to the cause; and the experiments on which it is founded prove a perfect continuity from a pulling force to a smaller force in the same direction, and from the less force to zero, and from zero of pulling force to different degrees of push or positive pressure, or negative pull. Experimental proof merely of the *continuity of the phenomena through zero of force* suffices to show that, for *infinitely small* positive or negative pulls, positive or negative elongation is simply proportional to the positive or negative pull; or, in other words, positive or negative contraction is proportional to the positive or negative pressure producing it. But now must be invoked minutely accurate experimental measurement to find how nearly the law of simple proportionality holds through finite ranges of contraction and elongation. The answer happily for mathematicians and engineers is *that Hooke's law is fulfilled, as accurately as any experiments hitherto made can tell*, for all metals and hard solids each through the whole range within its limits of elasticity; and for woods, cork, india-rubber, jellies, when the elongation is not more than two or three per cent., or the angular distortion not more than a few hundredths of the radian (or not more than about two or three degrees). The same law holds for the condensation of liquids up to the highest pressures under which their compressibility has hitherto been accurately measured. [A decided but small deviation from Hooke's law has been found in steel pianoforte wire under combined influence of torsion and longitudinal pull by Mr M'Farlane in experiments made for the present article after this section was in type. See section 81.]

Boyle's law of the "spring of air" shows that the augmentation of density of a gas is simply proportional to the augmentation of the pressure, through the very wide ranges of pressure through which that law is approximately enough fulfilled. Hence the infinitesimal diminution of volume produced by a given infinitesimal augmentation of pressure varies as the square of the volume, and the *proportionate* diminution of volume (that is to say, the ratio of the diminution of volume to the volume) is proportional to the volume, or inversely proportional to the density. Andrews's experiments on the compressibility of a fluid, such as carbonic acid, at temperatures slightly above the critical temperature, and of the gas and of liquids at temperatures slightly below the critical temperature, are intensely interesting, not merely in respect to the *natural history* of elasticity, but as opening vistas into the philosophy of molecular action.

We cannot expect to find any law of simple proportionality between stress and change of dimensions, or proportionate change of dimensions, in the case of any elastic or semi-elastic "soft" solids, such as cork on one hand or india-rubber or jellies on the other, when strained to large angular distortions, or to large proportionate changes of dimensions. The exceedingly imperfect elasticity of all these solids, and the want of definiteness of the substance of many of them, renders accurate experimenting unavailable for obtaining any very definite or consistent numerical results; but it is interesting to observe roughly the forces required to produce some of the great strains of which they are capable without any total break down of elastic quality; for instance, to hang weights successively on an india-rubber band and measure the elongations. This any one may readily do, and may be surprised to find the enormous increase of resistance to elongation presented by the attenuated band before it breaks.

38. *Homogeneousness defined.*—A body is called homogeneous when any two equal, similar parts of it with corresponding lines parallel and turned towards the same parts, are undistinguishable from one another by any difference in quality. The perfect fulfilment of this condition, without any limit as to the smallness of the parts, though conceivable, is not generally regarded as probable, for any of the real solids or fluids known to us, however seemingly homogeneous. It is held by all naturalists that there is a *molecular structure*, according to which, in *compound* bodies such as water, ice, rock-crystal, &c., the constituent substances lie side by side, or arranged in groups of finite dimensions, and even in bodies called *simple* (that is those not known to be chemically resolvable into other substances) there is no ultimate homogeneousness. In other words, the prevailing belief is that every kind of matter with which we are acquainted has a more or less *coarse-grained* texture, whether (as great masses of solid brick-work or stone-building, or as natural sandstone or granite rocks) having visible molecules, or (as seemingly homogeneous metals, or continuous crystals, or liquids, or gases) having molecules too small to be directly visible, or measurable but not *undiscoverably* small,—really, it is to be believed, of dimensions to be accurately determined in future advances of science. Practically the definition of *homogeneousness* may be applied on a very large scale to masses of building or coarse-grained conglomerate rock, or on a more moderate scale to blocks of common sandstone, or on a very small scale to seemingly homogeneous metals;[1] or on a scale of extreme, undiscovered fineness, to vitreous bodies, continuous crystals, solidified gums, as india-rubber, gum-arabic, &c., and fluids.

39. *Isotropic and Æolotropic Substances defined.*—The substance of a homogeneous solid is called *isotropic* when a spherical portion of it, tested by any physical agency, exhibits no difference in quality however it is turned. Or, which amounts to the same, a cubical portion cut from any position in an isotropic body, exhibits the same qualities relatively to each pair of parallel faces. Or two equal and similar portions cut from *any* positions in the body, not subject to the condition of parallelism (section 38), are undistinguishable from one another. A substance which is not isotropic, but exhibits differences of quality in different directions, is called *æolotropic*.[2] The remarks of section 38 relative to homogeneousness in the aggregate, and the supposed ultimately heterogeneous texture of all substances, however seemingly homogeneous, indicate corresponding limitations and non-rigorous practical interpretations of isotropy and æolotropy.

40. *Isotropy and Æolotropy of different sets of properties.*—The substance of a homogeneous solid may be isotropic in one quality or class of qualities, but æolotropic in others. Or a transparent substance may transmit light at different velocities in different directions through it (that is, be *doubly-refracting*), and yet a cube of it may (and does in many natural crystals) show no sensible difference in its absorption of white light transmitted across it perpendicularly to any of its three pairs of faces. Or (as a crystal which exhibits *dichroism*) it may be sensibly æolotropic relatively to the absorption of light, but not sensibly double-refracting, or it may be dichroic and doubly-refracting, and yet it may conduct heat equally in all directions. Still, as a rule, a homogeneous substance which is æolotropic for one quality must be more than infinitesimally æolotropic for every quality which has directional character admitting of a corresponding æolotropy.

41. *Moduluses of Elasticity.*—A modulus of elasticity is the number obtained by dividing the number expressing a stress[3] by the number expressing the strain[4] which it produces. A modulus is called a principal modulus when the stress is such that it produces a strain of its own type.

(1.) An isotropic solid has two principal moduluses—a *modulus of compression* and a *rigidity.*

(2.) A crystal of the cubic class (fluor-spar, for instance) has three principal moduluses,—*one modulus of compression* and *two rigidities.*

(3.) An æolotropic solid having (what *no natural crystal* has, but what a *drawn wire* has) perfect isotropy of physical qualities relative to all lines perpendicular to a certain axis of its substance has three principal moduluses,—*two* determinable from its different compressibilities along and perpendicular to the axis, or from one compressibility and the "Young's modulus" (section 42) of an axial bar of the substance, or determinable from two compressibilities; and *one rigidity* determinable by measurement of the torsional rigidity of a round axial bar of the substance.

(4.) A crystal of Iceland spar has four principal moduluses,—three like those of case (3), and another rigidity depending on (want of complete circular symmetry, and) possession of *triple* symmetry of form, involving sextuple elastic symmetry, round the crystalline axis.

(5.) A crystal of the rectangular parallelepiped (or "tessaral") class has six distinct principal moduluses which, when the directions of the principal axes are known, are determinable by six single observations,—three, of the three (generally unequal) compressibilities along the three axes; and three, of the three rigidities (no doubt generally unequal) relatively to the three simple distortions of the parallelepiped, in any one of which one pair of parallel rectangular faces of the parallelepiped become oblique parallelograms.

(6.) An æolotropic solid generally has six principal moduluses,[1] which, when a piece of the solid is presented without information, and without any sure indication from its appearance of any particular axis or axes of symmetry of any kind, require just twenty-one independent observations for the determination of the fifteen quantities specifying their types, and the six numerical values of the moduluses themselves.

42. "*Young's Modulus,*" or *Modulus of Simple Longitudinal Stress.*—Thomas Young called *the modulus of elasticity* of an elastic solid the amount of the end-pull or end-thrust required to produce any infinitesimal elongation or contraction of a wire, or bar, or column of the substance multiplied by the ratio of its length to the elongation or contraction. In this definition the definite article is clearly misapplied. There are, as we have seen, two moduluses of elasticity for an isotropic solid,—one measuring elasticity of bulk, the other measuring elasticity of shape. An interesting and instructive illustration of the confusion of ideas so often rising in physical science from faulty logic is to be found in "An Account of an Experiment on the Elasticity of Ice: By Benjamin Bevan, Esq., in a letter to Dr Thomas Young, Foreign Sec. R. S." and in Young's "Note" upon it, both published in the *Transactions of the Royal Society* for 1826. Bevan gives an interesting account of a well-designed and well-executed experiment on the flexure of a bar, 3·97 inches thick, 10 inches broad, and 100 inches long, of ice on a pond near Leighton Buzzard (the bar remaining attached by one end to the rest of the ice, but being cut free by a saw along its sides and across its other end), by which he obtained a fairly accurate determination of "the modulus of ice";[2] and says that he repeated the experiment in various ways on ice bars of various dimensions, some remaining attached by one end, others completely detached, and found results agreeing with the first as nearly "as the admeasurement of the thickness could be ascertained." He then proceeds to compare "the modulus of ice" which he had thus found with "the modulus of water," which he quotes from Young's *Lectures* as deduced from Canton's experiments on the compressibility of water. Young in his "Note" does *not* point out that the two moduluses were essentially different, and that *the modulus of his definition*, the modulus determinable from the flexure of a bar, is essentially zero for every fluid. We now call "Young's modulus" the particular modulus of elasticity defined as above by Young, and so avoid all confusion.

43. *Modulus of Rigidity.*—The "modulus of rigidity" of an isotropic solid is the amount of tangential stress divided by the deformation it produces,—the former being measured in units of force per unit of the area to which it is applied in the manner indicated by the annexed diagram (fig. 3), and the latter by the variation of each of the four right angles reckoned in fraction of the radian. By drawing either diagonal of the square in the diagram we see that the distorting stress represented by it gives rise to a normal traction on every surface of the substance perpendicular to the square and parallel to one of its diagonals, and an equal normal *pressure* on every surface of the solid perpendicular to the square and parallel to the other diagonal; and that the amount of each of these normal forces[3] per unit of area is equal to the amount per unit area of the tangential forces which the diagram indicates. The corresponding[4] geometrical proposition, also easily proved, is as follows: A strain compounded of a simple *extension* in one set of parallels, and a simple contraction of equal amount in any other set perpendicular to those, is the same as a simple shear in either of the two sets of planes cutting the two sets of parallels at 45°, and the numerical measuring of this shear or simple distortion is equal to *double* the amount of the elongation or contraction, each reckoned per unit of length.

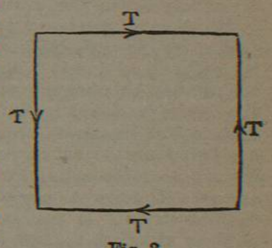

Fig. 3.

Hence we have another definition of "modulus of rigidity" equivalent to the preceding:—The modulus of rigidity of an isotropic substance is the amount of normal traction or pressure per unit of area, divided by *twice* the amount of elongation in the direction of the traction or of contraction in the direction of the pressure, when a piece of the substance is subjected to a stress producing uniform distortion.

44.[5] *Conditions fulfilled in Elastic Isotropy.*—To be elastically isotropic, a spherical or cubical portion of any solid, if subjected to uniform normal pressure (positive or negative) all round, must, in yielding, experience no deformation, and therefore must be equally compressed (or dilated) in all directions. But, further, a cube cut from any position in it, and acted on by *tangential* or distorting stress in planes parallel to two pairs of its sides, must experience simple deformation, or "shearing" parallel to either pair of these sides, unaccompanied by condensation or dilatation,[6]

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[1] Which, however, we know, as proved by Deville and Van Troost, are porous enough at high temperatures to allow very free percolation of gases. Helmholtz and Root find percolation of platinum by hydrogen at ordinary temperature (*Berl. Sitzungsbericht*).

[2] Thomson and Tait's *Natural Philosophy*, section 676.

[3] Mathematical Theory, below chap. i.

[4] *Ibid.*

[1] Mathematical Theory, chap. xvi.

[2] The result is given in the Table of Moduluses, sec. 77, below

[3] The directions of these forces are called the "axes" of the stress. The corresponding directions in the corresponding strain are called the axes of the strain.

[4] Mathematical Theory, chap. vi.

[5] This, with several of the following sections, 44–51, is borrowed, with but slight change, from the first edition of Thomson and Tait's *Natural Philosophy*, by permission of the authors.

[6] It must be remembered that the changes of figure and volume we are concerned with are so small that the principle of superposition is

and the same in amount for all the three ways in which a stress may be thus applied to any one cube, and for different cubes taken from any different positions in the solid. Hence the elastic quality of a perfectly elastic, homogeneous, isotropic solid is fully defined by two elements,—its resistance to distortion and its resistance to compression. The first has been already considered (section 43). The second is measured by the amount of uniform pressure in all directions per unit area of its surface required to produce a stated very small compression. The numerical reckoning of the first is the compressing pressure divided by the diminution of the bulk of a portion of the substance which, when uncompressed, occupies the unit volume. It is sometimes called the "*elasticity of bulk*," or sometimes the "*modulus of bulk-elasticity*," sometimes the *resistance to compression*. Its reciprocal, or the amount of compression on unit of volume divided by the compressing pressure, or, as we may conveniently say, the compression per unit of volume per unit of compressing pressure, is commonly called the *compressibility*.

45. *Strain produced by a single Longitudinal Stress (subject of Young's Modulus).*—Any stress whatever may[1] be made up of simple longitudinal stresses. Hence, to find the relation between any stress and the strain produced by it, we have only to find the strain produced by a single longitudinal stress, which, for an isotropic solid, we may do at once thus:—A simple longitudinal stress P is equivalent to a uniform dilating tension $\frac{1}{3}P$ in all directions, compounded with two distorting stresses, each equal to $\frac{1}{3}P$, and having a common axis in the line of the given longitudinal stress, and their other two axes any two lines at right angles to one another and to it. The diagram (fig. 4), drawn in a plane through one of these latter lines and the former, sufficiently indicates the synthesis,—the only forces not shown being those perpendicular to its plane.

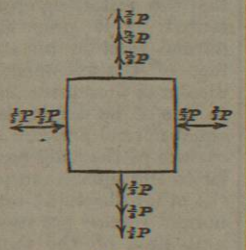

Fig. 4.

Hence if n denote the *rigidity*, and k the *modulus of compression*, or the *modulus of bulk-elasticity* (being the same as the reciprocal of the compressibility), the effect will be an equal dilatation in all directions, amounting, per unit of volume, to

$$\frac{\frac{1}{3}P}{k} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

compounded with two equal distortions, each amounting to

$$\frac{\frac{1}{3}P}{n} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2),$$

and having (section 43, footnote) their axes in the directions just stated for the axes of the distorting stresses.

46. The dilatation and two shears thus determined may be conveniently reduced to simple longitudinal strains by following the indications of section 43, thus:—

The two shears together constitute an elongation amounting to $\frac{\frac{1}{3}P}{n}$ in the direction of the given force P, and equal contraction amounting to $\frac{\frac{1}{6}P}{n}$ in all directions perpendicular to it. And the cubic dilatation $\frac{\frac{1}{3}P}{k}$ implies a lineal dilatation, equal in all directions, amounting to $\frac{1}{9}\frac{P}{k}$.

On the whole, therefore, we have

$$\left.\begin{array}{l}\text{linear elongation} = P\left(\frac{1}{3n}+\frac{1}{9k}\right),\ \text{in the direction of the applied stress, and}\\ \text{linear contraction} = P\left(\frac{1}{6n}-\frac{1}{9k}\right),\ \text{in all directions perpendicular to the applied stress.}\end{array}\right\}\ (3)$$

47. Hence "Young's Modulus" $=\frac{9nk}{3k+n}$, and when the ends of a column, bar, or wire of isotropic material are acted on by equal and opposite forces, it experiences a lateral lineal contraction equal to $\frac{3k-2n}{2(3k+n)}$ of the longitudinal dilatation, each reckoned as usual per unit of lineal measure. One specimen of the fallacious mathematics referred to in chap. xvi. of the mathematical theory below is a celebrated conclusion of Navier's and Poisson's that the ratio of lateral contraction to elongation by pull without transverse force is $\frac{1}{4}$. This would require the rigidity to be $\frac{3}{5}$ of the resistance to compression, for all solids; which was first shown to be false by Stokes[2] from many obvious observations, proving enormous discrepancies from it in many well-known bodies, and rendering it most improbable that there is any approach to a constancy of ratio between rigidity and resistance to compression in any class of solids. Thus clear elastic jellies and india-rubber present familiar specimens of isotropic homogeneous solids which, while differing very much from one another in rigidity ("stiffness"), are probably all of very nearly the same compressibility as water, which is about $\frac{1}{21000}$ per atmosphere. Their resistance to compression, measured by the reciprocal of this, is obviously many hundred times the absolute amount of the rigidity of the stiffest of those substances. A column of any of them, therefore, when pressed together or pulled out, within its limits of elasticity, by balancing forces applied to its ends (or an india-rubber band when pulled out), experiences no sensible change of volume, though very sensible change of length. Hence the proportionate extension or contraction of any transverse diameter must be sensibly equal to half the longitudinal contraction or extension; and such substances may be practically regarded as incompressible elastic solids in interpreting all the phenomena for which they are most remarkable. Stokes gave reasons for believing that metals also have in general greater resistance to compression, in proportion to their rigidities, than according to the fallacious theory, although for them the discrepancy is very much less than for the gelatinous bodies. This probable conclusion was soon experimentally demonstrated by Wertheim, who found the ratio of lateral to longitudinal change of lineal dimensions, in columns acted on solely by longitudinal force, to be about $\frac{1}{3}$ for glass and brass; and by Kirchhoff, who, by a well-devised experimental method, found ·387 as the value of that ratio for brass, and ·294 for iron. For copper it is shown to lie between ·226 and ·441, by experiments[3] quoted below, measuring the torsional and longitudinal rigidities of copper wires.

48. All these results indicate rigidity *less* in proportion to the compressibility than according to Navier's and Poisson's theory. And it has been supposed by many naturalists who have seen the necessity of abandoning that theory as inapplicable to ordinary solids that it may be regarded as the proper theory for an ideal *perfect solid*, and as indicating an amount of rigidity not quite reached in any real substance, but approached to in some of the

applicable; so that if any distorting stress produced a condensation, an opposite distorting stress would produce a dilatation, which is a violation of the isotropic condition.

[1] Mathematical Theory, chap. viii.

[2] "On the Friction of Fluids in Motion, and the Equilibrium and Motion of Elastic Solids," *Trans. Camb. Phil. Soc*, April 1845. See also *Camb. and Dub. Math. Jour.*, March 1848.

[3] "On the Elasticity and Viscosity of Metals" (W. Thomson), *Proc. R. S.*, May 1865.

most rigid of natural solids (as, for instance, iron). But it is scarcely possible to hold a piece of cork in the hand without perceiving the fallaciousness of this last attempt to maintain a theory which never had any good foundation. By careful measurements on columns of cork of various forms (among them, cylindrical pieces cut in the ordinary way for bottles), before and after compressing them longitudinally in a Bramah's press, we have found that the change of lateral dimensions is insensible both with small longitudinal contractions and return dilatations, within the limits of elasticity, and with such enormous longitudinal contractions as to $\frac{1}{5}$ or $\frac{1}{3}$ of the original length. It is thus proved decisively that cork is much more rigid, while metals, glass, and gelatinous bodies are all less rigid, in proportion to resistance to compression, than the supposed "perfect solid"; and the practical invalidity of the theory is experimentally demonstrated. By obvious mechanism of jointed bars a solid may be designed which shall swell laterally when pulled, and shrink laterally when compressed, in one direction, and which shall be homogeneous in the same sense (article 40) as crystals and liquids are called homogeneous.

49. *Modulus of Simple Longitudinal Strain.*—In sections 45, 46, we examined the effect of a simple longitudinal stress in producing elongation in its own direction, and contraction in lines perpendicular to it. With stresses substituted for strains, and strains for stresses, we may apply the same process to investigate the longitudinal and lateral tractions required to produce a simple longitudinal strain (that is, an elongation in one direction, with no change of dimensions perpendicular to it) in a rod or solid of any shape.

Thus a simple longitudinal strain e is equivalent to a cubic dilatation e without change of figure (or linear dilatation $\frac{1}{3}e$ equal in all directions), and two distortions consisting each of dilatation $\frac{1}{3}e$ in the given direction and contraction $\frac{1}{3}e$ in each of two directions perpendicular to it and to one another. To produce the cubic dilatation e alone requires (section 44) a normal traction ke equal in all directions. And, to produce either of the distortions simply, since the measure (section 43) of each is $\frac{2}{3}e$, requires a distorting stress equal to $n\times\frac{2}{3}e$, which consists of tangential tractions each equal to this amount, positive (or drawing outwards) in the line of the given elongation, and negative (or pressing inwards) in the perpendicular direction. Thus we have in all

$$\left.\begin{array}{l}\text{normal traction} = (k+\frac{4}{3}n)e,\ \text{in the direction of the given strain, and}\\ \text{normal traction} = (k-\frac{2}{3}n)e,\ \text{in every direction perpendicular to the given strain.}\end{array}\right\}\ (4).$$

Hence the modulus of simple longitudinal strain is $k+\frac{4}{3}n$.

50. *Weight-Modulus and Length of Modulus.*—Instead of reckoning moduluses in units of force per unit of area, it is sometimes convenient to express them in terms of the weight of unit bulk of the solid. A modulus thus reckoned, or, as it is called by some writers, the length of the modulus, is of course found by dividing the weight-modulus by the weight of the unit bulk. It is useful in many applications of the theory of elasticity, as, for instance, in this result, which is proved in the elementary dynamics of waves in an elastic solid or fluid (chap. xvii. of the Mathematical Theory, below):—the velocity of transmission of longitudinal[1] vibrations (as of sound) along a bar of cord, or of waves of simple distortion, or of simple longitudinal extension and contraction in a homogeneous isotropic solid, or of sound waves in a fluid, is equal to the velocity acquired by a body in falling from a height equal to half the length of the proper modulus[2] for the case;—that is, the Young's Modulus $\left(\frac{9kn}{3k+n}\right)$ for the first case, the modulus of rigidity (n) for the second, the modulus of simple longitudinal strain ($k+\frac{4}{3}n$) for the third, the modulus of compression k for the fourth. Remark that for air the static "length-modulus of compression" at constant temperature is the same as what is often technically called the "height of the homogeneous atmosphere."

51. In reckoning moduluses there must be a definite understanding as to the unit in terms of which the force is measured, which may be either the *kinetic unit* or the gravitation unit for a specified locality, that is, the weight in that locality of the unit of mass. Experimenters hitherto have stated their results in terms in the gravitation unit, each for his own locality,—the accuracy hitherto attained being scarcely in any cases sufficient to require corrections for the different intensities of gravity in the different places of observation.

The most useful and generally convenient specification of the modulus of elasticity of a substance is in grammes-weight per square centimetre. This has only to be divided by the specific gravity of the substance to give the *length of the modulus*. British measures, however, being still unhappily sometimes used in practical and even in scientific statements, we too often meet with reckonings of the modulus in pounds per square inch or per square foot, in tons per square inch, or of length of the modulus in feet or in British statute miles.

The reckoning most commonly adopted in British treatises on mechanics and practical statements is pounds per square inch. The modulus thus stated must be divided by the weight of 12 cubic inches of the solid, or by the product of its specific gravity into ·4335,[3] to find the length of the modulus in feet.

To reduce from pounds per square inch to grammes per square centimetre, multiply by 70·31, or divide by ·014223. French engineers generally state their results in kilogrammes per square millimetre, and so bring them to more convenient numbers, being $\frac{1}{100000}$ of the incon-

[1] It is to be understood that the vibrations in question are so much spread out through the *length* of the body that inertia does not sensibly influence the transverse contractions and dilatations which (unless the substance have in this respect the peculiar character presented by cork, section 48) take place along with them.

[2] In sections 73–76 we shall see that changes of temperature produced by the varying stresses cause changes of temperature which, in ordinary solids, render the velocity of transmission of longitudinal vibrations sensibly greater than that calculated by the rule stated in the text, if we use the *static modulus* as understood from the definition there given; and it will be shown how to take into account the thermal effect by using a definite *static modulus*, or *kinetic modulus*, according to the circumstances of any case that may occur.

[3] This decimal being the weight in pounds of 12 cubic inches of water. The one great advantage of the French metrical system is that the mass of the unit volume (1 cubic centimetre) of water at its temperature of maximum density (3°·945 C.) is unity (1 gramme) to a sufficient degree of approximation for almost all practical purposes. Thus, according to this system, the density of a body and its specific gravity mean one and the same thing; whereas on the British no-system the density is expressed by a number found by multiplying the specific gravity by one number or another, according to the choice of a cubic inch, pint, quart, wine gallon, imperial gallon, cubic foot, cubic yard, or cubic mile that is made for the unit of volume; and the grain, scruple, gunmaker's drachm, apothecary's drachm, ounce Troy, ounce avoirdupois, pound Troy, pound avoirdupois, stone (Imperial, Ayrshire, Lanarkshire, Dumbartonshire), stone for hay, stone for corn, quarter (of a hundredweight), quarter (of corn), hundredweight, or ton that is chosen for unit of mass. It is a remarkable phenomenon, belonging rather to moral and social than to physical science, that a people tending naturally to be regulated by common sense should voluntarily condemn themselves, as the British have so long done, to unnecessary hard labour in every action of common business or scientific work related to measurement, from which all the other nations of Europe have emancipated themselves. Professor W. H. Miller, of Cambridge, concludes, from a very trustworthy comparison of standards by Kupffer, of St Petersburg, that the weight of a cubic decimetre of water at temperature of maximum density is 1000·013 grammes.

veniently large numbers expressing moduluses in grammes weight per square centimetre, but it is much better to reckon in millions of grammes per square centimetre.

52. "Resilience" is a very useful word, introduced about forty years ago (when the *doctrine of energy* was beginning to become practically appreciated) by Lewis Gordon, first professor of engineering in the university of Glasgow, to denote the quantity of work that a spring (or elastic body) gives back when strained to some stated limit and then allowed to return to the condition in which it rests when free from stress. The word "resilience" used without special qualification may be understood as meaning *extreme resilience*, or the work given back by the spring after being strained to the extreme limit within which it can be strained again and again without breaking or taking a permanent set. In all cases for which Hooke's law of simple proportionality between stress and strain holds, the resilience is obviously equal to the work done by a constant force of half the amount of the extreme force acting through a space equal to the extreme deflection.

53. When force is reckoned in "gravitation measure," resilience per unit of the spring's mass is simply the height that the spring itself, or an equal weight, could be lifted against gravity by an amount of work equal to that given back by the spring returning from the stressed condition.

54. Let the elastic body be a long homogeneous cylinder or prism with flat ends (a bar as we may call it for brevity), and let the stress for which its resilience is reckoned be *positive* normal pressures on its ends. The resilience per unit mass is equal to the greatest height from which the bar can fall with its length vertical, and impinge against a perfectly hard horizontal plane without suffering stress beyond its limits of elasticity. For in this case (as in the case of the direct impact of two equal and similar bars meeting with equal and opposite velocities, discussed in Thomson and Tait's *Natural Philosophy*, section 303), the kinetic energy of the translational motion preceding the impact is, during the first half of the collision, wholly converted into potential energy of elastic force, which during the second half of the collision is wholly reconverted into kinetic energy of translational motion in the reverse direction. During the whole time of the collision the stopped end of the bar experiences a constant pressure, and at the middle of the collision the whole substance of the bar is for an instant at rest in the same state of compression as it would have permanently if in equilibrium under the influence of that pressure and an equal and opposite pressure on the other end. From the beginning to the middle of the collision the compression advances at a uniform rate through the bar from the stopped end to the free end. Every particle of the bar which the compression has not reached continues moving uniformly with the velocity of the whole before the collision until the compression reaches it, when it instantaneously comes to rest. The part of the bar which at any instant is all that is compressed remains at rest till the corresponding instant in the second half of the collision.

55. From our preceding view of a bar impinging against an ideal perfectly rigid plane, we see at once all that takes place in the real case of any rigorously direct longitudinal collision between two equal and similar elastic bars with flat ends. In this case the whole of the kinetic energy which the bodies had before collision reappears as purely translational kinetic energy after collision. The same would be approximately true of any two bars, provided the times taken by a pulse of simple longitudinal stress to run through their lengths are equal. Thus if the two bars be of the same substance, or of different substances having the same value for Young's modulus, the lengths must be equal, but the diameters may be unequal. Or if the Young's modulus be different in the two bars, their lengths must (Math. Theory, chap. xvii.) be inversely as the square roots of its values. To all such cases the laws of "collision between two perfectly elastic bodies," whether of equal or unequal masses, as given in elementary dynamical treatises, are applicable. But in every other case part of the translational energy which the bodies have before collision is left in the shape of vibrations after collision, and the translational energy after collision is accordingly less than before collision. The losses of energy observed in common elementary dynamical experiments on collision between solid globes of the same substance are partly due to this cause. If they were wholly due to it they would be independent of the substance, when two globes of the same substance are used. They would bear the same proportion to the whole energy in every case of collision between two equal globes, or again, in every case of collision between two globes of any stated proportion of diameters, provided in each case the two which collide are of the same substances; but the proportion of translational energy converted into vibrations would not be the same for two equal globes as for two unequal globes. Hence when differences of proportionate losses of energy are found in experiments on different substances, as in Newton's on globes of glass, iron, or compressed wool, this must be due to imperfect elasticity of the material. It is to be expected that careful experiments upon hard well-polished globes striking one another with such gentle forces as not to produce even at the point of contact any stress approaching to the limit of elasticity, will be found to give results in which the observed loss of translational energy can be almost wholly accounted for by vibrations remaining in the globes after collision.

56. *Examples of Resilience.—Example* 1.—In respect to simple longitudinal pull, the extreme resilience of steel pianoforte wire of the gauge and quality referred to in section 22 above (calculated by multiplying the breaking weight into half the elongation produced by it according to the experimental data of section 22) is 6066 metre-grammes (gravitation measure) per ten metres of the wire. Or, whatever the length of the wire, its resilience is equal to the work required to lift its weight through 179 metres.

Example 2.—The torsional resilience of the same wire, twisted in either direction as far as it can be without giving it any notable permanent set, was found to be equal to the work required to lift its weight through 1·3 metres.

Example 3.—The extreme resilience of a vulcanized india-rubber band weighing 12·3 grammes was found to be equal to the work required to lift its weight through 1200 metres. This was found by stretching it by gradations of weights up to the breaking weight, representing the results by aid of a curve, and measuring its area to find the integral work given back by the spring after being stretched by a weight just short of the breaking weight.

57. *Flexure of a Beam or Rod.*—In the problem of simple flexure a bar or uniform rod or wire, straight when free from stress, is kept in a circular form by equal opposing couples properly applied to its ends. The parts of the bar on the convex side of the circle must obviously be stretched longitudinally, and those on the concave side contracted longitudinally, by the flexure. It is not obvious, however, what are the conditions affecting the lateral shrinkings and swellings of ideal filaments into which we may imagine the bar divided lengthwise. Earlier writers had assumed without proof that each filament, bent as it is in its actual position in the bar, is elongated or contracted by the same amount as it would be if it were detached, and subjected to the same end pull or end compression with its sides quite free to shrink or

expand, but they had taken no account of the lateral shrinking or swelling which the filament must really experience in the bent bar. The subject first received satisfactory mathematical investigation from St Venant.[1] He proved that the old supposition is substantially correct, with the important practical exception of the flat spring referred to in section 59 below. His theory shows that, in fact, if we imagine the whole rod divided parallel to its length into infinitesimal filaments, each of these shrinks or swells laterally with sensibly the same freedom as if it were separated from the rest of the substance and subjected to end pull or end compression, lengthening or shortening it in a straight line to the same extent as it is really lengthened or shortened in the circular arc which it becomes in the bent rod. He illustrates the distortion of the cross section by which these changes of lateral dimensions are necessarily accompanied in the annexed diagram (fig. 5), in which either the whole normal section of a rectangular beam, or a rectangular area in the normal section of a beam of any figure, is represented in its strained and unstrained figures, with the central point O common to the two. The flexure is in planes perpendicular to YOY_1, and is concave upwards (or towards X),—G, the centre of curvature, being in the direction indicated, but too far to be included in the diagram. The straight sides AC, BD, and all straight lines parallel to them, of the unstrained rectangular area become concentric arcs of circles concave in the opposite direction, their centre of curvature H being (articles 47, 48) for rods of india-rubber or gelatinous substance, or of glass or metal, from 2 to 4 times as far from O on one side as G is on the other. Thus the originally plane sides AC, BD of a rectangular bar become anticlastic[2] surfaces, of curvatures $\frac{1}{\rho}$ and $\frac{-\sigma}{\rho}$, in the two principal sections, if σ denote the ratio of lateral shrinking to longitudinal extension. A flat rectangular, or a square, rod of india-rubber [for which σ amounts (section 47) to very nearly $\frac{1}{2}$, and which is susceptible of very great amounts of strain without utter loss of corresponding elastic action] exhibits this phenomenon remarkably well.

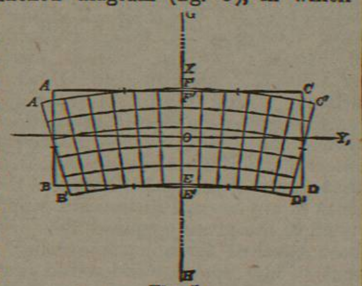

Fig. 5.

58. *Limits to the bending of Rods or Beams of hard solid substance.*—For hard solids, such as metals, stones, glasses, woods, ivory, vulcanite, papier-maché, elongations and contractions to be within the limits of elasticity must generally (section 23) be less than $\frac{1}{100}$. Hence the breadth or thickness of the bar in the plane of curvature must generally be less than $\frac{1}{100}$ of the diameter of curvature in order that the bending may not break it, or give it a permanent bend, or strain it beyond its "limits of elasticity."

59. *Exceptional case of Thin flat Spring, too much bent to fulfil conditions of section* 57.—St Venant's theory shows that a farther condition must be fulfilled if the ideal filaments are to have the freedom to shrink or expand as explained in section 57. For unless the *breadth* AC of the bar (or diameter perpendicular to the plane of flexure) be very small in comparison with the mean proportional between the radius OH and the thickness AB the distances from YY_1 to the corners A′, C′, would fall short of the half thickness, OE, and the distances to B′, D′, would exceed it, by differences comparable with its own amount. This would give rise to sensibly less and greater shortenings and stretchings in the filaments towards the corners than those supposed in the ordinary calculation of flexural rigidity (article 61), and so vitiate the result. Unhappily, mathematicians have not hitherto succeeded in solving, possibly not even tried to solve, the beautiful problem thus presented by the flexure of a broad very thin band (such as a watch spring) into a circle of radius comparable with a third proportional to its thickness and its breadth.

60. But, provided the radius of curvature of the flexure is not only a large multiple of the *greatest* diameter, but also of a third proportional to the diameters in and perpendicular to the plane of flexure; then, however great may be the ratio of the greatest diameter to the least, the preceding solution is applicable; and it is remarkable that the necessary distortion of the normal section (illustrated in the diagram of article 57) does not sensibly impede the free lateral contractions and expansions in the filaments, even in the case of a broad thin lamina (whether of precisely rectangular section, or of unequal thicknesses in different parts).

61. *Flexural Rigidities of a Rod or Beam.*—The couple required to give unit curvature in any plane to a rod or beam is called its flexural rigidity for curvature in that plane. When the beam is of circular cross section and of isotropic material, the flexural rigidity is clearly the same, whatever be the plane of flexure through the axis, and the plane of the bending couple coincides with the plane of flexure. It might be expected that in a round bar of æolotropic material, such as a wooden rod with the annual woody layers sensibly plane and parallel to a plane through its axis, would show different flexural rigidities in different planes,—in the case of wood, for example, different according as the flexure is in a parallel or perpendicular to the annual layers. This is not so, however; on the contrary, it is easy to show, by an extension of St Venant's theory, that in the case of the wooden rod the flexural rigidity is equal in all planes through the axis, and that the plane of flexure always agrees with the plane of the bending couple, and to prove generally that the flexure of a bar of æolotropic substance, and composed it may be of longitudinal filaments of heterogeneous materials, is precisely the same as if it were isotropic, and that its flexural rigidities are calculated by the same rule from its Young's modulus, provided that the æolotropy is not such as (section 81) to give rise to alteration of the angle between the length and any diameter perpendicular to the length when weight is hung on the rod, or on any longitudinal filament cut from it. Excluding then all cases in which there is any such oblique æolotropy, we have a very simple theory for the flexure of bars of any substance, whether isotropic or æolotropic, and whether homogeneous or not homogeneous through the cross section.

62. *Principal Flexural Rigidities and Principal Planes of Flexure of a Beam.*—The flexural rigidity of a rod is generally not equal in different directions, and the plane of flexure does not generally coincide with the plane of the bending couple. Thus a flat ruler is much more easily bent in a plane perpendicular to its breadth than in the plane of its breadth; and if we apply opposing couples to its two ends in any plane through its axis not either perpendicular or parallel to its breadth, it is obvious that the plane in which the flexure takes place will be more inclined to the plane of the breadth than to the plane of the bending couple. Very elementary statical theory, founded on St

[1] *Mémoires des Savants Etrangers*, 1855, "De la Torsion des Prismes, avec des considérations sur leur Flexion." &c.

[2] See Thomson and Tait's *Natural Philosophy*, vol. i. § 128.

Venant's conclusions of section 57, shows that, whatever the shape and the distribution of matter in the cross section of the bar, there are two planes at right angles to one another such that if the bar be bent in either of these planes the bending couple will coincide with the plane of flexure. These planes are called principal planes of flexure, and the rigidities of the bar for flexure in these planes are called its principal flexural rigidities. When the principal flexural rigidities are known the flexure of the bar in any plane oblique to the principal planes is readily found by supposing it to be bent in one of the principal planes and simultaneously in the other, and calculating separately the couples required to produce these two component flexures. The positions of the principal planes of flexure, the relative flexural rigidities, and the law of elongation and contraction in different parts of the cross section, are found according to the following simple rules :—

(1.) Imagine an infinitely thin plane disc of the same shape and size as the cross section loaded with matter in simple proportion to the Young's modulus in different parts of the cross section. Let the quantity of matter per unit area on any point of the disc be equal to the Young's modulus on the corresponding point of the rod when the material is heterogeneous : on the other hand, when the material is homogeneous it is more convenient to call the quantity of matter unity per unit area of the disc. Considering different axes in the plane of the disc through its centre of inertia, find the two principal axes of greatest and least moments of inertia, and find the moments of inertia round them.

(2.) In whatever plane the bar be bent it will experience neither elongation nor contraction in the filament which passes through the centres of inertia of the cross sections found according to rule (1), nor in the diameter of the cross section perpendicular to the plane of flexure.

(3.) Thus all the particles which experience neither elongation nor contraction lie in a surface cutting the plane of flexure perpendicularly through the centres of inertia of the cross sections. All the material on the outside of this cylindrical surface is elongated, and all on the interior is contracted, in simple proportion to distance from it : the amount of the elongation or contraction being in fact equal to distance from this neutral surface divided by the radius of its curvature.

(4.) Hence it is obvious that the portions of the solid on the two sides of any cross section must experience mutual normal force, pulling them towards one another in the stretched part, and pressing them from one another in the condensed part, and that the amount of this negative or positive normal pressure per unit of area must be equal to the Young's modulus at the place, multiplied into the ratio of its distance from the neutral line of the cross section to the radius of curvature.

The sum of these positive and negative forces over the whole area of the cross section is zero in virtue of condition (2). Their couple resultant has its axis perpendicular to the plane of curvature when this line is either of the principal axes (3) of the cross section; and its moment is clearly equal to the moment of inertia of the material disc (1) divided by the radius of curvature. Hence the principal flexural rigidities are simply equal to the principal moments of inertia of this disc; and the principal flexural planes are the planes through its principal axes and the length of the bar; or taking the quantity of matter per unit area of the disc unity for the case of a homogeneous bar, we have the rule that the principal rigidities are equal to the product of the Young's modulus into the principal moments of inertia of the cross sectional areas, and the principal planes of flexure are the longitudinal planes through the principal axes of this area.

63. *Law of Torsion.*—One of the most beautiful applications of the general equations of internal equilibrium of an elastic solid hitherto made is that of M. de St Venant to "the torsion of prisms." In this work the mathematical methods invented by Fourier for the solution of problems regarding conduction of heat have been most ingeniously and happily applied by St Venant to the problem of torsion. To reproduce St Venant's mathematical investigation here would make this article too long (it occupies 227 quarto pages of the *Mémoires des Savants Étrangers*); but a statement of some of the chief results is given (sections 65–72), not only on account of their strong scientific interest, but also because they are of great practical value in engineering; and the reader is referred to Thomson and Tait's *Natural Philosophy*, sections 700–710, for the proofs and for further details regarding results, but much that is valuable and interesting is only to be found in St Venant's original memoir.

64. *Torsion Problem stated and Torsional Rigidity defined.*—To one end of a long, straight prismatic rod, wire, or solid or hollow cylinder of any form, a given couple is applied in a plane perpendicular to the length, while the other end is held fast: it is required to find the degree of twist produced, and the distribution of strain and stress throughout the prism. The moment of the couple divided by the amount of the twist per unit length is called the torsional rigidity of the rod or prism. This definition is founded simply on the extension of Hooke's law to torsion discovered experimentally by Coulomb, according to which a rod or wire when twisted within limits of torsional elasticity exerts a reactive couple in simple proportion to the angle through which one end is turned relatively to the other. The internal conditions to be satisfied in the torsion problem are that the resultant action between the substance on the two sides of any normal section is a couple, in the normal plane, equal to the given couple. This problem has not hitherto been attacked for æolotropic solids. Even such a case as that of the round wooden rod (section 61) with annual layers sensibly parallel to a plane through its length, will, when twisted, experience a distribution of strain complicated much by its æolotropy. The following statements of results are confined to rods of isotropic material.

65. *Torsion of Circular Cylinder.*—For a solid or hollow circular cylinder, the solution (given first, we believe, by Coulomb) obviously is that each circular normal section remains unchanged in its own dimensions, figure, and internal arrangement (so that every straight line of its particles remains a straight line of unchanged length), but is turned round the axis of the cylinder through such an angle as to give a uniform *rate of twist* equal to the applied couple divided by the product of the moment of inertia of the circular area (whether annular or complete to the centre) into the modulus of rigidity of the substance.

For, if we suppose the distribution of strain thus specified to be actually produced, by whatever application of stress is necessary, we have, in every part of the substance, a simple shear parallel to the normal section, and perpendicular to the radius through it. The elastic reaction against this requires, to balance it (section 43), a simple distorting stress consisting of forces in the normal section, directed as the shear, and others in planes through the axis, and directed parallel to the axis. The amount of the shear is, for parts of the substance at distance r from the axis, equal obviously to τr, if τ be the rate of twist reckoned in radians per unit of length of the cylinder. Hence the amount of the tangential force in either set of planes is $n\tau$ per unit of area, if n be the rigidity of the substance. Hence there is no force between parts of the substance lying on the two sides of any element of any circular cylinder coaxal with the bounding cylinder or cylinders; and consequently no force is required on the cylindrical boundary to maintain the supposed state of strain. And the mutual action between the parts of the substance on the two sides of any normal plane section consists of force in this plane, directed perpendicular to the radius through each point, and amounting to $n\tau r$ per unit of area. The moment of this distribution of force round the axis of the cylinder is (if $d\sigma$ denote an element of the area) $n\tau\iint d\sigma r^2$, or the product of $n\tau$ into the moment of inertia of the area round the perpendicular to its plane through its centre, which is therefore equal to the moment of the couple applied at either end.

66. *Prism of any shape constrained to a Simple Twist.*—Farther, it is easily proved that if a cylinder or prism of any shape be compelled to take exactly the state of strain above specified (section 65) with the line through the centres of inertia of the normal sections, taken instead of the axis of the cylinder, the mutual action between the parts of it on the two sides of any normal section will be a couple of which the moment will be expressed by the same formula, that is, the product of the rigidity, into the rate of twist, into the moment of inertia of the section round its centre of inertia. But for any other shape of prism than a solid or symmetrical hollow circular cylinder, the supposed state of strain requires, besides the terminal opposed couples, force parallel to the length of the prism, distributed over the prismatic boundary, in proportion to the distance PE along the tangent, from each point of the surface, to the point in which this line is cut by a perpendicular to it from O the centre of inertia of the normal section. To prove this let a normal section of the prism be represented in the annexed diagram (fig. 6). Let PK, representing the shear at any point P, close to the prismatic boundary, be resolved into PN and PT along the normal and tangent respectively. The whole shear PK being equal to τr its component PN is equal to $\tau r \sin \omega$ or $\tau . PE$. The corresponding component of the required stress is $n\tau . PE$, and involves equal forces in the plane of the diagram, and in the plane through TP perpendicular to it, each amounting to $n\tau . PE$ per unit of area.

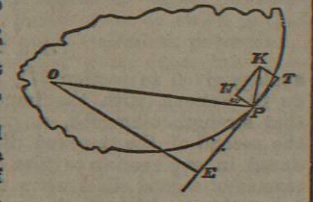

Fig. 6

An application of force equal and opposite to the distribution thus found over the prismatic boundary, would of course alone produce in the prism, otherwise free, a state of strain which, compounded with that supposed above, would give the state of strain actually produced by the sole application of balancing couples to the two ends. The result, it is easily seen, consists of an increased twist, together with a warping of naturally plane normal sections, by infinitesimal displacements perpendicular to themselves, into certain surfaces of anticlastic curvature, with equal opposite curvatures. In bringing forward this theory, St Venant not only pointed out the falsity of the supposition admitted by several previous writers, and used in practice fallaciously by engineers, that Coulomb's law holds for other forms of prism than the solid or hollow circular cylinder, but he discovered fully the nature of the requisite correction, reduced the determination of it to a problem of pure mathematics, worked out the solution for a great variety of important and curious cases, compared the results with observation in a manner satisfactory and interesting to the naturalist, and gave conclusions of great value to the practical engineer.

67. "*Hydrokinetic Analogue to Torsion Problem.*[1]—We take advantage of the identity of mathematical conditions in St Venant's torsion problem, and a hydrokinetic problem first solved a few years earlier by Stokes,[2] to give the following statement, which will be found very useful in estimating deficiencies in torsional rigidity below the amount calculated from the fallacious extension of Coulomb's law :—

"Conceive a liquid of density n completely filling a closed infinitely light prismatic box of the same shape within as the given elastic prism and of length unity, and let a couple be applied to the box in a plane perpendicular to its length. The *effective* moment of inertia of the liquid[3] will be equal to the correction by which the torsional rigidity of the elastic prism, calculated by the false extension of Coulomb's law, must be diminished to give the true torsional rigidity.

"Farther, the actual *shear* of the solid, in any infinitely thin plate of it between two normal sections, will at each point be, when reckoned as a differential sliding (section 43) parallel to their planes, equal to and in the same direction as the velocity of the liquid relatively to the containing box."

68. *Solution of Torsion Problem.*—To prove these propositions and investigate the mathematical equations of the problem, the process followed in Thomson and Tait's *Natural Philosophy*, section 706, is first to show that the conditions of sections 63 are verified by a state of strain compounded of (1) a simple twist round the line through the centres of inertia, and (2) a distortion of each normal section by infinitesimal displacements perpendicular to its plane; then find the interior and surface equations to determine this warping; and lastly, calculate the actual moment of the couple to which the mutual action between the matter on the two sides of any normal section is equivalent.

69. St Venant's treatise abounds in beautiful and instructive graphical illustrations of his results, from which the following are selected :—

(1.) *Elliptic Cylinder.*—The plain and dotted curvilineal arcs are (fig. 7) "contour lines" (*coupes topographiques*) of the section as warped by torsion; that is to say, lines in which it is cut by a series of parallel planes, each perpendicular to the axis. The arrows indicate the direction of rotation in the part of the prism *above* the plane of the diagram.

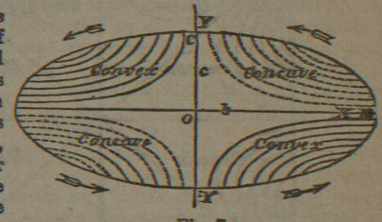

Fig. 7.

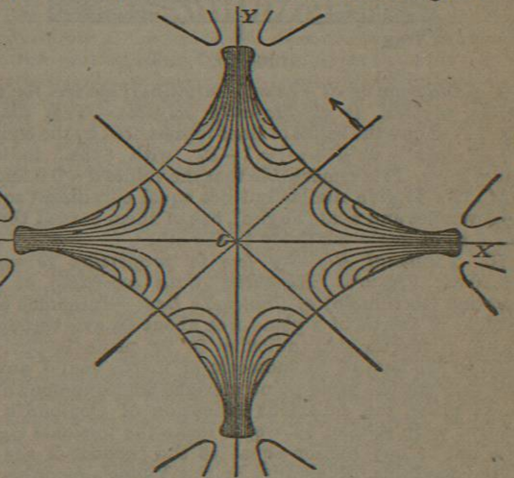

Fig. 8.

[1] Extracted from Thomson and Tait, sections 704, 705

[2] "On some cases of Fluid Motion,"—*Camb. Phil. Trans.*, 1843.

[3] "That is, the moment of inertia of a rigid solid which, as will be proved in vol. ii., may be fixed within the box, if the liquid be removed, to make its motions the same as they are with the liquid in it."