^2)^{\frac{3}{2}}}.$$

The maximum deflection takes place at a point whose distance from Q is to the depth of the mass as $1:\surd 2$, and its amount is

$$\frac{2}{3\sqrt{3}}\frac{\mu}{h^2}.$$

If, for instance, the disturbing mass were a sphere a mile in diameter, the excess of its density above that of the surrounding country being equal to half the density of the earth, and the depth of its centre half a mile, the greatest deflection would be 5″, and the greatest value of y only two inches. Thus a large disturbance of gravity may arise from an irregularity in the mathematical surface whose actual magnitude, as regards height at least, is extremely small.

The effect of the disturbing mass μ on the vibrations of a pendulum would be a maximum at Q; if ν be the number of seconds of time gained per diem by the pendulum at Q, and σ the number of seconds of angle in the maximum deflection, then it may be shown that

$$\frac{\nu}{\sigma}=\frac{\pi\sqrt{3}}{10};$$

[1] In the *Philosophical Transactions* for 1855 and 1859 will be found Archdeacon Pratt's calculations of the attractions of the Himalayas and the mountain region beyond them, and the consequent deflection of the plumb-line at various stations in India; the subject, which presents many anomalies and difficulties, is very fully gone into in his treatise on the figure of the earth. His computed deflections are vastly greater than anything brought to light by observation.

so that the number of seconds of time by which at the maximum the pendulum is accelerated is about half the number of seconds of angle in the maximum deflection.

Principles of Calculation.

Let α, α' be the mutual azimuths of two points P, Q on a spheroid, k the chord line joining them, μ, μ' the angles made by the chord with the normals at P and Q, ϕ, ϕ', ω their latitudes and difference of longitude, and $\frac{x^2+y^2}{a^2}+\frac{y^2}{b^2}-1=0$ the equation of the surface; then if the plane xz passes through P the co-ordinates of P and Q will be

$$x=\frac{a}{\Delta}\cos\phi, \qquad x'=\frac{a}{\Delta'}\cos\phi'\cos\omega,$$

$$y=0 \qquad y'=\frac{a}{\Delta'}\cos\phi'\sin\omega,$$

$$z=\frac{a}{\Delta}(1-e^2)\sin\phi, \qquad z'=\frac{a}{\Delta'}(1-e^2)\sin\phi',$$

where $\Delta=(1-e^2\sin^2\phi)^{\frac{1}{2}}$, $\Delta'=(1-e^2\sin^2\phi')^{\frac{1}{2}}$, and e is the eccentricity. Let f, g, h be the direction cosines of the normal to that plane which contains the normal at P and the point Q, and whose inclinations to the meridian plane of P is $=\alpha$; let also l, m, n and l', m', n' be the direction cosines of the normal at P, and of the tangent to the surface at P which lies in the plane passing through Q, then since the first line is perpendicular to each of the other two and to the chord k, whose direction cosines are proportional to $x'-x$, $y'-y$, $z'-z$, we have these three equations

$$f(x'-x)+gy'+h(z'-z)=0$$
$$fl+gm+hn=0$$
$$fl'+gm'+hn'=0.$$

Eliminate f, g, h from these equations, and substitute

$$l=\cos\phi \qquad l'=-\sin\phi\cos\alpha$$
$$m=0 \qquad m'=\sin\alpha$$
$$n=\sin\phi \qquad n'=\cos\phi\cos\alpha,$$

and we get

$$(x'-x)\sin\phi+y'\cot\alpha-(z'-z)\cos\phi=0.$$

The substitution of the values of x, z, x', y', z' in this equation will give immediately the value of $\cot\alpha$; and if we put ζ, ζ' for the corresponding azimuths on a sphere, or on the supposition $e=0$, the following relations exist

$$\cot\alpha-\cot\zeta= e^2\frac{\cos\phi}{\cos\phi'}\frac{Q}{\Delta}$$

$$\cot\alpha'-\cot\zeta'=-e^2\frac{\cos\phi'}{\cos\phi}\frac{Q}{\Delta'}$$

$$\Delta'\sin\phi-\Delta\sin\phi'=\sin\omega\, Q.$$

If from Q we let fall a perpendicular on the meridian plane of P, and from P let fall a perpendicular on the meridian plane of Q, then the following equations become geometrically evident:

$$k\sin\mu\sin\alpha=\frac{a}{\Delta'}\cos\phi'\sin\omega$$

$$k\sin\mu'\sin\alpha'=\frac{a}{\Delta}\cos\phi\sin\omega.$$

Now in any surface $u=0$ we have

$$k^2=(x'-x)^2+(y'-y)^2+(z'-z)^2$$

$$\cos\mu=\frac{(x'-x)\frac{du}{dx}+(y'-y)\frac{du}{dy}+(z'-z)\frac{du}{dz}}{k\left(\frac{du^2}{dx^2}+\frac{du^2}{dy^2}+\frac{du^2}{dz^2}\right)^{\frac{1}{2}}}$$

$$\cos\mu=\frac{(x'-x)\frac{du}{dx'}+(y'-y)\frac{du}{dy'}+(z'-z)\frac{du}{dz'}}{k\left(\frac{du^2}{dx'^2}+\frac{du^2}{dy'^2}+\frac{du}{dz'^2}\right)^{\frac{1}{2}}}$$

in the present case, if we put

$$1-\frac{xx'}{a^2}-\frac{zz'}{b^2}=U,$$

then

$$\frac{k^2}{a^2}=2U-e^2\left(\frac{z'-z}{b}\right)^2$$

$$\cos\mu=\frac{a}{k}\Delta U; \qquad \cos\mu'=\frac{a}{k}\Delta' U$$

Let u be such an angle that

$$(1-e^2)^{\frac{1}{2}}\sin\phi=\Delta\sin u$$
$$\cos\phi=\Delta\cos u,$$

then on expressing x, x', z, z' in terms of u and u',

$$U=1-\cos u\cos u'\cos\omega-\sin u\sin u';$$

also, if v be the third side of a spherical triangle, of which two sides are $\frac{1}{2}\pi-u$ and $\frac{1}{2}\pi-u'$ and the included angle ω, using a subsidiary angle ψ such that

$$\sin\psi\sin\frac{v}{2}=e\sin\frac{u'-u}{2}\cos\frac{u'+u}{2},$$

we obtain finally the following equations:—

$$k=2a\cos\psi\sin\frac{v}{2}$$

$$\cos\mu=\Delta\sec\psi\sin\frac{v}{2}$$

$$\cos\mu'=\Delta'\sec\psi\sin\frac{v}{2}$$

$$\sin\mu\sin\alpha=\frac{a}{k}\cos u'\sin\omega$$

$$\sin\mu'\sin\alpha'=\frac{a}{k}\cos u\sin\omega.$$

These determine rigorously the distance, and the mutual zenith distances and azimuths, of any two points on a spheroid whose latitudes and difference of longitude are given.

By a series of reductions from the equations containing ζ, ζ' it may be shown that

$$\alpha+\alpha'=\zeta+\zeta'+\frac{e^4}{4}\omega(\phi'-\phi)^2\cos^4\phi_0\sin\phi_0+\ldots,$$

where ϕ_0 is the mean of ϕ and ϕ', and the higher powers of e are neglected. A short computation will show that the small quantity on the right-hand side of this equation can never amount even to the ten thousandth part of a second, which is, practically speaking, zero; consequently the sum of the azimuths $\alpha+\alpha'$ on the spheroid is equal to the sum of the spherical azimuths, whence follows this very important theorem (known as Dalby's theorem). If ϕ, ϕ' be the latitudes of two points on the surface of a spheroid, ω their difference of longitude, α, α' their reciprocal azimuths,

$$\tan\frac{\omega}{2}=\frac{\cos\frac{\phi'-\phi}{2}}{\sin\frac{\phi'+\phi}{2}}\cot\frac{\alpha+\alpha'}{2}.$$

The vertical plane at P passing through Q and the vertical plane at Q passing through P cut the surface of the spheroid in two distinct curves. The greatest distance apart of these curves is, if α_0 = the mean azimuth of PQ,

$$\frac{e^2}{16}\frac{s^3}{a^2}\cos^2\phi_0\sin 2\alpha_0.$$

This is a very small quantity; for even in the case of a line of 100 miles in length having a mean azimuth $\alpha=45°$ in the latitude of Great Britain, it will only amount to half an inch, whilst for a line of fifty miles it cannot exceed the sixteenth part of an inch. The geodesic line joining P and Q lies wholly between these two curves.[1] If we designate by P′, Q′ the two curves (the former being that in the vertical plane through P), then, neglecting quantities of the order $e^2\theta^3$, where θ is the angular distance of P and Q at the centre of the earth, the geodesic curve makes with P′ at P an angle equal to the angle it makes with Q′ at Q, each of these angles being a third of the angle of intersection of P′ and Q′. The difference of length of the geodesic line and either of the curves P′, Q′ is, s being the length of either,

$$\frac{s}{360}e^4\theta^4\cos^4\phi_0\sin^2 2\alpha_0.$$

At least this is an approximate expression. Supposing the angle PQ to be as much as 10°, this quantity would be less than one hundredth of an inch.

An idea of the course of a geodesic line may be gathered from the following example. Let the line be that joining Cadiz and St Petersburg, whose approximate positions are

	Cadiz.	St Petersburg.
Lat.	36° 22′ N.	59° 56′ N.
Long.	6° 18′ W.	30° 17′ E.

If G be the point on the geodesic corresponding to F on that one of the plane curves which contains the normal at Cadiz (by "corresponding" we mean that F and G are on a meridian) then G is to the north of F; at a quarter of the whole distance from Cadiz GF is 458 feet, at half the distance it is 637 feet, and at three quarters it is 473 feet. The azimuth of the geodesic at Cadiz differs 20″ from that of the vertical plane, which is the astronomical azimuth. The azimuth of a geodesic line cannot be observed, so that the line does not enter of necessity into practical geodesy, although many formulæ connected with its use are of great simplicity and elegance. The geodesic line has always held a more important place in the science of geodesy among the mathematicians of France, Germany, and Russia than has been assigned to it in the operations of the English and Indian triangulations. Although the observed angles of a triangulation are not geodesic angles, yet in the calculation of the distance and reciprocal bearings of two points which are far apart, and are connected by a long chain of triangles, we may fall upon the geodesic line in this manner:—

[1] See a paper "On the course of Geodesic Lines on the Earth's Surface" in the *Philosophical Magazine* for 1870.

If A. Z be the points, then to start the calculation from A, we obtain by some preliminary calculation the approximate azimuth of Z, or the angle made by the direction of Z with the side AB or AC of the first triangle. Let P_1 be the point where this line intersects BC; then, to find P_2, where the line cuts the next triangle side CD, we make the angle BP_1P_2 such that $BP_1P_2+BP_1A=180°$. This fixes P_2, and P_3 is fixed by a repetition of the same process; so for P_4, P_5.... Now it is clear that the points P_1, P_2, P_3 so computed are those which would be actually fixed by an observer with a theodolite, proceeding in the following manner. Having set the instrument up at A, and turned the telescope in the direction of the computed bearing, an assistant places a mark P_1 on the line BC, adjusting it till bisected by the cross-hairs of the telescope at A. The theodolite is then placed over P_1, and the telescope turned to A; the horizontal circle is then moved through 180°. The assistant then places a mark P_2 on the line CD, so as to be bisected by the telescope, which is then moved to P_2, and in the same manner P_3 is fixed. Now it is clear that the series of points P_1, P_2, P_3 approaches to the geodesic line, for the plane of any two consecutive elements $P_{n-1}P_n$, P_nP_{n+1} contains the normal at P_n.

From the formulæ which we have given above, expressing the mutual relations of two points P, Q on a spheroid, we may obtain the following solution of the problem: Given the latitude ϕ of P, with the azimuth α and distance s of Q, to determine the latitude and longitude of Q and the back azimuth α'.

Let

$$\theta=\frac{s}{a}\Delta$$

$$\zeta=\frac{e^2\theta^2}{4(1-e^2)}\cos^2\phi\sin 2\alpha$$

$$\zeta'=\frac{e^2\theta^3}{6(1-e^2)}\cos^2\phi\cos^2\alpha;$$

ζ, ζ' are always very minute quantities even for the longest distances; then, putting $\kappa=90°-\phi$,

$$\tan\frac{\alpha'+\zeta-\omega}{2}=\frac{\sin\frac{1}{2}(\kappa-\theta-\zeta')}{\sin\frac{1}{2}(\kappa+\theta+\zeta')}\cot\frac{\alpha}{2}$$

$$\tan\frac{\alpha'+\zeta+\omega}{2}=\frac{\cos\frac{1}{2}(\kappa-\theta-\zeta')}{\cos\frac{1}{2}(\kappa+\theta+\zeta')}\cot\frac{\alpha}{2}$$

$$\phi-\phi=\frac{s}{\rho}\frac{\sin\frac{1}{2}(\alpha'+\zeta-\alpha)}{\sin\frac{1}{2}(\alpha'+\zeta+\alpha)}\left(1+\frac{\theta^2}{12}\cos^2\frac{\alpha'-\alpha}{2}\right);$$

here ρ is the radius of curvature of the meridian for the mean latitude $\frac{1}{2}(\phi+\phi')$. These formulæ are approximate only, but they are sufficiently precise even for very long distances.

Meridian Arcs.

The length of the arc of meridian between the latitudes ϕ_1 and ϕ_2 is

$$s=\int_{\phi_1}^{\phi_2}\rho d\phi=a\int_{\phi_1}^{\phi_2}\frac{(1-e^2)d\phi}{(1-e^2\sin^2\phi)^{\frac{3}{2}}};$$

instead of using the excentricity, put the ratio of the axes $=1-n:1+n$, then

$$s=\int_{\phi_1}^{\phi_2}\frac{b(1+n)(1-n^2)}{(1+2n\cos 2\phi+n^2)^{\frac{3}{2}}}.$$

This, after integration, gives

$$\frac{s}{b}=\left(1+n+\frac{5}{4}n^2+\frac{5}{4}n^3\right)a_0-\left(3n+3n^2+\frac{21}{8}n^3\right)a_1+\left(\frac{15}{8}n^2+\frac{15}{8}n^3\right)a_2 -\left(\frac{35}{24}n^3\right)a_3,$$

where

$$a_0=\phi_2-\phi_1$$
$$a_1=\sin(\phi_2-\phi_1)\cos(\phi_2+\phi_1)$$
$$a_2=\sin 2(\phi_2-\phi_1)\cos 2(\phi_2+\phi_1)$$
$$a_3=\sin 3(\phi_2-\phi_1)\cos 3(\phi_2+\phi_1)$$

The part of s which depends on n^3 is very small; in fact, if we calculate it for the longest arc measured, the Russian arc, it amounts to only an inch and a half, therefore we omit this term, and put for $\frac{s}{b}$ the value

$$\left(1+n+\frac{5}{4}n^2\right)a_0-\left(3n+3n^2\right)a_1+\left(\frac{15}{8}n^2\right)a_2$$

Now, if we suppose the observed latitudes to be affected with errors, and that the true latitudes are ϕ_1+x_1, ϕ_2+x_2; and if further we suppose that n_1+dn is the true value of $a-b:a+b$, and that n_1 itself is merely a very approximate numerical value, we get, on making these substitutions and neglecting the influence of the corrections x on the *position* of the arc in latitude, *i.e.*, on $\phi_1+\phi_2$,

$$\frac{s}{b}=\left(1+n_1+\frac{5}{4}n_1^2\right)a_0-\left(3n_1+3n_1^2\right)a_1+\left(\frac{15}{8}n_1^2\right)a_2$$
$$+\left\{\left(1+\frac{5}{2}n_1\right)a_0-\left(3+6n_1\right)a_1+\left(\frac{15}{4}n_1\right)a_2\right\}dn$$
$$+\left\{1+n_1-3n_1\frac{da_1}{da_0}\right\}da_0;$$

here $da_0=x_2-x_1$; and as b is only known approximately, put $b_1=b(1+u)$, then we get, after dividing through by the coefficient of da_0, which is $=1+n_1-3n_1\cos(\phi_2-\phi_1)\cos(\phi_2-\phi_1)$, an equation of the form $x_2=x_1+h+fu+gv$, where for convenience we put v for dn.

Now in every measured arc there are not only the extreme stations determined in latitude, but also a number of intermediate stations, so that if there be $i+1$ stations there will be i equations

$$x_2=x_1+f_1u+g_1v+h_1$$
$$x_3=x_1+f_2u+g_2v+h_2$$
$$\vdots$$
$$x_i=x_1+f_iu+g_iv+h_i.$$

In combining a number of different arcs of meridian, with the view of determining the figure of the earth, each arc will supply a number of equations in u and v and the corrections to its observed latitudes. Then, according to the method of least squares, those values of u and v are the most probable which render the sum of the squares of *all* the errors x a minimum. The corrections x which are here applied arise not from errors of observation only. The mere uncertainty of a latitude, as determined with modern instruments, does not exceed a very small fraction of a second as far as errors of observation go, but no accuracy in observing will remove the error that may arise from local attraction. This, as we have seen, may amount to some seconds, so that the corrections x to the observed latitudes are attributable to local attraction. Archdeacon Pratt, in his treatise on the figure of the earth, objects to this mode of applying least squares first used by Bessel; but certainly Bessel was right, and the objection is groundless.

Comparisons of Standards.

In determining the figure of the earth from the arcs of meridian measured in different countries, one source of uncertainty was, until the last few years, the want of comparisons between the standards of length in which the arcs were expressed. This has been removed by the very extensive series of comparisons recently made at Southampton (see *Comparisons of Standard of Length of England, France, Belgium, Prussia, Russia, India, and Australia, made at the Ordnance Survey Office, Southampton*, 1866, and a paper in the *Philosophical Transactions* for 1873, by Lieut.-Col. A. R. Clarke, C.B., R.E., on the further comparisons of the standards of Austria, Spain, the United States, Cape of Good Hope, and Russia). These direct comparisons, which were carried out with the highest attainable precision, are of very great value. The length of the toise has three independent determinations, viz., through the Russian standard double toise, the Prussian toise, and the Belgium toise,—giving for the length of the toise, expressed in terms of the standard yard of England

through the Russian standard	....	6·39453216 ft
„ „ Prussian „	....	6·39453703 ft.
„ „ Belgian „	....	6·39453215 ft

By combining all the different comparisons made in England and on the Continent on these bars, by the method of least squares, the final value of the toise is

6·39453348 ft. (log = 0·8058088656),

from which the greatest divergence of the three separate results specified above is only half a millionth of a toise, corresponding to ten feet in the earth's radius. From the known ratio of the toise and the metre, 864000 : 443296, we get for the metre

3·28086933 ft. (log = 0·5159889356).

That the close agreement between the determinations of the toise is not due to chance will be seen from the fact that the comparisons of the Prussian toise with the English standard involved 2340 micrometer readings and 520 thermometer readings, extending over twenty-five days, the probable error of the resulting length of the toise being ± 0·00000015 yard. The probable error of the determination of the Belgian toise is ± 0·00000027; that of the Russian double toise ± 0·00000031. With regard to the metre, there is an independent determination resulting from the comparison of the platinum metre of the Royal Society,—a large number of observations giving for the length of the metre 3·28087206 feet, which differs from the former result by about one millionth part. But this determination, involving the expansion of the bar for 30° of temperature, and being dependent on some old observations of Arago, cannot be allowed any weight in modifying the result obtained through the toises. The Russian standard, compared at Southampton, was that on which the length of their base lines and therefore their whole arc depends.

Calculation of the Semiaxes.

We now bring together the results of the various meridian arcs, omitting many short arcs which have been used in previous determinations, but which on account of their smallness have little influence in the result aimed at.

The data of the French arc from Formentera to Dunkirk are—

Stations.	Astronomical Latitudes. °	′	″	Distance of Parallels. Feet.
Formentera	38	39	53·17	...
Mountjouy	41	21	44·96	982671·04
Barcelona	41	22	47·90	988701·92
Carcassonne	43	12	54·30	1657287·93
Pantheon	48	50	47·98	3710827·13
Dunkirk	51	2	8·41	4509790·84

The latitude of Formentera as here given is taken from the observations of M. Biot, recorded and computed in the third volume of his *Traité Élémentaire d'Astronomie physique.*

The latitude of the Pantheon, given in the *Base du Système Métrique Décimal* (ii. 413), is 48° 50′ 48″·86. In the *Annales de l'Observatoire Imperial de Paris*, vol. viii. page 317, we find the latitude of south face of the observatory determined as 48° 50′ 11″·71. The Pantheon being 35″·38 north of this, we thus get a second determination of its latitude. The mean is that given above.

The distance of the parallels of Dunkirk and Greenwich, deduced from the recent extension of the triangulation of England into France, in 1862, is 161407·3 feet, which is 3·9 feet greater than that obtained from Captain Kater's triangulation, and 3·2 feet less than the distance calculated by Delambre from General Roy's triangulation. The following table shows the data of the English arc with the distances in standard feet from Formentera.

	°	′	″	Feet.
Formentera	...			...
Greenwich	51	28	38·30	4671198·3
Arbury	52	13	26·59	4943837·6
Clifton	53	27	29·50	5394063·4
Kellie Law	56	14	53·60	6413221·7
Stirling	57	27	49·12	6857323·3
Saxavord	60	49	37·21	8086820·7

The latitude assigned in this table to Saxavord is not the directly observed latitude, which is 60° 49′ 38″·58, for there are here a cluster of three points, whose latitudes are astronomically determined; and if we transfer, by means of the geodesic connection, the latitude of *Gerth of Scaw* to Saxavord, we get 60° 49′ 36″·59; and if we similarly transfer the latitude of *Balta*, we get 60° 49′ 36″·46. The mean of these three is that entered in the above table.

For the Indian arc in long. 77° 40′ we have the following data:—

	°	′	″	Feet.
Punnœ	8	9	31·132	...
Putchapolliam	10	59	42·276	1029174·9
Dodagoontah	12	59	52·165	1756562·0
Namthabad	15	5	53·562	2518376·3
Daumergida	18	3	15·292	3591788·4
Takalkhera	21	5	51·532	4697329·5
Kalianpur	24	7	11·262	5794695·7
Kaliana	29	30	48·322	7755835·9

The data of the Russian arc (long. 26° 40′) taken from M. Struve's work are as below:—

	°	′	″	Feet.
Staro Nekrassowka	45	20	2·94	...
Wodolui	47	1	24·98	616529·81
Ssuprunkowzi	48	45	3·04	1246762·17
Kremenetz	50	5	49·95	1737551·48
Belin	52	2	42·16	2448745·17
Nemesch	54	39	4·16	3400312·63
Jacobstadt	56	30	4·97	4076412·28
Dorpat	58	22	47·56	4762421·43
Hogland	60	5	9·84	5386135·39
Kilpi-maki	62	38	5·25	6317905·67
Tornea	65	49	44·57	7486789·97
Stuor-oivi	68	40	58·40	8530517·90
Fuglenœs	70	40	11·23	9257921·06

From the arc measured by Sir Thomas Maclear in long. 18° 30′, we have

	°	′	″	Feet.
North End	29	44	17·66	...
Heerenlogement Berg.	31	58	9·11	811507·7
Royal Observatory	33	56	3·20	1526386·8
Zwart Kop	34	13	32·13	1632583·3
Cape Point	34	21	6·26	1678375·7

And, finally, for the Peruvian arc, in long. 281° 0′,

	°	′	″	Feet.
Tarqui	−3	4	32·068	...
Cotchesqui	0	2	31·387	1131036·3.

Having now stated the data of the problem, we may either seek that ellipsoid which best represents the observations, or we may restrict the figure to one of revolution. It will be convenient to commence with the supposition of an ellipsoidal figure, as on so doing we can, by a slight alteration in the equations of minimum, obtain also the required figure of revolution. It may be remarked that, whatever the real figure may be, it is certain that if we presuppose it an ellipsoid, the arithmetical process will bring out an ellipsoid, which ellipsoid will agree better with all the observed latitudes than any spheroid would, therefore we do not *prove* that it is an ellipsoid; to prove this, arcs of longitude would be required. There is no doubt such arcs will be shortly forthcoming, but as yet they are not available.

The first thing that occurs to one in considering an ellipsoidal earth is the question, What is a meridian curve? It may be defined in different ways: a point moving on the surface in the direction astronomically determined as "north" might be said to trace a meridian; or we may define it as the locus of those points which have a constant longitude, whose zeniths lie in a great circle of the heavens, having its poles in the equator; we adopt this definition. Let a, b, c be the semiaxes, c being the polar semiaxis. The equation of the ellipsoid being

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1\,,$$

if P be any point on the surface, the direction cosines of the normal at P are proportional to

$$\frac{du}{dx},\ \frac{du}{dy},\ \frac{du}{dz},\ \text{or}\ \frac{x}{a^2},\ \frac{y}{b^2},\ \frac{z}{c^2}\,;$$

and if $\frac{1}{2}\pi-\phi$ be the angle between this normal and the minor axis, so that ϕ is the latitude of P, we have

$$\sin\phi=\frac{\frac{z}{c^2}}{\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}\right)^{\frac{1}{2}}}\,.$$

Hence the equation to a "parallel" in which the latitude ϕ is constant is

$$\frac{x^2}{a^4}+\frac{y^2}{b^4}-\frac{z^2}{c^4}\cot^2\phi=0\,.$$

So that in an ellipsoidal earth the parallel is no longer a plane curve. Let longitude be reckoned from the plane of xz. As there are two species of latitude, astronomical and geocentric, so there are in the ellipsoidal earth two species of longitude, geocentric (called u) and astronomical (called ω). Conceive a line passing through the origin in the plane of the equator and directed to a point whose longitude is $\frac{1}{2}\pi+\omega$. The direction cosines of that line are—$\sin\omega$, $\cos\omega$, and 0. Those points of the surface whose normals are at right angles to this line are in the meridian whose longitude is ω; the condition of perpendicularity is expressed by

$$\frac{x\sin\omega}{a^2}+\frac{y\cos\omega}{b^2}=0\,;$$

and this, in fact, is the equation of the meridian, which is still on the ellipsoidal hypothesis a plane curve. The geocentric and astronomical longitudes are connected by the relation

$$a^2\tan u=b^2\tan\omega\,.$$

This meridian curve is an ellipse whose minor semi-axis is c, and of which the semi-axis major is some quantity r intermediate between a and b, such that

$$\frac{1}{r^2}=\frac{\cos^2 u}{a^2}+\frac{\sin^2 u}{b^2}\,.$$

Take two quantities i, k, such that $a^2(1-i)=b^2(1+i)=k^2$, then $k^2=r^2(1-i\cos 2u)$; and take n such that

$$n=\frac{r-c}{r+c}\,,$$

and substitute the value of r, neglecting the square of i; this gives

$$n=\frac{k-c}{k+c}+\frac{i}{4}\cos 2u\,.$$

Now we have to determine not only the three semi-axes a, b, c, but the longitude of a. Let u_1 be the longitude of one of the measured meridian arcs, u_0 the longitude of a, then, for that arc,

$$n=\frac{k-c}{k+c}+\frac{i}{4}\cos 2(u_1-u_0)$$

$$=\frac{k-c}{k+c}+p\cos 2u_1+q\sin 2u_1\,,$$

where $4p=i\cos 2u_0$, $4q=i\sin 2u_0$.

The normal at P does not pass through the axis of rotation, so that the observed latitudes on an ellipsoid are not exactly the quantities which should be used in the ordinary method of expressing the length of a meridian arc in terms of the latitudes. But it may be shown that this consideration may be neglected.

The data we have collected form 35 equations between the 40 x-corrections to the observed latitudes, and the four unknown quantities determining the elements of the ellipsoid. Suppose n_1 to be an approximate value of the ratio $k-c : k+c$, so that

$$\frac{k-c}{k+c}=n_1+r\,,$$

where r is a small correction to n_1 and suppose c_1 to be an approximate value of c so that $c=c_1(1+t)$, then the four unknown quantities are p, q, r, t. The result of making the sum of the squares of the 40 corrections a minimum is

	Feet.		Metres.
a =	20926350	=	6378294·0
b =	20919972	=	6376350·4
c =	20853429	=	6356068·1

$$\frac{a-c}{c}=\frac{1}{285\cdot 97}\,;\quad \frac{b-c}{c}=\frac{1}{313\cdot 38}$$

$$\frac{a-b}{c}=\frac{1}{3269\cdot 5}$$

Longitude of a............15° 34′ East.

The meridian of the greater axis passes, in the Eastern Hemisphere, through Spitzbergen, the Straits of Messina, Lake Chad in North Africa, and along the west coast of South Africa,—nearly corresponding to the meridian which passes over the greatest quantity of land in that hemisphere. In the Western Hemisphere it passes through Behring's Straits and through the centre of the Pacific Ocean. The meridian (105° 34′ E.) of the minor axis of the equator passes near North-east Cape on the Arctic Sea, through Tong-king and the Straits of Sunda, and corresponds nearly to the meridian which passes over the greatest amount of land in Asia; and in the Western Hemisphere it passes through Smith Sound, the west of Labrador, Montreal, between Cuba and Hayti, and along the west coast of South America, nearly coinciding with the meridian that passes over the greatest amount of land in that hemisphere.

The length of the meridian quadrant passing through Paris, in the ellipsoidal figure given above, is 10001472·5 metres, showing that the length of the ideal French standard is considerably in error as representing the ten-millionth part of the quadrant. The minimum quadrant, in longitude 105° 34′, has a length of 10000024·5 metres. The probable error of the longitude of the major axis of the equator given above is of course large, as much perhaps as ± 15°.

It has been objected to this figure of three unequal axes that it does not satisfy, in the proportions of the axes, the conditions brought out in Jacobi's theorem. Admitting this, it has to be noted, on the other hand, that Jacobi's theorem contemplates a homogeneous fluid, and this is certainly far from the actual condition of our globe, indeed the irregular distribution of continents and oceans suggests as possible a sensible divergence from a perfect surface of revolution.

If we limit the figure to being an ellipsoid of revolution, we get rid in our equations of two unknown quantities, and the result may be expressed thus:—

	Feet.		Metres.
a =	20926062	=	6378206·4
c =	20855121	=	6356503·8

$c : a = 293\cdot 98 : 294\cdot 98$.

As might be expected, the sum of the squares of the 40 latitude corrections, viz., 153·99, is greater in this figure than in that of three axes, where it amounts to 138·30. In the Indian arc the largest corrections are at Dodagoontah, + 3″·87, and at Kalianpur, − 3″·68. In the Russian arc the largest corrections are + 3″·76, at Tornea, and − 3″·31, at Staro Nekrassowka. Of the whole 40 corrections, 16 are under 1″·0, 10 between 1″·0 and 2″·0, 10 between 2″·0 and 3″·0, and 4 over 3″·0. For the ellipsoidal figure the probable error of an observed latitude is ± 1″·42; for the spheroidal it would be very slightly larger. This quantity may be taken therefore as approximately the probable amount of local deflection.

In 1860, the Russian Government, at the instance of M. Otto Struve, imperial astronomer at St Petersburg, invited the co-operation of the Governments of Prussia, Belgium, France, and England, to the important end of connecting their respective triangulations so as to form a continuous chain under the parallel of 52° from the island of Valentia on the south-west coast of Ireland, in longitude 10° 20′ 40″ W., to Orsk on the river Ural in Russia. This grand undertaking was at once set in action, but up to the present