ELAND (*Boselaphus oreas*) is the largest and most valuable member of the antelope family. It is fully equal to the horse in size, standing six feet high at the shoulders, and measuring nine feet from the nose to the root of the tail. In robustness of build it resembles the ox, and forms the type of the bovine subdivision of antelopes. Its neck is thick, and is furnished with a prominent dewlap, fringed with long hair. Except on the ridge of the back the fur is short, and is usually of a reddish fawn colour above and white beneath. Its horns are about 20 inches in length, nearly straight, and in the male are surrounded throughout the greater part of their length with a spiral wreath; in the female they are more slender, and the spiral ridge is indistinct or absent. The eland is a native of South Africa, where it roams in considerable herds over the open plains, "rejoicing," says a recent traveller, "in the belts of shaded hillocks, and in the isolated groves of *Acacia capensis*, which, like islands in the ocean, are scattered over many of the stony and gravelly plains of the interior." It is slow in its movements as compared with the other antelopes, and is readily captured, while in disposition it is exceedingly gentle, and thus seems eminently adapted for domestication. It breeds readily in confinement, and herds of elands have already been introduced into various parks in Britain. Its flesh is highly prized as an article of food, resembling beef, it is said, in grain and colour, but being more delicate and better flavoured. The eland is remarkable for the quantity of fat which it takes on, exceeding in this respect all other large game. The carcase of a single individual weighs from 1500 to 2000 ℔s. The eland was formerly abundant in the neighbourhood of Cape Town, but is now rarely found within the colony, and should man not succeed meanwhile in domesticating it, there is reason to fear that a valuable source of animal food will be lost to him by the speedy extermination of the eland.

EL-ARAISH, L'ARAISH, or in French LARACHE, a town of Morocco on the Atlantic coast, about 45 miles S. of Tangier, is picturesquely situated on a rocky height to the south of the embouchure of the Wady Loukhus or Lixus. It is the seat of a military governor, and has a number of well-kept though practically useless defences. The impress of Spanish occupation is still evident, and all the main points described in the 17th century by Pidon de Saint Olon can easily be distinguished—such as the church, the fort of St Jacques, the castle of St Etienne with its four cupolas, the Jew's Tower, and the castle of Notre Dame d'Europe, now the Kasba or citadel. The market-place is surrounded with arcades of monolithic sandstone pillars. In spite of the bar at the entrance of the river preventing the passage of all vessels of more than 150 tons, the port is one of the most frequented on that part of the coast. The exports, gradually increasing in value, consist mainly of millet, drà, and other cereals, canary-seed, beans, pease, cork, and wool. In 1875, 136 vessels entered and cleared, 26 being British and 58 Spanish. The population of the town at the same date was estimated at 5000, of whom nearly 4000 were Mahometans, about 1000 Spanish-speaking Jews, and 60 Christians.

Though the name of El-Araish is comparatively modern, and is mentioned neither by El-Bikri nor by Edrisi, it seems not improbable from a passage in Scylax that the site of the town was occupied by a Libyan settlement at an early date; and about 3½ miles up the river there still exist on the hill of Tchemmish very considerable ruins of the Punico-Roman city of Lixus. The modern town was finally taken from the Portuguese in 1689 by Mulei Ismael after a five months' siege; in 1785 it was attacked by the French, and in 1829 saw the destruction of the Morocco fleet by the Austrians. A convent in connection with the Spanish mission was maintained till 1822.

See Barth, *Wanderungen durch die Kustenländer des Mittelmeeres*, 1849; Rohlf's *Adventures in Morocco*, 1874; Tissot. "Itinéraire de Tanger à R'bat," in *Bull. de la Soc. de Géogr.*, 1876.

ELASTICITY

1. ELASTICITY of matter is that property in virtue of which a body requires force to change its bulk or shape, and requires a continued application of the force to maintain the change, and springs back when the force is removed, and, if left at rest without the force, does not remain at rest except in its previous bulk and shape. The elasticity is said to be perfect when the body always requires the same force to keep it at rest in the same bulk and shape and at the same temperature through whatever variations of bulk, shape, and temperature it be brought. A body is said to possess some degree of elasticity if it requires any force to keep it in any particular bulk or shape. It is convenient to discuss elasticity of bulk and elasticity of shape sometimes separately and sometimes jointly.

2. Every body has some degree of elasticity of bulk. If a body possesses any degree of elasticity of shape it is called a solid; if it possesses no degree of elasticity of shape it is called a fluid.

3. All fluids possess elasticity of bulk to perfection. Probably so do all homogeneous solids, such as crystals and glasses. It is not probable that any degree of fluid pressure (or pressure acting equally in all directions) on a piece of common glass, or rock crystal, or of diamond, or on a crystal of bismuth, or of copper, or of lead, or of silver, would make it denser after the pressure is removed, or put it into a condition in which at any particular intermediate pressure it would be denser than it was at that pressure before the application of the extreme pressure. Malleable metals and alloys, on the other hand, may have their densities considerably increased and diminished by hammering and by mere traction. By compression between the dies used in coining, the density of gold may be raised from 19·258 to 19·367, and the density of copper from 8·535 to 8·916;[1] and Mr M'Farlane's experiments quoted below (section 78), show a piece of copper wire decreasing in density from 8·91 to 8·835 after successive simple tractions, by which its length was increased from 287 centimetres to 317 centimetres, while its modulus of rigidity decreased from 443 to 426 million grammes per square centimetre. Later experiments, recently made for this article by the same experimenter, have shown *augmentation* of density from 8·85 to 8·95, produced by successive tractions which elongated a piece of copper wire from weighing 16·4 grammes per metre to weighing 13·5 grammes per metre, the wire having been first annealed by heating it to redness in sand, and allowing it to cool slowly. Augmentation of density by traction is a somewhat surprising result, but not altogether so when we consider that the wire had been reduced to an abnormally small density by the previous thermal treatment (the "annealing"). The common explanation of these changes of density in metals, which attributes them to porosity, is probably true; by porosity being understood a porous structure with such vast numbers of the ultimate molecules in the portions of the solid substance between pores or interstices that these portions may be called homogeneous in the sense that a crystal or a liquid can be called homogeneous (compare section 40 below).

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[1] *Seventh Annual Report of the Deputy-Master of the Mint*, p. 43, quoting as authority Percy's *Metallurgy of Copper*. London, 1861.

4. The elasticity of shape of many solids is not perfect: it is not known whether it is perfect for any. It might be expected to be perfect for glass and rock crystal and diamond and other hard, brittle, homogeneous substances; but experiment proves that at all events for glass it is not so, and shows on the contrary a notable degree of imperfection in the torsional elasticity of glass fibres. It might be expected that in copper and soft iron and other plastic metals the elasticity of shape would be very imperfect; experiment shows, on the contrary, that in copper, brass, soft iron, steel, platinum, provided the distortion does not exceed a certain limit in each case, elasticity of shape is remarkably perfect, much more perfect than in glass. It is quite probable that even in the softer metals—zinc, tin, lead, cadmium, potassium, sodium, &c.—the elasticity of shape may be as perfect as in the metals mentioned above, but within narrower limits as to degree of distortion. Accurate experiment is utterly wanting, to discover what is the degree of imperfection, if any, of the elasticity of any metal or alloy, when tested within sufficiently narrow limits of distortion.

5. The "viscosity of metals" described below (sections 21–25) does not demonstrate any imperfectness of elasticity according to the definition of section 1, which is purely statical. The viscosity of solids *may* (for all we yet know by experiment) depend, as does the viscosity of fluids, upon a resistance varying with the velocity of the change, and *vanishing when the velocity of the change is zero*, that is to say, when the body is at rest in any configuration; if so, the elasticity of the substance concerned is perfect within the limits of the experiment in question. If, on the other hand (as the discovery of elastic fatigue described below seems to indicate may be to some degree the case), the loss of energy from the vibrations in the experiments described is due to a dependence of the elastic resilient force upon previous conditions of the substance in respect to strain, the "viscosity" would be continuous with a true imperfectness of static elasticity. Here, then, we have a definite question which can be answered by experiment only:—Consider a certain definite stress applied to a solid substance; as, for example, a certain "couple" twisting a wire or rod; or a certain weight pulling it out, or compressing it lengthwise; or a certain weight placed on the middle of a beam supported by trestles under its ends. Let it be applied and removed a great many times, and suppose it to be seen that after each application and removal of the stress the body comes to rest in exactly the same configuration as after the previous application or removal of the stress. If now the body be left to itself with the stress removed, and if it be found to remain at rest in the same configuration for minutes, or hours, or days, or years after the removal of the stress, a part of the definition of perfect elasticity is fulfilled. Or, again, if the stress be applied, and kept applied with absolute constancy, and if the body remain permanently in a constant configuration, another item of the definition of perfect elasticity is proved. When any such experiment is made on any metal, unless some of the softer metals (section 4) is to be excepted, there is certainly very little *if any* change of configuration in the circumstances now supposed. The writer believes, indeed, that nothing of the kind has hitherto been discovered by experiment, provided the stress has been considerably less than that which would break or give a notable permanent twist, or elongation, or bend, to the body, that is to say, provided the action has been kept decidedly within the limits of the body's elasticity as commonly understood (sections 7–20). Mr J. T. Bottomley, with the assistance of a grant of money from the British Association, has commenced making arrangements for secular experiments on the elasticity of metals, in the tower of the university of Glasgow, to answer this question in respect to permanence or non-permanence through minutes, or hours, or days, or years, or centuries. If several gold wires are hung side by side, one of them bearing the smallest weight that will keep it approximately straight, another wire $\frac{1}{20}$ of the breaking weight, another wire $\frac{2}{20}$ of the breaking weight, and so on; the one of them bearing $\frac{19}{20}$ of the breaking weight will probably, in the course of a few hours or days, show very sensible elongation. Will it go on becoming longer and longer till it breaks, or will the time-curve of its elongation be asymptotic? Even with considerably less than $\frac{19}{20}$ of the breaking weight there will probably be a continually augmenting elongation, but with asymptotic time-curve indicating a limit beyond which the elongation never goes, but which it infinitely nearly reaches in an infinite time. It is not probable that a gold wire stretched by $\frac{1}{10}$ of its present breaking weight, or by $\frac{1}{4}$ of its present breaking weight, or even by $\frac{1}{2}$ of its present breaking weight, would break in a thousand or in a million years. The existence of gold ornaments which have been found in ancient tombs and cities, and have preserved their shapes for thousands of years without running down glacier-wise (as does brittle pitch or sealing-wax in the course of a few years in moderately warm climates), seems to prove that for gold (and therefore leaves no doubt also for many other metals) the time-curve is asymptotic, if indeed there is any slow change of shape at all after the application of a moderate stress well within the limits of elasticity. Egyptian and Greek statues, Etruscan vases, Egyptian obelisks, and other stone monuments with their engraved hieroglyphics, flint implements and boulders, and mountains with the geological evidence we have of their antiquity, prove for stones, and pottery, and rocks of various kinds, a permanence for thousands and millions of years of resistance to distorting stress.

6. The complete fulfilment of the definition of perfect elasticity is not proved by mere permanence of the extreme configurations assumed by the substance when a stated amount of the stress is alternately applied and removed. This condition might be fulfilled, and yet the amount of elastic force might be different with the same palpable configuration of the body during gradual augmentation and during gradual diminution of the stress. That it is so in fact is proved by the discovery of viscosity referred to below; but it is not yet proved that if, after increasing the stress to a certain definite amount, the body is brought to rest in the same palpable configuration as before, the amounts of stress required to hold it in this configuration are different in the two cases. If they are (section 1) the elasticity is imperfect; if they are not the elasticity is perfect within the limits of the experiment (compare section 36 below).

7. LIMITS OF ELASTICITY—*Elasticity of Shape.*—The degree of distortion within which elasticity of shape is found is essentially limited in every solid. Within sufficiently narrow limits of distortion every solid shows elasticity of shape to some degree—some solids to perfection, so far as we know at present. When the distortion is too great, the body either breaks or receives a permanent bend (that is, such a molecular disturbance that it does not return to its original figure when the bending force is removed). If the first notable dereliction from perfectness of elasticity is a breakage, the body is called brittle,—if a permanent bend, plastic or malleable or ductile. The metals are generally ductile; some metals and metallic alloys and compounds of metals with small proportions of other substances, are brittle; some of them brittle only in certain states of temper, others it seems essentially brittle. The steel of before the days of Bessemer and Siemens is a remarkable instance. When slowly cooled from a bright

red heat, it is remarkably tough and ductile When heated to redness and cooled suddenly by being plunged in oil or water or mercury, it becomes exceedingly brittle and hard (glass-hard, as it is called), and to ordinary observation seems incapable of taking a permanent bend (though probably careful observation would prove it not quite so). The definition of steel used to be *approximately pure iron capable of being tempered glass-hard, and again softened to different degrees by different degrees of heat.* Now, the excellent qualities of iron made by Bessemer's and Siemens's processes are called steel, and are reckoned best when *incapable of being tempered glass-hard*, the possibility of brittleness supervening in the course of any treatment which the metal may meet with in its manufacture being an objection against the use of what was formerly called steel for ship's plates, ribs, stringers, &c., and for many applications of land engineering, even if the material could be had in sufficient abundance.

8. LIMITS OF ELASTICITY (CONTINUED)—*Elasticity of Bulk.*—If we reckon by the amount of pressure, there is probably no limit to the elasticity of bulk in the direction of increase of pressure for any solid or fluid; but whether continued augmentation produces continued diminution of bulk towards zero without limit, or whether for any or every solid or fluid there is a limit towards which it may be reduced in bulk, but smaller than which no degree of pressure, however great, can condense it, is a question which cannot be answered in the present state of science. Would any pressure, however tremendous, give to gold a density greater than 19·6, or to copper a density greater than 9·0, after the pressure is removed (section 3 above)? But whether the body be fluid or a continuous non-porous solid, it probably recovers the same density, however tremendously it may have been pressed, and probably shows perfect elasticity of bulk (section 3 above) through the whole range of positive pressure from zero to infinity, provided the pressure has been equal in all directions like fluid pressure. As for negative pressure, we have no knowledge of what limit, if any, there may be to the amount of force which can be applied to a body pulling its surface out equally in all directions. The question of how to apply the negative pressure is inextricably involved with that of the body's power to resist. The upper part of the mercury of a barometer adhering to the glass above the level corresponding to the atmospheric pressure is a familiar example of what is called negative pressure in liquids. Water and other transparent liquids show similar phenomena, another of which is the warming of water above its boiling point in an open glass or metal vessel varnished with shellac. Attempts to produce great degrees of this so-called negative pressure are baffled by what seems an instability of the equilibrium which supervenes when the negative pressure is too much augmented. It is a very interesting subject for experimental inquiry to find how high mercury or water or any other liquid can be got to stand above the level corresponding to the atmospheric pressure in a tall hermetically sealed tube, and how many degrees a liquid can, with all precautions, be warmed above its boiling point. In each case it seems to be by a minute bubble forming and expanding somewhere at the boundary of the liquid, where it is in contact with the containing vessel, that the possible range of the negative pressure is limited, judging from what we see when we carefully examine a transparent liquid; or the surface of separation between mercury and glass, in any such experiment. The contrast of the amounts of negative pressure practically obtainable or obtained hitherto in such experiments on liquids (which are at the most those corresponding to the weight of a few metres of the substance), with that obtainable in the case of even the weakest solids, is remarkable; and as for the strongest, consider for instance (sec. 22 below) 17 nautical miles of steel pianoforte wire hanging by one end. When a cord, or rod, or wire of any solid substance hangs vertically, the negative pressure (for example, 23,000 atmospheres in the case just cited) in any transverse section is equal to the weight of the part hanging below it. It is an interesting question not to be answered by any experiment easily made or even devised,—How much would the longitudinal pull which can be applied to a cord, rod, or wire without breaking it be augmented (probably augmented, but possibly diminished) by lateral pull applied all round the sides so as to give equal negative pressure in all directions?

9. LIMITS OF ELASTICITY (CONTINUED)—*Elasticity of Shape for Distortions not Uniform through the Substance, and for Compound Distortions; and Elasticity corresponding to Co-existent Distortion and Change of Bulk*:—

Example 1.—A round wire twisted, or a cylindrical shaft transmitting revolutional motive in machinery, presents, as we shall see (sec. 64), an instance of simple distortion, but to different degrees in different parts of the substance, increasing from the axis where it is zero, uniformly to the surface where it is greatest.

Example 2.—Elongation of a wire or rod by direct pull, is (sec. 23) an instance of a compound distortion co-existing with a rarefaction of the substance, both distortion and rarefaction uniform throughout.

Example 3.—Shortening of a column by end pressure is an instance of a similar compound distortion combined with condensation of the substance, both distortion and condensation uniform throughout.

Example 4.—Flexure of a round wire or of a bar, or beam, or girder, of any shape of normal section, by opposite bending couples applied at the two ends, is an instance in which one-half of the substance is stretched, and the other half shortened with exactly the same combination of distortions and changes of bulk as in examples 2 and 3. The strain is uniform along the length of the bar, but varies in the cross section in simple proportion to distance from a certain line (sec. 62) through the centre of gravity of the sectional area, which, in the case of a round bar, is the diameter perpendicular to the plane of curvature.

The limits of elasticity in the cases of these four examples are subjects of vital importance in practical mechanics, and a vast amount of careful and accurate observation and experiment, which has given much valuable practical information regarding them, has been gone through by engineers, in their necessary dealings with questions regarding strength of materials. Still there is great want of definite scientific information on the subject of limits of elasticity generally, and particularly on many elementary questions (section 21 below), which force themselves upon us when we endeavour to analyze the molecular actions concerned in such cases as the four examples now before us. Some principles of much importance for guidance in practical as well as theoretical deductions from observations and experiments on this subject were set forth twenty-nine years ago by Professor James Thomson, in an article published in the *Cambridge and Dublin Mathematical Journal* for November 1848. Nothing is to be gained either in clearness or brevity by any other way of dealing with it than reproducing it *in extenso*. It is accordingly given here, with a few changes made in it with its author's concurrence.

It constitutes the following sections, 10–20.

"*On the strength of materials, as influenced by the existence or non-existence of certain mutual strains*[1] *among the particles composing them.*. By James Thomson, M.A., College, Glasgow.

10. "My principal object in the following paper is to show that the absolute strength of any material composed of a substance possessing ductility (and few substances, if any, are entirely devoid of this property) may vary to a great extent, according to the state of tension or relaxation in which the particles have been made to exist when the material as a whole is subject to no external strain.

11. "Let, for instance, a round bar of malleable iron, or a piece of iron wire, be made red hot, and then be allowed to cool. Its

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[1] [*Note added Nov.* 1877.] More nearly what is now called *stress* than what is now called *strain* is meant by "strain" in this article, which was written before Rankine's introduction of the word stress, and distinct definition of the word strain (see chap. I. of Mathematical Theory below).

particles may now be regarded as being all completely relaxed. Let, next, one end of the bar be fixed, and the other be made to revolve by torsion, till the particles at the circumference of the bar are strained to the utmost extent of which they can admit, without undergoing a permanent alteration in their mutual connexion.[1] In this condition, equal elements of the cross section of the bar afford resistances proportional to the distances of the elements from the centre of the bar; since the particles are displaced from their positions of relaxation through spaces which are proportional to the distances of the particles from the centre. The couple which the bar now resists, and which is equal to the sum of the couples due to the resistances of all the elements of the section, is that which is commonly assumed as the measure of the torsional strength of the bar. For future reference, this couple may be denoted by L, and the angle through which it has twisted the loose end of the bar by Θ.

12. "The twisting of the bar may, however, be carried still farther, and during the progress of this process the outer particles will yield in virtue of their ductility, those towards the interior successively reaching their elastic limits, until, when the twisting has been sufficiently continued, all the particles in the section, except those quite close to the centre, have been strained beyond their elastic limits. Hence, if we suppose[2] that no change in the hardness of the substance composing the material has resulted from the sliding of its particles past one another, and that therefore all small elements of the section of the bar afford the same resistance, no matter what their distances from the centre may be, it is easy to prove that the total torsional resistance of the bar is $\frac{4}{3}$ of what it was in the former case; or, according to the notation already adopted, it is[3] now $\frac{4}{3}$ L.

13. "If, after this, all external stress be removed from the bar, it will assume a position of equilibrium, in which the outer particles will be strained in the direction opposite to that in which it was twisted, and the inner ones in the same direction as that of the twisting, the two sets of opposite couples thus produced among the particles of the bar balancing one another. It is easy to show that the line of separation between the particles strained in one direction and those in the other is a circle whose radius is $\frac{3}{4}$ of the radius of the bar. The particles in this line are evidently subject to no strain[4] when no external couple is applied. The bar with its new molecular arrangement may now be subjected, *as often as we please*,[5] to the couple $\frac{4}{3}$ L without undergoing any farther alteration. Its strength to resist torsion, in the direction of the couple L has therefore been considerably increased. Its strength to resist torsion in the opposite direction has, however, by the same process, been much diminished; for as soon as its free extremity has been made to revolve backwards through an angle[6] of $\frac{2}{3}$ Θ from the position of equilibrium, the particles of the circumference will have suffered the utmost distortion of which they can admit without undergoing permanent alteration. Now, it is easy to prove that the couple required to produce a certain angle of torsion is the same in the new state of the bar as in the old.[7] Hence the ultimate strength of the bar when twisted backwards is represented by a couple amounting to only $\frac{2}{3}$ L. But, as we have seen, it is $\frac{4}{3}$ L when the wire is twisted forwards. That is, then, *The wire in its new state has twice as much strength to resist torsion in one direction as it has to resist it the other.*

14. "Principles quite similar to the foregoing, are applicable in regard to beams subjected to cross strain. As, however, my chief object at present is to point out the existence of such principles, to indicate the mode in which they are to be applied, and to show their great practical importance in the determination of the strength of materials, I need not enter fully into their application in the case of cross strain. The investigation in this case closely resembles that in the case of torsion, but is more complicated on account of the different ultimate resistances afforded by any material to tension and to compression, and on account of the numerous varieties in the form of section of beams which for different purposes it is found advisable to adopt. I shall therefore merely make a few remarks on this subject.

15. "If a bent bar of wrought iron or other ductile material be straightened, its particles will thus be put into such a state that its strength to resist cross strain, in the direction towards which it has been straightened, will be very much greater than its strength to resist it in the opposite direction, each of these two resistances being entirely different from that which the same bar would afford were its particles all relaxed when the entire bar is free from external strain. The actual ratios of these various resistances depend on the comparative ultimate resistances afforded by the substance to compression and extension, and also, in a very material degree, on the form of the section of the bar. I may, however, state that in general the variations in the strength of a bar to resist cross strain, which are occasioned by variations in its molecular arrangement, are much greater even than those which have already been pointed out as occurring in the strength of bars subjected to torsion.

16. "What has already been stated is quite sufficient to account for many very discordant and perplexing results which have been arrived at by different experimenters on the strength of materials. It scarcely ever occurs that a material is presented to us, either for experiment or for application to a practical use, in which the particles are free from great mutual strains. Processes have already been pointed out by which we may at pleasure produce certain peculiar strains of this kind. These, or other processes producing somewhat similar strains, are used in the manufacture of almost all materials. Thus, for instance, when malleable iron has received its final conformation by the process termed *cold swaging*, that is, by hammering it till it is cold, the outer particles exist in a state of extreme compression, and the internal ones in a state of extreme tension. The same seems to be the case in cast iron when it is taken from the mould in which it has been cast. The outer portions have cooled first, and have therefore contracted, while the inner ones still continued expanded by heat. The inner ones then contract as they subsequently cool, and thus they, as it were, pull the outer ones together. That is, in the end the outer ones are in a state of compression and the inner ones in the opposite condition.

17. "The foregoing principles may serve to explain the true

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[1] "I here assume the existence of a definite '*elastic limit*,' or a limit within which, if two particles of a substance be displaced, they will return to their original relative positions when the disturbing force is removed. The opposite conclusion, to which Mr Hodgkinson seems to have been led by some interesting experimental results, will be considered at a more advanced part of this paper."

[2] [*Note added October* 1877.] This supposition may be true for some solids; it is certainly not true for solids generally. A piece of copper or of iron taken in a soft and unstrained condition certainly becomes "harder" when strained beyond its first limits of elasticity, that is to say, its limits of elasticity become wider; and a similar result will probably be found in ductile metals generally. Thus the resistance of the outer elements will be greater than those of the inner elements in the case described in the text, until the torsion has been pushed so far as to bring about the greatest hardness in all the elements at any considerable distance from the axis. It may be that before this condition has been attained the hardening of the outer elements will have been overdone, and they may have begun to lose strength, and to have become friable and fissured. The principle set forth in the text is not, however, vitiated by the incorrectness of a supposition introduced merely for the sake of numerical illustration.

[3] "To prove this, let r be the radius of the bar, η the utmost force of a unit of area of the section to resist a strain tending to make the particles slide past one another, or to resist a shearing strain, as it is commonly called. Also, let the section of the bar be supposed to be divided into an infinite number of concentric annular elements,—the radius of any one of these being denoted by x and its area by $2\pi x dx$.

"Now, when only the particles at the circumference are strained to the utmost, and when, therefore, the forces on equal areas of the various elements are proportional to the distances of the elements from the centre, we have $\eta\frac{x}{r}$ for the force of a unit of area at the distance of x from the centre. Hence the total tangential force of the element is

$$=2\pi x dx\,.\,\eta\frac{x}{r},$$

and the couple due to the same element is

$$=x\,.\,2\pi x dx\,.\,\eta\frac{x}{r}=2\pi\eta\frac{1}{r}\,.\,x^3dx;$$

and therefore the total couple, which has been denoted above by L, is

$$=2\pi\eta\frac{1}{r}\int_0^r x^3dx,$$

that is

$$L=\tfrac{1}{2}\pi\eta r^3 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a).$$

Next, when the bar has been twisted so much that all the particles in its section afford their utmost resistance, we have the total tangential force of the element $2\pi x dx\,.\,\eta$, and the couple due to the same element

$$=x\,.\,2\pi x dx\,.\,\eta=2\pi\eta\,.\,x^2dx.$$

Hence the total couple due to the entire section is

$$=2\pi\eta\int_0^r x^2dx=\tfrac{2}{3}\pi\eta r^3.$$

But this quantity is $\frac{4}{3}$ of the value of L in formula (a). That is, the couple which the bar resists in this case is $\frac{4}{3}$ L, or $\frac{4}{3}$ of that which it resisted in the former case."

[4] "Or at least they are subject to *no strain of torsion*, either in the one direction or in the other; though they *may* be subject to a strain of compression or extension in the direction of the length of the bar." [That they are so is proved by experiments made for the present article by Mr Thomas Gray in October 1877]. "This, however, does not fall to be considered in the investigation of the text."

[5] "This statement, if not strictly, is at least extremely nearly true, since from the experiments made by Mr Fairbairn and Mr Hodgkinson on cast-iron (see various *Reports of the British Association*), we may conclude that the metals are influenced only in an extremely slight degree by time. Were the bars composed of some substance, such as sealing wax, or hard pitch, possessing a sensible amount of viscidity, the statement in the text would not hold good."

[6] [*Note added October* 1877.] This assumes that the limits of elasticity in a substance which has already been strained beyond its limits of elasticity are equal on the two sides of the shape which it has when in equilibrium without disturbing force—a supposition which may be true or may not be true. Experiment is urgently needed to test it; for its truth or falseness is a matter of much importance in the theory of elasticity.

[7] "To prove this, let the bar be supposed to be divided into an infinite number of elementary concentric tubes (like the so-called annual rings of growth in trees). To twist each of these tubes through a certain angle, the same couple will be required, whether the tube is already subject to the action of a couple of any moderate amount in either direction or not. Hence, to twist them all, or, what is the same thing, to twist the whole bar, through a certain angle, the same couple will be required whether the various elementary tubes be or be not relaxed, when the bar as a whole is free from external strain."

cause of an important fact observed by Mr Eaton Hodgkinson in his valuable researches in regard to the strength of cast iron (*Report of the British Association* for 1837, p. 362).[1] He found, that, contrary to what had been previously supposed. a strain, however small in comparison to that which would occasion rupture, was sufficient to produce a *set*. or permanent change of form, in the beams on which he experimented. Now this is just what should be expected in accordance with the principles which I have brought forward: for if, for some of the causes already pointed out, various parts of a beam previously to the application of an external force have been strained to the utmost. when, by the application of such force. however small, they are still farther displaced from their positions of relaxation. they must necessarily undergo a permanent alteration in their connexion with one another, an alteration permitted by the ductility of the material; or, in other words, the beam as a whole must take a set.

18. "In accordance with this explanation of the fact observed by Mr Hodgkinson, I do not think we are to conclude with him that 'the maxim of loading bodies within the elastic limit has no foundation in nature.' It appears to me that the defect of elasticity, which he has shown to occur even with very slight strains, exists only when the strain is applied for the first time; or, in other words, that if a beam has already been subjected to a considerable strain, it may again be subjected to any smaller strain in the same direction without its taking a set. It will readily be seen, however, from Mr Hodgkinson's experiments, that the term 'elastic limit,' as commonly employed, is entirely vague, and must tend to lead to erroneous results.

19. "The considerations adduced seem to me to show clearly that there really exist *two elastic limits* for any material, between which the displacements or deflexions, or what may in general be termed the changes of form, must be confined, if we wish to avoid giving the material a set, or, in the case of variable strains, if we wish to avoid giving it a continuous succession of sets which would gradually bring about its destruction; that these two elastic limits are usually situated one on the one side and the other on the opposite side of the position which the material assumes when subject to no external strain, though they may be both on the same side of this position of relaxation;[2] and that they may therefore with propriety be called the *superior* and the *inferior limit* of the change of form of the material for the particular arrangement which has been given to its particles; that these two limits are not *fixed* for any given material, but that, if the change of form be continued beyond either limit, two new limits will, by means of an alteration in the arrangement of the particles of the material, be given to it in place of those which it previously possessed; and lastly, that the processes employed in the manufacture of materials are usually such as to place the two limits in close contiguity with one another, thus causing the material to take in the first instance a set from any strain, however slight, while the interval which may afterwards exist between the two limits, and also, as was before stated, the actual position assumed by each of them are determined by the peculiar strains which are subsequently applied to the material.

20. "The introduction of new, though necessary, elements into the consideration of the strength of materials may, on the one hand, seem annoying from rendering the investigations more complicated. On the other hand, their introduction will really have the effect of obviating difficulties, by removing erroneous modes of viewing the subject, and preventing contradictory or incongruous results from being obtained by theory and experiment. In all investigations, in fact, in which we desire to attain or to approach nearly to truth, we must take facts as they actually are, not as we might be tempted to wish them to be for enabling us to dispense with examining processes which are somewhat concealed and intricate but are not the less influential from their hidden character."

21. Passing now to homogeneous matter (sec. 38), homogeneously strained (chap. ii. of Math. Theory below), we are met by physical questions of great interest regarding limits of elasticity. Supposing the solid to be homogeneously distorted in any particular way to nearly the limit of its elasticity for this kind of distortion. will the limits be widened or narrowed by the superposition of negative or positive pressure equal in all directions producing a dilatation or a condensation? It seems probable that a dilatation would narrow the limits of elasticity, and a condensation widen them. This, however, is a mere guess: experiment alone can answer the question. Take again a somewhat less simple case. A wire is stretched by a weight to nearly its limits of longitudinal elasticity; a couple twisting it is applied to its lower end—Will this either cause the weight to run down and give the wire a permanent set, or break it? Probably,—yes; but experiment only can decide. The corresponding question with reference to a column loaded with a weight *may* have the same answer, but not necessarily so. Experiment again is wanting. A wire hanging stretched by a light weight, merely to steady it, is twisted to nearly its limit of torsional elasticity by a couple of given magnitude applied to its lower end: the stretching weight is increased—Will this cause it to yield to the couple and take a permanent set? Probably,—yes. [Certainly *yes*, for steel piano-forte wire experimented on by Mr M'Farlane to answer this question since it was first put in type for the present article.] If so, then the limits of torsional elasticity of a wire bearing a heavy weight are widened by diminishing or taking off the weight; and no doubt it will follow continuously that a column twisted by opposing couples at its two ends will have its limits of torsional elasticity widened by the application of forces to its two ends, pressing them towards one another. Experiments to answer these questions would certainly reward the experimenter with definite and interesting results.

22. NARROWNESS OF LIMITS OF ELASTICITY—*Solids.*—The limit of elasticity of metals, stones, crystals, woods, are so narrow that the distance between any two neighbouring points of the substance never alters by more than a small proportion of its own amount without the substance either breaking or experiencing a permanent set, and therefore the angle between two lines meeting in any point of the substance and passing always through the same matter is never altered by more than a small fraction of the radian,[3] before the body either breaks or takes a permanent set. By far the widest limits of elasticity hitherto discovered by experiment, for any substance except cork, india-rubber, jellies, are those of steel pianoforte wire. Take, for example, the piano-forte wire at present in use for deep-sea soundings. It is No. 22 of the Birmingham wire gauge, its density is 7·727, it weighs 0·34 gramme per centimetre, or 6·298 kilogrammes per nautical mile of 1852·3 metres. and therefore its sectional area and diameter are ·0044 square centimetre and ·0244 centimetre. It bears a weight of 106 kilogrammes. which is equal in weight to about 31 kilometres of its length, and when this weight is alternately hung on and removed the length of the wire varies by $\frac{1}{86}$ of its amount. While this elongation takes place there is a lateral shrinking, as we shall see (section 47), of from $\frac{1}{4}$ to $\frac{3}{10}$ of the same amount.

23. Consider now in the unstrained wire two lines through the substance of the wire at right angles to one another in any plane through or parallel to the axis of the wire in directions equally inclined to this line. When the wire is pulled lengthwise the two vertical angles bisected by the length of the wire become acute, and the other two obtuse by a small difference, as illustrated in the diagram (fig. 2), where the continuous lines represent a portion of the unpulled wire, and the dotted lines the same portion of the wire when pulled. The change in each of the angles would be $\frac{1}{86}$ of the radian in virtue of the elongation were there no lateral shrinking, and about $\frac{1}{330}$ of the radian in virtue of the lateral shrinking were there no elongation. The whole change experienced by each of the right angles is therefore actually (section 37) $\frac{1}{86}+\frac{1}{330}$, or about $\frac{1}{68}$ of the radian, or 0°·84. This is an extreme case. In all other cases of metals, stones, glasses, crystals, the substance either breaks or takes a permanent bend, probably before it experiences any so great angular distortion as a degree; and except in the case of steel we may roughly regard the limits of elasticity as being something between $\frac{1}{1000}$ and $\frac{1}{100}$ in respect to the linear elongation or contraction, and from $\frac{1}{50}$ of a degree to half a degree in respect to angular distortion.

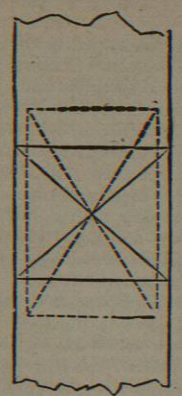

Fig. 2.

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[1] "For further information regarding Mr Hodgkinson's views and experiments see his communications in the *Transactions of the Sections of the British Association* for the years 1843 (p. 23) and 1844 (p. 25), and a work by him, entitled *Experimental Researches on the Strength and other Properties of Cast Iron*, 8vo., 1846."

[2] "Thus if the section of a beam be of some such form as that shown in either of the accompanying figures, the one rib or the two ribs, as the case may be, being very weak in comparison to the thick part of the beam, it may readily occur that the two elastic limits of deflexion may be situated both on the same side of the position assumed by the beam when free from external force. For if the beam has been supported at its extremities and loaded at its middle till the rib A B has yielded by its ductility so as to make all its particles exert their utmost tension, and if the load be now gradually removed, the particles at B may come to be compressed to the utmost before the load has been entirely removed."

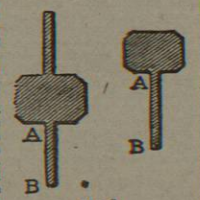

Fig. 1.

[3] The *radian* is the angle whose arc is equal to radius; it is equal to 57°.29......

24. On the other hand, gelatinous substances, such as india-rubber and elastic jellies, have very wide limits of elasticity. A vulcanized india-rubber band, for instance, is capable of being stretched, again and again, to eight times its length, and returning always to nearly its previous condition when the stress is removed. A shape of transparent jelly presents a beautiful instance of great degrees of distortion with seemingly very perfect elasticity. All these instances, india-rubber and jellies, show with great changes of shape but slight changes of bulk. They have, in fact, all, as nearly as experiment has hitherto been able to determine, the same compressibility as water.

25. Cork, another body with very wide limits of elasticity (very imperfect elasticity it is true) is singular, among bodies seemingly homogeneous to the eye, in its remarkably easy compressibility. It is, in fact, the only seemingly homogeneous solid which shows to the unaided eye any sensible change of bulk under any practically applicable forces. A small homogeneous piece torn out of a cork may, by merely pressing it between the fingers, be readily compressed to half its bulk, and a large slab of cork in a Bramah press may be compressed to $\frac{1}{10}$ of its bulk. An ordinary bottle cork loaded with a small piece of metal presents a very interesting appearance in an Oersted glass compressing vessel; first floating, and when compressed to 20 or 30 atmospheres sinking, and shrivelling in bulk very curiously; then on the pressure being removed, expanding again, but not quite to previous bulk, and floating up or remaining down according to the amount of its load.

The divergencies presented by cork and gelatinous bodies in opposite directions from the regular elasticity of hard solids form an interesting subject, to which we shall return later (section 48).

26. *Liquids.*—In respect to liquids, there are no limits of elasticity so far as regards the magnitude of the positive pressure applied or conceivably applicable; but in respect to the magnitude of negative pressure, and in respect to the magnitude of the change of bulk, whether by negative or positive pressure, there are probably very decided and not very wide limits. Thus water, though condensed $\frac{1}{13\cdot 5}$ of its bulk by 2000 atmospheres in Perkins's[1] experiments corrected roughly for the compression of his glass "piezometer," which is very nearly at the rate of $\frac{1}{21000}$ per atmosphere found (section 75 below) more accurately by subsequent experiments for moderate pressures up to 20 or 30 atmospheres, may be expected to be compressed by much less than $\frac{1}{3}$ of its volume under a pressure of 7000 atmospheres. How much it or any other liquid is condensed by a pressure of 10,000 atmospheres, or by 20,000 atmospheres, is an interesting subject for experimental investigation.

27. *Gases.*—In respect to rarefaction, and in respect to *proportionate* condensation, gases present enormously wider limits of elasticity than any liquids or solids,—in fact no limit in respect to dilatation, and in respect to condensation a definite limit only when the gas is below Andrews's "critical temperature." If the gas be kept at any temperature above that critical temperature, it remains homogeneous, however much it be condensed; and therefore for a fluid above the critical temperature there is, in respect to magnitude of pressure, no superior limit to its elasticity. On the other hand, if a fluid be kept at any constant temperature less than its critical temperature, it remains homogeneous, and presents an increasing pressure until a certain density is reached; when its bulk is further diminished it divides into two parts of less and greater density (the part of less density being called vapour, that of greater density being called liquid, if it is not solid) and presents no further increase of pressure until the vaporous part shrinks to nothing, and the whole becomes liquid (that is to say, homogeneous fluid at the greater of the two densities) or else becomes solid—the question whether the more dense part is liquid or solid depending on the particular temperature below the critical temperature at which the whole substance is kept during the supposed experiment.

28. The thermo-dynamic reasoning of Professor James Thomson, which showed the effect of change of pressure in altering the freezing point of a liquid, leads to analogous considerations regarding the effect of continuous increase or continuous decrease of pressure upon a mass consisting of the same substance partly in the liquid and partly in the solid state at one temperature. The three cases of transition from gas to liquid, from gas to solid, and from liquid to solid, present us with perfectly definite limits of elasticity,—the only perfectly definite limits of elasticity in nature of which we have any certain knowledge.

29. *Viscosity of Fluids and Solids.*—Closely connected with limits of elasticity, and with imperfectness of elasticity, is viscosity, that is to say, resistance to change of shape depending on the velocity of the change. The full discovery of the viscosity of liquids and gases is due originally to Stokes; and his hypothesis that in fluids the force of resistance is in simple proportion to the velocity of change of shape has been subsequently confirmed by the experimental investigations of Helmholtz, Maxwell, Meyer, Kundt, and Warburg. The definition of a fluid given in section 2 above may, by section 1, be transformed into the following:—A fluid is a body which requires no force to keep it in any particular shape, or—A fluid is a body which exercises no permanent resistance to a change of shape. The resistance to a change of shape presented by a fluid, evanescent as it is when the shape is not being changed (or vanishing when the velocity of the change vanishes), is essentially different from that permanent resistance to change of shape, the manifestation of which in solids constitutes elasticity of shape as defined in section 1. Maxwell's admirable kinetic theory of the viscosity of gases points to a full explanation of viscosity, whether of gases, liquids, or solids, in the consideration of configurations and arrangements of relative motions of molecules, permanent in a solid under distorting stress, and temporary in fluids or solids while the shape is being changed, in

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[1] *Transactions of Royal Society*, June 1826, "On the Progressive Compression of Water by high degrees of force, with some trials of its effects on other liquids," by J. Perkins. Communicated by W. H. Wollaston, M.D., V.P.R.S.

virtue of which elastic force in the quiescent solid, and viscous resistance to change of shape in the non-quiescent fluid or solid, are produced.

30. *Viscosity of Metals and Fatigue of their Elasticity.*—Experimental exercises performed by students in the physical laboratory of the university of Glasgow, during the session 1864–65, brought to light some very remarkable and interesting results, proving a loss of energy in elastic vibrators (sometimes as much as two or three per cent. of energy lost in the course of a single vibration in one direction) incomparably greater than anything that could be due to imperfections in their elasticity (section 1), and showing also a very remarkable fatigue of elasticity, according to which a wire which had been kept vibrating for several hours or days through a certain range came to rest much quicker when left to itself than when set in vibration after it had been at rest for several days and then immediately left to itself. Thus it was found that the rates of subsidence of the vibrations of the several wires experimented on were generally much less rapid on the Monday mornings, when they had been at rest since the previous Friday, than on other days of the week, or than after several series of experiments had been made on a Monday. The following statement (sections 31–34) is extracted from a short article by W. Thomson, in the *Proceedings of the Royal Society* for May 18, 1865, containing some of the results of these observations.

31. "*Viscosity.*—By induction from a great variety of observed phenomena, we are compelled to conclude that no change of volume or of shape can be produced in any kind of matter without dissipation of energy. Even in dealing with the *absolutely perfect* elasticity of volume presented by every fluid, and possibly by some solids, as for instance homogeneous crystals, dissipation of energy is an inevitable result of every change of volume, because of the accompanying change of temperature, and consequent dissipation of heat by conduction or radiation. The same cause gives rise necessarily to some degree of dissipation in connection with every change of shape of an elastic solid. But estimates founded on the thermodynamic theory of elastic solids, which I have given elsewhere,[1] have sufficed to prove that the loss of energy due to this cause is small in comparison with the whole loss of energy observed in many cases of vibration. I have also found, by vibrating a spring alternately in air of ordinary pressure and in the exhausted receiver of an air-pump, that there is an internal resistance to its motions immensely greater than the resistance of the air. The same conclusion is to be drawn from the observation made by Kupffer in his great work on the elasticity of metals, that his vibrating springs subsided much more rapidly in their vibrations than rigid pendulums supported on knife-edges. The subsidence of vibrations is probably more rapid in glass than in some of the most elastic metals, as copper, iron, silver, aluminium;[2] but it is much more rapid than in glass, marvellously rapid indeed, in some metals (as for instance zinc),[3] and in india-rubber and even in homogeneous jellies.

32. "The *frictional resistance* against change of shape must in every solid be infinitely small when the change of shape is made at an infinitely slow rate, since, if it were finite for an infinitely slow change of shape, there would be infinite rigidity, which we may be sure does not exist in nature. Hence there is in elastic solids a *molecular friction* which may be properly called *viscosity of solids*, because, as being an internal resistance to change of shape depending on the rapidity of the change, it must be classed with fluid molecular friction, which by general consent is called *viscosity of fluids*. But, at the same time, it ought to be remarked that the word viscosity, as used hitherto by the best writers, when solids or heterogeneous semi-solid semi-fluid masses are referred to, has not been distinctly applied to molecular friction, especially not to the molecular friction of a highly elastic solid within its limits of high elasticity, but has rather been employed to designate a property of slow continual yielding through very great, or altogether unlimited, extent of change of shape, under the action of continued stress. It is in this sense that Forbes, for instance, has used the word in stating that 'viscous theory of glacial motion,' which he demonstrated by his grand observations on glaciers. As, however, he and many other writers after him have used the words plasticity and plastic, both with reference to homogeneous solids (such as wax or pitch even though also brittle, soft metals, &c.) and to heterogeneous semi-solid semi-fluid masses (as mud, moist earth, mortar, glacial ice, &c.), to designate the property common to all those cases of experiencing, under continued stress, either quite continued and unlimited change of shape, or gradually very great change at a diminishing (asymptotic) rate through infinite time, and as the use of the term *plasticity* implies no more than does *viscosity* any physical theory or explanation of the property, the word viscosity is without inconvenience left available for the definition I propose.

33. "To investigate the viscosity of metals, I have in the first place taken them in the form of round wires, and have chosen torsional vibrations, after the manner of Coulomb, for observation, as being much the easiest way to arrive at definite results. In every case one end of the wire was attached to a rigid vibrator with sufficient firmness (thorough and smooth soldering I find to be always the best plan when the wire is thick enough); and the other to a fixed rigid body, from which the wire hangs, bearing the vibrator at its lower end. I arranged sets of observations to be made for the separate comparison of the following cases:—

(*a*) "The same wire with different vibrators of equal weights to give equal stretching-tractions but different moments of inertia (to test the relation between viscous resistances against motions with different velocities through the same range and under the same stress).

(*b*) "The same wire with different vibrators of equal moments of inertia but unequal weights (to test the effect of different longitudinal tractions on the viscous resistance to torsion under circumstances similar in all other respects).

(*c*) "The same wire and the same vibrator, but different initial ranges in successive experiments (to test an effect unexpectedly discovered, by which the subsidence of vibrations from any amplitude takes place at very different rates according to the immediately previous molecular condition, whether of quiescence or of recurring changes of shape through a wider range).

(*d*) "Two equal and similar wires, with equal and similar vibrators, one of them kept as continually as possible in a state of vibration, from day to day; the other kept at rest, except when vibrated in an experiment once a day (to test the effect of continued vibration on the viscosity of a metal).

34. "*Results.*—(*a*) It was found that the loss of energy in a single vibration through one range was greater the greater the velocity (within the limits of the experiments); but the difference between the losses at low and high speeds was *much less* than it would have been had the resistance been, as Stokes has proved it to be, in fluid friction, approximately as the rapidity of the change of shape. The irregularities in the results of the experiments which up to this time I have made seem to prove that much smaller vibrations (producing less absolute amounts of distortion in the parts of the wires most stressed) must be observed before any simple law of relation between molecular friction and velocity can be discovered.

(*b*) "When the weight was increased, the viscosity was always at first much increased; but then day after day it gradually diminished and became as small in amount as it had been with the lighter weight. It has not yet been practicable to continue the experiments long enough in any case to find the limit to this variation.

(*c*) "The vibration subsided in aluminium wires much more rapidly from amplitude 20 to amplitude 10, when the initial amplitude was 40, than when it was 20. Thus, with a certain aluminium wire, and vibrator No. 1 (time of vibration one way 1·757 second), the number of vibrations counted were in three trials—

	Vibrations.		
Subsidence from 40 initial amplitude to 20 …….	56	64	64
And from 20 (in course of the same experiments) to 10……………………	96	98	96

The same wire and the same vibrator showed—

Subsidence from 20 initial amplitude to 10 (average of four trials)…………… 112 vibrations.

Again, the same wire, with vibrator No. 2[1] (time of vibration one way 1·236), showed in two trials—

	Vibrations.	
Subsidence from 40 initial amplitude to 20…….	54	52
And continued from 20 to 10 …………………	90	90

Again, same wire and vibrator,—

From initial amplitude 20 to 10 . . 103 (mean of eight trials).

This remarkable result suggested the question (*d*).

(*d*) "In a wire which was kept vibrating nearly all day, from day to day, after several days very much more molecular friction was found than in another kept quiescent except during each experiment. Thus two equal and similar pieces of copper wire were put up about the 26th of April, hanging with equal and similar lead weights, the upper and lower ends of the two wires being similarly fixed by soldering. No. 2 was more frequently vibrated than No. 1 for a few days at first, but no comparison of viscosities was made till May 15. Then

No. 1 subsided from 20 initial range to 10 in 97 vibrations.

No. 2 gave the same subsidence in 77 vibrations.

During the greater part of May 16 and 17, No. 2 was kept vibrating and No. 1 quiescent, and late on May 17 experiments with the following results were made:—

			Time per Vibration.
No. 1 subsided from 20 to 10 after	99 vibrations in	237 secs.,	2·4
„ „ „	98 „	235 „	2·4
„ „ „	98 „	235 „	2·4
No. 2 subsided from 20 to 10 after	58 vibrations in	142 „	2·45
„ „ „	60 „	147 „	2·45
„ „ „	57 „	139 „	2·45
„ „ „	60 „	147 „	2·45"

[Addition, May 27, after the reading of the paper.]—No. 1 has been kept at rest from May 17, while No. 2 has been kept oscillating more or less every day till yesterday, May 26, when both were oscillated, with the following results:—

			Time per Vibration.
No. 1 subsided from 20 to 10 after	100 vibrations in	242 secs.,	2·42
No. 2 „ „	44 or 45 vibrations………		2·495

35. The investigation was continued with much smaller degrees of maximum angular distortion, to discover, if possible, the law of the molecular friction, the existence of which was demonstrated by these experiments. Two questions immediately occurred:—What is the law of subsidence of range in any single series of oscillations, the vibrator being undisturbed by external force? and (question (*a*) of § 33 above) what is the relation between the law of subsidence in two sets of oscillations having different periods, with the same elastic body in the same circumstances of elastic force, as for instance the same or similar metallic wires with equal weights hung upon them, performing torsional oscillations in different times on account of the moments of inertia of the suspended masses being different?

36. So far as the irregularities depending on previous conditions of the elastic substance allowed any simple law to be indicated, the experimental answer to the first question for degrees of angular distortion much smaller than the palpable limits of elasticity was the COMPOUND INTEREST LAW, that is to say,—*The diminutions of range per equal intervals of time or per equal numbers of oscillations bore a constant proportion to the diminishing range;* or, *The differences of the logarithms of the ranges were proportional to the intervals of time.*

The only approach to an answer to the second question yet obtained is that the proportionate losses of amplitude in the different cases are not such as they would be if the molecular resistance were simply proportional to the velocity of change of shape in the different cases. If the molecular friction followed this simple law, the proportionate diminutions of range per period would be inversely as the periods, or per equal intervals of time they would be inversely as the squares of the periods. Instead of the proportion being so, the loss was greater with the longer periods than that calculated according to the law of square roots from its amount in the shorter periods. It was in fact as it would be if the result were wholly or partially due to imperfect elasticity, or "elastische Nach-wirkung"—elastic after-working—as the Germans call it (compare section 6 above). To form a rough idea of the results, irrespectively of the ultimate molecular theory (which is to be looked for in the proper extension of Maxwell's kinetic theory of viscosity of gases), consider a perfectly elastic vesicular solid, whether like a sponge with communications between the vesicles, or with each vesicle separately inclosed in elastic solid: imagine its pores and interstices filled up with a viscous fluid, such as oil. Static experiments on such a solid will show perfect elasticity of bulk and shape; kinetic experiments will show losses of energy such as are really shown by vibrators of india-rubber, jelly, glass, metals, or other elastic homogeneous solids, but more regular, and following more closely the compound interest law for single series and the law of relation to square roots of periods stated above for sets of oscillations in different periods. In short, according to Stokes's law of viscosity of fluids, our supposed vesicular vibrator would follow the law of subsidence of a simple vibrator experiencing a resistance simply proportional to the velocity of its motion, while no such simple law is applicable to the effects of the internal molecular resistance in a vibrating elastic solid.

37. *Hooke's Law.*—A law expressed by Hooke with Latin terseness in the words *Ut tensio sic vis* is the foundation of the mathematical theory of the elasticity of hard solids. By *tensio* here is meant not force (as is generally meant by the English word tension), but an elongation produced by force. In English, then, Hooke's law is that elongation (understood of an elastic solid) is proportional to the force producing it. It is, of course, to be extended continuously from elongation to contraction in respect to the effect, and from pull to push in respect

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[1] "On the Thermo-elastic Properties of Solids," *Quarterly Journal of Mathematics*, April, 1855.

[2] We have no evidence that the precious metals are more elastic than copper, iron, or brass. One of the new bronze pennies gives quite as clear a ring as a two-shilling silver piece tested in the usual manner.

[3] Torsional vibrations of a weight hung on a zinc wire subside so rapidly, that it has been found scarcely possible to count more than twenty of them in one case experimented on.

[4] Those who believe in the existence of indivisible, infinitely strong and infinitely rigid, very small bodies (finite hard atoms!) deny this.

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[1] Of same weight as No. 1, but different moment of inertia.