(2.) *Contour lines for St Venant's "étoile à quatre points arrondis."*—This diagram (fig. 8) shows the contour lines, in all respects as in case (1), for the case of a prism having for section the figure indicated. The portions of curve outside the continuous closed curve are merely indications of mathematical extensions irrelevant to the physical problem.

(3.) *Contour lines of normal section of triangular prism, as warped by torsion*, shown as in case (1) (fig. 9.

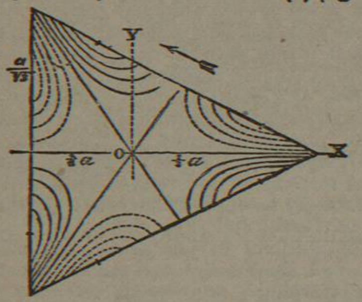

Fig. 9.

(4.) *Contour lines of normal sections of square prisms as warped by torsion* (fig. 10).

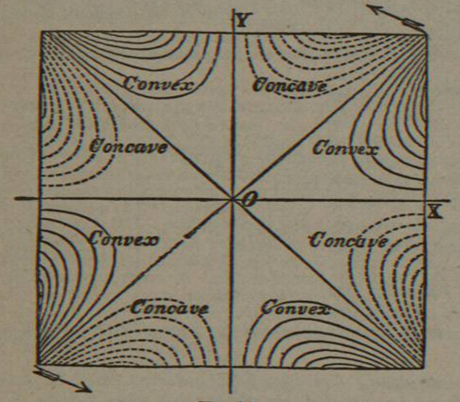

Fig. 10.

(5.) *Diagram of St Venant's curvilineal squares for which torsion problem is algebraically solvable.*—This diagram (fig. 11) shows the series of lines represented by the equation $x^2+y^2-a(x^4-6x^2y^2+y^4)=1-a$, with the indicated values for a. It is remarkable that the values $a=0{\cdot}5$ and $a=-\frac{1}{2}(\sqrt{2}-1)$ give similar but not equal curvilineal squares (hollow sides and acute angles), one of them turned through half a right angle relatively to the other.

70. *Torsional Rigidity less in proportion to sum of principal Flexural Rigidities than according to false extension* (section 66) *of Coulomb's Law.*—Inasmuch as the moment of inertia of a plane area about an axis through its centre of inertia perpendicular to its plane is obviously equal to the sum of its moments of inertia round any two axes through the same point at right angles to one another in its plane, the fallacious extension of Coulomb's law, referred to in section 66, would make the torsional rigidity of a bar of any section equal to the product of the ratio of the modulus of rigidity to the Young's modulus into the sum of its flexural rigidities (section 61) in any two planes at right angles to one another through its length. The true theory, as we have seen (section 67), always gives a torsional rigidity less than this. How great the deficiency may be expected to be in cases in which the figure of the section presents projecting angles, or considerable prominences (which may be imagined from the hydrokinetic

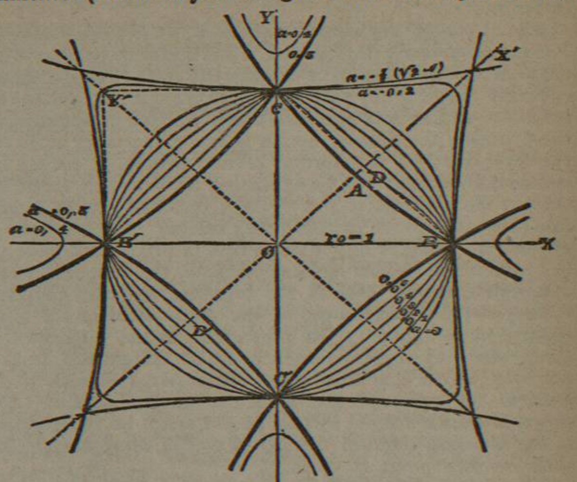

Fig. 11.

analogy given in section 67), has been pointed out by M. de St Venant, with the important practical application, that strengthening ribs, or projections (see, for instance, the second of the annexed diagrams), such as are introduced in engineering to give stiffness to beams, have the reverse of a good effect when *torsional* rigidity or strength is an object, although they are truly of great value in increasing the flexural rigidity, and giving strength to bear ordinary strains, which are always more or less flexural. With remarkable ingenuity and mathematical skill he has drawn beautiful illustrations of this important practical principle from his algebraic and transcendental solutions.

(1.) Rectilineal square.	(2.) Square with curved corners and hollow sides; being curve, $a=0{\cdot}4$, of § 788 (3).	(3.) Square with acute angles and hollow sides.	(4.) Star with four rounded points, being a curve of the eighth degree [§ 707 (35)].	(5.) Equilatera triangle.

·84346.	·8186.	·7783.	5374.	·60000.
·88326	·8666.	·8276.	·6745.	·72552.

Fig. 12.—Diagrams showing torsional rigidities.

Thus, for an equilateral triangle, and for the rectilineal and three curvilineal squares shown in the diagrams (fig. 12), he finds for the torsional rigidities the values stated. The number immediately below the diagram indicates in each case the fraction which the true torsional rigidity is of the old fallacious estimate (section 66),—the latter being the product of the rigidity of the substance into the moment of inertia of the cross section round an axis perpendicular to its plane through its centre of inertia. The second number indicates in each case the fraction which the torsional rigidity is of that of a solid circular cylinder of the same sectional area.

71. *Places of greatest Distortion in Twisted Prisms.*—M. de St Venant also calls attention to a conclusion from his solutions which to many may be startling, that in his simpler cases the places of greatest distortion are those points of the boundary which are nearest to the axis of the twisted prism in each case, and the places of least distortion those farthest from it. Thus in the elliptic cylinder the substance is most strained at the ends of the smaller principal diameter, and least at the ends of the greater. In the equilateral triangular and square prisms there are longitudinal lines of maximum strain through the middles of the sides. In the oblong rectangular prism there are two lines of greater maximum strain through the middles of the broader pair of sides, and two lines of less maximum strain through the middles of the narrow sides. The strain is, as we may judge from the hydrokinetic analogy, excessively small, but not evanescent, in the projecting ribs of a prism of the figure shown in (2) of section 69. It is quite evanescent infinitely near the angle, in the triangular and rectangular prisms, and in each other case, as (5) of section 69, in which there is a finite angle, whether acute or obtuse, projecting outwards. This reminds us of a general remark we have to make, although consideration of space may oblige us to leave it without formal proof.

72. *Strain at Projecting Angles, evanescent; at Re-entrant Angles, infinite; Liability to Cracks proceeding from Re-entrant Angles, or any places of too sharp concave curvature.*—A solid of any elastic substance, isotropic or æolotropic, bounded by any surfaces presenting projecting edges or angles, or re-entrant angles or edges, however obtuse, cannot experience any finite stress or strain in the neighbourhood of a *projecting* angle (trihedral, polyhedral, or conical); in the neighbourhood of an edge, can only experience simple longitudinal stress parallel to the neighbouring part of the edge; and generally experiences infinite stress and strain in the neighbourhood of a *re-entrant* edge or angle; when influenced by any distribution of force, exclusive of surface tractions infinitely near the angles or edges in question. An important application of the last part of this statement is the practical rule, well known in mechanics, that every re-entering edge or angle ought to be rounded, to prevent risk of rupture, in solid pieces designed to bear stress. An illustration of these principles is afforded by the concluding example of torsion in Thomson and Tait's section 707; in which we have the complete mathematical solution of the torsion problem for prisms of fan-shaped sections, such as the annexed forms (fig. 13).

Fig. 13.

The solution shows that when the solid is continuous from the circular cylindrical surface to its axis, as in (4), (5), (6), the strain is zero or infinite according as the angle between the bounding planes of the solid is less than or greater than two right angles as in cases (4) and (6) respectively.

73. *Changes of Temperature produced by Compressions or Dilatations of a Fluid and Stresses of any kind in an Elastic Solid.*—From thermodynamic theory[1] it is concluded that cold is produced whenever a solid is strained by opposing, and heat when it is strained by yielding to, any elastic force of its own, the strength of which would diminish if the temperature were raised; but that, on the contrary, heat is produced when a solid is strained against, and cold when it is strained by yielding to, any elastic force of its own, the strength of which would increase if the temperature were raised. When the strain is a condensation or dilatation, uniform in all directions, a fluid may be included in the statement. Hence the following propositions:—

(1.) A cubical compression of any elastic fluid or solid in an ordinary condition causes an evolution of heat; but, on the contrary, a cubical compression produces cold in any substance, solid or fluid, in such an abnormal state that it would contract if heated while kept under constant pressure. Water below its temperature ($3^{\circ}{\cdot}9$ Cent.) of maximum density is a familiar instance. (See table of section 76.)

(2.) If a wire already twisted be suddenly twisted further, always, however, within its limits of elasticity, cold will be produced; and if it be allowed suddenly to untwist, heat will be evolved from itself (besides heat generated externally by any work allowed to be wasted, which it does in untwisting). It is assumed that the torsional rigidity of the wire is diminished by an elevation of temperature, as the writer of this article had found it to be for copper, iron, platinum, and other metals (compare section 78).

(3.) A spiral spring suddenly drawn out will become lower in temperature, and will rise in temperature when suddenly allowed to draw in. [This result has been experimentally verified by Joule ("Thermodynamic Properties of Solids," *Trans. Roy. Soc.*, 1858) and the amount of the effect found to agree with that calculated, according to the preceding thermodynamic theory, from the amount of the weakening of the spring which he found by experiment.]

(4.) A bar or rod or wire of any substance with or with out a weight hung on it, or experiencing any degree of end thrust, to begin with, becomes cooled if suddenly elongated by end pull or by diminution of end thrust, and warmed if suddenly shortened by end thrust or by diminution of end pull; except abnormal cases in which with constant end pull or end thrust elevation of temperature produces shortening; in every such case pull or diminished thrust produces elevation of temperature, thrust or diminished pull lowering of temperature.

(5.) An india-rubber band suddenly drawn out (within its limits of elasticity) becomes warmer; and when allowed to contract, it becomes colder. Any one may easily verify this curious property by placing an india-rubber band in slight contact with the edges of the lips, then suddenly extending it—it becomes very perceptibly warmer: hold it for some time stretched nearly to breaking, and then suddenly allow it to shrink—it becomes quite startingly colder, the cooling effect being sensible not merely to the lips but to the fingers holding the band. The first published statement of this curious observation is due to Gough (*Memoirs of the Literary and Philosophical Society of Manchester*, 2d series, vol. i. p. 288), quoted by Joule in his paper on "Thermodynamic Properties of Solids" (*Transactions of Royal Society*, 1858). The thermodynamic conclusion from it is that an india-rubber band, stretched by a constant weight of sufficient amount hung on it, must, when heated, pull up the weight, and, when cooled, allow the weight to descend: this Gough, independently of thermodynamic theory, had found to be actually the case. The experiment any one can make with the greatest ease by hanging a few pounds weight on a common india-rubber band, and taking a red-hot coal in a pair of tongs, or a red-hot poker, and moving it up and down close to the band. The way in which the weight rises when the red-hot body is near, and falls when it is removed, is quite startling. Joule experimented on the amount of shrinking per degree of elevation of temperature, with different weights hung on a band of vulcanized india-rubber, and found that they closely agreed with the amounts calculated by Thomson's theory from the heating effects of pull, and cooling effects of ceasing to pull, which he had observed in the same piece of india-rubber.

[1] W. Thomson on "Thermo-elastic Properties of Matter," in *Quarterly Journal of Mathematics*, April 1855 (republished in *Phil. Mag.* 1877, second half year)

74. The thermodynamic theory gives one formula[1] by which the change of temperature in every such case may be calculated when the other physical properties are known:—

$$\theta=\frac{tep}{JK\rho};$$

where θ denotes the elevation of temperature produced by the sudden application of a stress p;
t, the temperature of the substance on the absolute thermodynamic scale,[2] the change of temperature θ being supposed to be but a very small fraction of t;
e, the geometrical effect (expansion or other strain) produced by an elevation of temperature of one degree when the body is kept under constant stress;
K, the specific heat of the substance per unit mass under constant stress;
ρ, the density;
and J, Joule's equivalent (taken as 42400 centimetres).

In using the formula for a fluid, p must be normal pressure equal in all directions, or normal pressure on a set of parallel planes, or tangential traction on one or other of the two sets of mutually perpendicular parallel planes which (section 43) experience tangential traction when the body is subjected to a simple distorting stress; or, quite generally, p may be the proper numerical reckoning (Mathematical Theory, chap. x.) of any stress, simple or compound. When p is pressure uniform in all directions, e must be expansion of bulk, whether the body expands equally in all directions or not. When p is pressure perpendicular to a set of parallel planes, e must be expansion in the direction opposed to this pressure, irrespectively of any change of shape not altering the distance between the two planes of the solid perpendicular to the direction of p. When p is a simple tangential stress, reckoned as in section 43, e must be the change, reckoned in fraction of the radian, of the angle, infinitely nearly a right angle, between the two sets of parallel planes in either of which there is the tangential traction denoted by p. In each of these cases p is reckoned simply in units of force per unit of area. Quite generally p may be any stress, simple or compound, and e must be the component (Math. Th., chaps. viii. and ix.) relatively to the type of p, of the strain produced by an elevation of temperature of one degree when the body is kept under constant stress. The constant stress for which K and e are reckoned ought to be the mean of the stresses which the body experiences with and without p. Mathematically speaking, p is to be infinitesimal, but practically it may be of any magnitude moderate enough not to give any sensible difference in the value of either K or e, whether the "constant stress" be with p or without p, or with the mean of the two: thus for air p must be a small fraction of the whole pressure, for instance a small fraction of one atmosphere for air at ordinary pressure; for water or watery solutions of salts or other solids, for mercury, for oil, and for other known liquids p may, for all we know, amount to twenty atmospheres or one hundred atmospheres without transgressing the limits for which the preceding formula is applicable. When the law of variation of K and e with pressure is known, the differential formula is readily integrated to give the integral amount of the change of temperature produced by greater stress than those for which the differential formula is applicable. For air and other permanent gases Boyle's law of compression and Charles's law of thermal expansion supply the requisite data with considerable accuracy up to twenty or thirty atmospheres. The result is expressed by the formula

$$\frac{t+\theta}{t}=\left(\frac{P+p}{P}\right)^{\frac{k-1}{k}} \quad . \quad . \quad . \quad . \quad (1.)$$

where k denotes the ratio of the thermal capacity, pressure constant, to the thermal capacity, volume constant, of the gas, a number which thermodynamic theory proves to be approximately constant for all temperatures and densities, for any fluid approximately fulfilling Boyle's and Charles's laws;
P and t the initial pressure and temperature of the gas;
p the sudden addition to the pressure;
and, as before, θ the elevation of temperature.

For the case of p a small fraction of P the formula gives

$$\theta=(k-1)\frac{p}{P}\,t \quad . \quad . \quad . \quad . \quad (2.)$$

It is by an integration of this formula that (1) is obtained.

For common air the value of k is very approximately 1·41. Thus if a quantity of air be given at 15° C. ($t=$ 289°) and the ordinary atmospheric pressure, and if it be compressed gradually up to 32 atmospheres, or dilated to $\frac{1}{32}$ of an atmosphere, and perfectly guarded against gain or loss of heat from or to without, its temperature at several different pressures, chosen for example, will be according to the following table of excesses of temperature above the primitive temperature, calculated by (1).

TABLE SHOWING EFFECTS OF PRESSURE ON TEMPERATURE.

Air given at temperature 15° *Cent.* (289° *absolute*).

Value of P+p.	Elevation of temperature produced by compression.	Value of P+p.	Lowering of temperature produced by dilatation.
2	95°	$\frac{1}{2}$	71°
4	221	$\frac{1}{4}$	125
8	389	$\frac{1}{8}$	166
16	612	$\frac{1}{16}$	196
32	911	$\frac{1}{32}$	219

But we have no knowledge of the effect of pressures of several thousand atmospheres in altering the expansibility or specific heat in liquids, or in fluids which at less heavy or at ordinary pressures are "gases."

75. When change of temperature, whether in a solid or a fluid is produced by the application of a stress, the corresponding modulus of elasticity will be greater in virtue of the change of temperature than what may be called the static modulus defined as above, on the understanding that the temperature if changed by the stress is brought back to its primitive degree before the measurement of the strain is performed. The modulus calculated on the supposition that the body, neither losing nor gaining heat during the application of the stress and the measurement of its effect, retains the whole change of temperature due to the stress, will be called for want of a better name the kinetic modulus, because it is this which must (as in Laplace's celebrated correction of Newton's calculation of the velocity of sound) be used in reckoning the elastic forces concerned in waves and vibrations in almost all practical cases. To find the ratio of the kinetic to the static modulus remark that $e\theta$, according to the notation of section 74, is the diminution of the strain due to the change of temperature θ. Hence if M denote the static modulus (section 41), the strain actually produced by it when the body is not allowed either to gain or lose heat is $\frac{p}{M}-e\theta$, or, with θ replaced by its value according to the formula of section 74,

$$\frac{p}{M}-e\frac{tep}{JK\rho}.$$

Dividing p by this expression we find for the kinetic modulus

$$M'=\frac{1}{\frac{1}{M}-\frac{te^2}{JK\rho}}.$$

Hence

$$\frac{M'}{M}=\frac{1}{1-\frac{te^2M}{JK\rho}}.$$

76. For any substance, fluid or solid, it is easily proved, without thermodynamic theory, that

$$\frac{M'}{M}=\frac{K}{N};$$

where K denotes the thermal capacity of a stated quantity of the substance under constant stress, and N its thermal capacity under constant strain (or thermal capacity when the body is prevented from change of shape or change of volume). For permanent gases, and generally for fluids approximately fulfilling Boyle's and Charles's laws as said above, k is proved by thermodynamic theory to be approximately constant. Its value for all gases for which it has been measured differs largely from unity, and probably also for liquids generally (except water near its temperature of maximum density).

On the other hand, for solids whether the stress considered be uniform compression in all directions or of any other type, the value of $\frac{M'}{M}$ or $\frac{K}{N}$ differs but very little from unity; and both for solids and liquids it is far from constant at different temperatures (in the case of water it is zero at 3°·9 Cent., and varies as the square of the difference of the temperature from 3°·9 at all events for moderate differences from this critical temperature, whether above or below it). The following tables show the value of $\frac{M'}{M}$ or $\frac{K}{N}$, and the value of θ by the formula of sec. 74, for different fluid and solid substances at the temperature 15° Cent. (289° absolute scale). The first table is for compression uniform in all directions; the second, necessarily confined to solids, is for the stress dealt with in "Young's Modulus," that is, normal pressure (positive or negative) on one set of parallel planes, with perfect freedom to expand or contract in all directions in these planes. A wire or rod pulled longitudinally is a practical application of the latter.

THERMODYNAMIC TABLE I.

Pressure equal in all directions—Ratio of Kinetic to Static Bulk-Modulus. Temperature 15° C. (289° *absolute*) J = 42400 *centimetres.*

Substance.	Density.	Thermal Capacity per unit mass =K.	Expansibility =e.	Elevation of Temperature produced by a pressure of one gramme per square centimetre $=\frac{te}{JK\rho}$.	Static Bulk-Modulus in grammes per square centimetre =M.	Deduced value of $\frac{M'}{M}$ or $\frac{K}{N}$ $=\left(1-\frac{te^2M}{JK\rho}\right)^{-1}$
Air	·001225	·2375	·00346	·0824	1033	1·41
Distilled water	1·000	1·000	·00016	·000011	22·63×10⁶	1·0040
Alcohol	·795	·6148	·00106	·0000148	11·4×10⁶	1·22
Ether	·7005	·5157	·00155	·0000292	8·07×10⁶	1·577
Mercury	13·56	·0330	·00018	·0000274	552·5×10⁶	1·375
Glass, flint	2·942	·1770	·000026	·000000340	423×10⁶	1·00375
Brass, drawn	8·471	·09391	·0000545	·000000466	1063×10⁶	1·028
Iron	7·677	·1098	·0000395	·000000319	1485×10⁶	1·019
Copper	8·843	·0949	·0000545	·000000443	1717×10⁶	1·043

THERMODYNAMIC TABLE II.

Pressure parallel to one direction in a solid—Ratio of Kinetic to Static Young's Modulus. Temperature 15° C. (289° *absolute*).

Substance.	Density =ρ.	Thermal Capacity per unit mass =K.	Expansibility =e.	Lowering of Temperature produced by a pull of one gramme per square centimetre $=\frac{te}{JK\rho}$.	Static Young's Modulus in grammes per square centimetre =M.	Deduced value of $\frac{M'}{M}$ or $\frac{N}{K}$ $=\left(1-\frac{te^2M}{JK\rho}\right)^{-1}$
Zinc	7·008	·0927	·0000249	·000000308	873×10⁶	1·0080
Tin	7·404	·0514	·000022	·000000394	417×10⁶	1·00362
Silver	10·369	·0557	·000019	·000000224	736×10⁶	1·00315
Copper	8·933	·0949	·000018	·000000145	1245×10⁶	1·00325
Lead	11·215	·0293	·000029	·000000602	177×10⁶	1·00310
Glass	2·942	·177	·0000086	·000000113	614·4×10⁶	1·000600
Iron	7·553	·1098	·000013	·000000107	1861×10⁶	1·00259
Platinum	21·275	·0314	·0000086	·0000000778	1704×10⁶	1·00129

77. *Experimental Results.*—The following tables show determinations of moduluses of compression, of Young's modulus, and of moduluses of rigidity by various experimenters and various methods. It will be seen that the Young's moduluses obtained by Wertheim by vibrations, longitudinal or transverse, are generally in excess of those which he found by static extension; but the differences are enormously greater than those due to the heating and cooling effects of elongation and contraction (section 76), and are to be certainly reckoned as errors of observation. It is probable that his moduluses determined by static elongation are minutely accurate; the discrepancies of those found by vibrations are probably due to imperfections of the arrangements for carrying out the vibrational method:—

TABLE OF MODULUSES OF COMPRESSIBILITY.

Substance.	Moduluses of compressibility in grammes per square centimetre.	Temperature.	Authority.
Distilled water	22·63×10⁶	15°	Amaury and Descamps, *Comptes Rendus*, tome xvii. p. 1564 (1869).
Alcohol	12·4 ×10⁶	0°	
"	11·4 ×10⁶	15°	
Ether	9·5 ×10⁶	0°	
"	8·07×10⁶	14°	
Bisulphide of carbon	16·3 ×10⁶	14°	
Mercury	552·5 ×10⁶	15°	
Glass	423×10⁶	...	Everett's *Illustrations of the Centimetre-Gramme-Second System of Units.*
Another specimen	354×10⁶	...	
Steel	1876×10⁶	...	
Iron	1485×10⁶	...	
Copper	1717×10⁶	...	
Brass, different specimens	Mean 1063×10⁶	..	Wertheim, *Ann. de Chim.*, 1848.

TABLE OF MODULUSES OF RIGIDITY.

Substance.	Modulus of Rigidity in grammes per square centimetre.	Authority.
Glass, different specimens	Mean 150×10⁶	Wertheim, *Annales de Chimie*, 1848.
Brass, different specimens	Mean 350×10⁶	
Glass	243×10⁶	Everett's *Ill. of the Centimetre-Gramme-Second System of Units.*
Another specimen	240×10⁶	
Brass, drawn	373×10⁶	
Steel	834×10⁶	
Iron, wrought	785×10⁶	
Copper	542×10⁶	
	456×10⁶	

[1] W. Thomson, "Dynamical Theory of Heat" (§ 49), *Trans. R.S.E.*, March 1851, and "Thermoelastic Properties of Matter," *Quarterly Journal of Mathematics*, April 1855 (republished *Phil. Mag.* 1877, second half year).

[2] Ibid., Part vi. §§ 97, 100, *Trans. R.S.E.*, May 1854. According to the scale there defined on thermo dynamic principles, independently of the properties of any particular substance, t is found, by Joule and Thomson's experiments, to agree very approximately with temperature centigrade, with 274° added.

TABLE OF MODULUSES AND STRENGTHS.

Substance.	Density.	Young's Modulus. Grammes per square centimetre.	Young's Modulus. Length Modulus.	Tenacity in grammes per square centimetre	Length Modulus of Rupture in centimetres (or Tenacity in terms of Weight of Unit-Bulk.)	Extreme Elastic Elongation.	Resilience per cubic centimetre in centimetre-grammes.	Resilience per Unit Mass in centimetres.	Authority.	Method of Determination.
Iron or steel	..	Abt 2100×10^6	Abt 9,000,000 ft.	...	...	...	...	..	Dr T. Young	Probably flexure (Young's *Works*, vol. ii. p. 133).
Wood	...	105×10^6 to 280×10^6	4,000,000 to 10,000,000 ft.	...	...	...	...	...	,,	
Stone	...	Abt 350×10^6	Abt 5,000,000 ft.	...	...	...	...	...	,,	
Slate	...	910×10^6 to 1120×10^6	...	...	...	...	...	...	Rankine's "Rules and Tables."	
Ice	...	...	21,000,000 ft.	...	...	...	...	...	Bevan.	Flexure (see § 42).
Brass, cast	...	645×10^6	...	127×10^4	..	·00198	1256	...	Rankine's "Rules and Tables."	
,, wire	...	1001×10^6	...	343×10^4	...	·00344	5905	...		
Bronze, or gun metal	...	696×10^6	...	252×10^4	...	·00362	4562	...	,,	
Copper, cast	...	...	...	134×10^4	...	...	...	...	,,	
,, sheet	...	...	...	211×10^4	...	...	...	...	,,	
,, bolts	...	...	...	253×10^4	...	...	...	...	,,	
,, wires	...	1195×10^6	...	422×10^4	...	·0036	7480	...	,,	
Iron, cast	..	984×10^6 to 1610×10^6	...	94×10^4 to 204×10^4	...	·00116	879	...	,,	
Iron, wrought, plates	...	...	...	359×10^4	...	...	...	...	,,	
,, ,, bars and bolts	..	2040×10^6	...	422×10^4 to 492×10^4	...	·00224	5120	...	,,	
Steel, plates	...	...	...	563×10^4	...	...	...	...	,,	
,, bars	...	2040×10^6 to 2953×10^6	...	703×10^4 to 914×10^4	...	·00324	1310	...	,,	
Lead, sheet	...	51×10^6	...	23×10^4	...	·00451	518	...	,,	
Tin, cast	...	...	...	32×10^4	...	...	...	...	,,	
Zinc	...	...	...	$(49\text{ to }56)\times10^4$	...	...	...	...	,,	
Ash	...	113×10^6	...	120×10^4	...	·0106	6370	...	,,	
Beech	...	95×10^6	...	81×10^4	...	·00853	3455	...	,,	
Birch	...	116×10^6	...	105×10^4	...	·00905	4752	...	,,	
Cedar of Lebanon	...	34×10^6	...	80×10^4	...	·0235	9410	...	,,	
Fir, red pine	.	118×10^6	...	91×10^4	...	·00771	3510	...	,,	
Spruce	..	113×10^6	...	87×10^4	...	·0077	3347	...	,,	
Fir, larch	..	79×10^6	...	68×10^4	...	·00861	2927	...	,,	
Mahogany	...	88×10^6	...	105×10^4	...	·0120	6265	...	,,	
Oak, European	..	103×10^6	...	105×10^4	...	·0102	5352	...	,,	
Sycamore		73×10^6	...	91×10^4	...	·0125	5670	...	,,	
Teak, Indian		169×10^6	...	105×10^4	...	·00621	3262	...	,,	
Lead, cast	11·215	177×10^6	16×10^5 cms.	22×10^4	...	·0012	...	12	Wertheim.	By direct elong.
,,	...	198×10^6	...	...	...	...	..	...	,,	,, trans. vibr.
,,	...	199×10^6	...	...	...	...	..	...	,,	,, longitud. ,,
Tin, cast	7·404	$417{\cdot}2\times10^6$	56×10^5 cms.	$1{\cdot}6\times10^4$	...	·001	...	28	,,	,, trans. ,,
,,	...	464×10^6	...	...	...	...	..	...	,,	,, longitud. ,,
Cadmium, drawn	8·665	542×10^6	63×10^5 cms.	...	...	...	..	...	,,	,, trans. ,,
,,	...	609×10^6	...	...	...	...	...	...	,,	,, long. ,,
Gold, drawn	18·514	813×10^6	44×10^5 cms.	$(266\text{ to }284)\times10^4$	15×10^4	·0034	...	250	,,	,, direct elong.
,,	...	864×10^6	...	...	..	...	..	...	,,	,, trans. vib.
,,	...	860×10^6	...	...	...	...	...	...	,,	,, long. ,,
Silver, drawn	10·369	736×10^6	71×10^5 cms.	296×10^4	28×10^4	·0041	...	575	,,	,, direct elong.
,,	...	782×10^6	...	...	...	...	..	...	,,	,, trans. vib.
,,	...	758×10^6	...	...	...	...	..	...	,,	,, long. ,,
Zinc, common, drawn	7·008	873×10^6	124×10^5 cms.	158×10^4	23×10^4	·0018	...	204	,,	,, direct elong.
,,	...	879×10^6	...	...	...	...	...	...	,,	,, trans. vibr.
,,	...	955×10^6	...	...	...	...	...	...	,,	,, long. ,,
Palladium	11·35	1175×10^6	104×10^5 cms.	272×10^4	23×10^4	·0023	..	277	,,	,, direct elong.
,,	...	1239×10^6	...	...	...	...	.	...	,,	,, trans. vib.
Copper, drawn	8·933	1245×10^6	139×10^5 cms.	410×10^4	46×10^4	·0033	...	756	,,	,, direct elong.
,, ,,	...	1251×10^6	...	...	...	...	...	...	,,	,, trans. vib.
,, ,,	...	1254×10^6	...	...	...	...	...	...	,,	,, long. ,,
,, annealed	8·936	1052×10^6	118×10^5 cms.	316×10^4	35×10^4	·003	...	531	,,	,, direct elong.
,, ,,	...	1183×10^6	...	...	...	...	...	...	,,	,, trans. vib.
,, ,,	...	1254×10^6	...	...	...	...	...	...	,,	,, long. ,,
Platinum wire, fine	21·166	1593×10^6	75×10^5 cms.	350×10^4	17×10^4	·0022	...	182	,,	,, trans. ,,
,, ,,	...	1618×10^6	...	...	...	...	...	...	,,	,, long. ,,
,, ,, medium	21·275	1704×10^6	...	...	...	...	...	...	,,	,, direct elong.
,, ,,	...	1715×10^6	...	...	...	...	...	...	,,	,, trans. ,,
,, ,,	...	1716×10^6	...	...	...	...	...	...	,,	,, long. ,,
,, ,, thick	21·259	1581×10^6	...	...	...	...	...	...	,,	,, trans. vib.
,, ,,	...	1616×10^6	...	...	...	...	...	...	,,	,, long. ,,
Iron wire, common	7·553	1861×10^6	246×10^5 cms.	$(625\text{ to }651)\times10^4$	85×10^4	·0034	...	1450	,,	,, direct elong.
Steel, cast drawn	7·717	1955×10^6	...	838×10^4	108×10^4	..	...	...	,,	,, ,, ,,
,, ,,	...	1825×10^6	...	...	...	...	...	...	,,	,, trans. vib.
,, ,,	..	1982×10^6	...	...	...	...	...	...	,,	,, long. ,,
Steel wire, English, drawn	7·718	1881×10^6	244×10^5 cms.	$(859\text{ to }991)\times10^4$	125×10^4	·0050	...	2945	,,	,, direct elong.
,,	...	2071×10^6	...	...	...	...	...	...	,,	,, trans. vib.
,,	...	1944×10^6	...	...	...	...	...	...	,,	,, long. ,,
Steel wire, common, tempered blue	7·420	1804×10^6	243×10^5 cms.	...	...	...	...	...	,,	,, direct elong.
English steel, pianoforte wire	7·727	2049×10^6	265×10^5 cms.	2262×10^4	306×10^4	·0115	...	17600	D. M'Farlane	,, ,, ,,
Copper wire	8·9	1150×10^6 to 1200×10^6	...	..	...		...	...	,,	,, ,, ,,

78. A question of great importance in the physical theory of the elasticity of solids, "What changes are produced in the moduluses of elasticity by permanent changes in its molecular condition," has occupied the attention, no doubt, of every "naturalist" who has studied the subject, and valuable contributions to its answer by experiment had been given by Wertheim and other investigators, but solely with reference to Young's modulus. In 1865 an investigation of the effect on the torsional rigidity of wires of different metals, produced by stretching them longitudinally beyond their limits of elasticity, was commenced in the physical laboratory of the university of Glasgow in its old buildings in 1865. The following description of experiments and table of results is extracted from the paper by W. Thomson "On the Elasticity and Viscosity of Metals," already quoted (section 30), with reference to viscosity and fatigue of elasticity.

"To determine rigidities by torsional vibrations, taking advantage of an obvious but most valuable suggestion made to me by Dr Joule, I used as vibrator in each case a thin cylinder of sheet brass, turned true outside and inside (of which the radius of gravitation must be, to a very close degree of approximation, the arithmetic mean of the radii of the outer and inner cylindrical surfaces),[1] supported by a thin flat rectangular bar, of which the square of the radius of gravitation is one-third of the square of the distance from the centre to the corner. The wire to be tested passed perpendicularly through a hole in the middle of the bar, and was there firmly soldered. The cylinder was tied to the middle of the bar by light silk thread so as to hang with its axis vertical. Each wire, after having been suspended and stretched with just force enough to make it as nearly straight as was necessary for accuracy, was vibrated. Then it was stretched by hand (applied to the cross bar soldered to its lower end) and vibrated again, and stretched again, and so on till it broke." The experiments were performed with great care and accuracy by Mr Donald M'Farlane. "The results, as shown in the accompanying table, were most surprising."

The highest and lowest rigidities found for copper in the table are as follows:—

Highest rigidity 473×10^6, being that of a wire which had been softened by heating it to redness and plunging it into water, and which was found to be of density 8·91.

Lowest rigidity $393{\cdot}4\times10^6$, being that of a wire which had been rendered so brittle by heating it to redness surrounded by powdered charcoal in a crucible and letting it cool very slowly, that it could scarcely be touched without breaking it, and which had been found to be reduced in density by this process to as low as 8·674. The wires used were all commercial specimens—those of copper being all, or nearly all, cut from hanks supplied by the Gutta Percha Company, having been selected as of high electric conductivity, and of good mechanical quality, for submarine cables.

It ought to be remarked that the change of molecular condition produced by permanently stretching a wire or solid cylinder of metal is certainly a change from a condition which, if originally isotropic, becomes æolotropic as to some qualities,[2] and that the changed conditions may therefore be presumed to be æolotropic as to elasticity. If so, the rigidities corresponding to the direct and diagonal distortions (indicated by No. 1 and No. 2 in fig. 14) must in all probability become different from one another when a wire is permanently stretched, instead of being equal as they must be when its substance is isotropic. It becomes, therefore, a question of extreme interest to find whether rigidity No. 2 is not *increased* by this process, which, as is proved by the experiments above described, diminishes, to a very remarkable degree, the rigidity No. 1. The most obvious experiment, and indeed the only practicable experiment, adapted to answer this question, for a wire or round bar is that of Cagniard-Latour, in which an accurate determination of the difference produced in the *volume* of the substance is made by applying and removing longitudinal traction within its limits of elasticity. With the requisite apparatus, which must be much more accurate

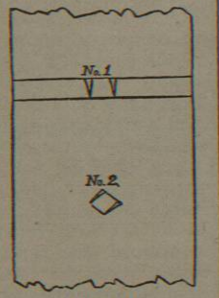

Fig. 14.

[1] It is exactly the square root of the mean of their squares.

[2] For example, see paper "On Electrodynamic Qualities of Metals," *Philosophical Transactions*, 1856, by W. Thomson.

than that of Cagniard-Latour, a most important and interesting investigation might be made. The results, along with an accurate determination of the Young's modulus for the particular case, give (sec. 47) the modulus of compression, and the rigidity No. 2. Regnault suggested the use of hollow instead of solid cylinders, to be subjected to longitudinal pull, and (after the manner of the bulb and tube of a thermometer) a capillary tube to aid in measuring changes of volume of the hollow; and Wertheim, adopting this excellent suggestion, obtained seemingly very accurate results for brass and glass, which are given in the tables of section 77.

Substance.	Length of Wire in centimetres.	Volume in cubic centimetres.	Density.	Moment of Inertia of Vibrator Wk^2.	Time of Vibration one way (or half period) in seconds. T.	Rigidity in grammes weight per square centimetre $\frac{2\pi^3 l^3 Wk}{gT^2V^2}$.
Aluminium [1]	60·3	1·1845	2·764	31771	1·14	241×10^6
Zinc [2]	304·9	2·351	7·105	31896	4·31	$359{\cdot}6\times10^6$
Brass	237·7	...	...	..	4·76	$410{\cdot}3\times10^6$
,,	248·3	...	..	...	5·456	$354{\cdot}8\times10^6$
,,	261·9	1·703	8·398	...	5·96	$350{\cdot}1\times10^6$
Copper	2435·0	15·30	8·91	38186	16·375	$448{\cdot}7\times10^6$
,,	,,	,,	,,	61412	20·77	$448{\cdot}4\times10^6$
Copper [3]	214·4	1·348	8·864	31771	5·015	$433{\cdot}0\times10^6$
,,	,,	,,	,,	61412	6·982	$431{\cdot}8\times10^6$
Copper [4]	143·7	·9096	8·674	..	3·381	$393{\cdot}4\times10^6$
Copper [5]	286·8	...	..	20612	4·245	$442{\cdot}9\times10^6$
,,	291			,,	4·375	$435{\cdot}6\times10^6$
,,	293			,,	4·417	$436{\cdot}2\times10^6$
,,	296·1			,,	4·500	$433{\cdot}8\times10^6$
,,	300·0	.	.	,,	4·588	$434{\cdot}0\times10^6$
,,	303·4	..	.	,,	4·646	$437{\cdot}8\times10^6$
,,	309·3	..	..	,,	4·833	$428{\cdot}6\times10^6$
,,	313·2	...	...	,,	4·931	$427{\cdot}5\times10^6$
,,	317·4	1·962	8·835	,,	5·040	$425{\cdot}9\times10^6$
Copper [6]	315·6	...	...	31771	8·155	$442{\cdot}3\times10^6$
,,	235·5	...	...	,,	9·425	$432{\cdot}2\times10^6$
,,	251·9	·827	8·872	,,	10·463	$428{\cdot}6\times10^6$
Copper [7]	253·2	1·580	8·91	...	5·285	$472{\cdot}9\times10^6$
,,	262·8	...	...	...	5·640	$464{\cdot}3\times10^6$
,,	270·4	.	..	...	5·910	$460{\cdot}4\times10^6$
,,	278·7	..	.	...	6·20	$458{\cdot}5\times10^6$
,,	287·9	...	...	...	6·5325	$455{\cdot}0\times10^6$
,,	297·5	..	...	...	6·8195	$451{\cdot}0\times10^6$
,,	308·8	...	...	..	7·3075	$448{\cdot}9\times10^6$
Copper [8]	256·5	1·6145	8·90	...	4·2226	$463{\cdot}5\times10^6$
,,	267·9	...	...	...	4·5625	$453{\cdot}3\times10^6$
,,	280·1	...	...	..	4·915	$446{\cdot}2\times10^6$
,,	292·2	...	...	...	5·240	$445{\cdot}5\times10^6$
,,	301·9		...	...	5·532	$438{\cdot}2\times10^6$
Soft Iron [9]	316·8	...	...	...	6·655	$791{\cdot}4\times10^6$
,,	322·1	..	...	...	6·88	$778{\cdot}3\times10^6$
,,	335·1	...	...	...	7·301	$779{\cdot}0\times10^6$
,,	347·4	...	...	...	7·768	$766{\cdot}6\times10^6$
,,	366·0	1·357	7·657	...	8·455	$756{\cdot}0\times10^6$
Platinum	39·4	·1745	20·805	20612	2·05	$622{\cdot}25\times10^6$
Gold	65·9	·1825	19·8	10902	..	281×10^6
Silver	75·7	·1185	10·21	10967	...	270×10^6

Remarks.

[1] Only forty vibrations from initial arc of convenient amplitude could be counted. Had been stretched considerably before this experiment.

[2] So viscous that only twenty vibrations could be counted. Broke in stretching.

[3] A piece of the preceding stretched.

[4] The preceding made red-hot in a crucible filled with powdered charcoal and allowed to cool slowly, became very brittle: a part of it with difficulty saved for the experiment.

[5] Another piece of the long (2435 centims.) wire; stretched by successive simple tractions.

[6] A finer gauge copper wire; stretched by successive tractions.

[7] A finer gauge copper wire, softened by being heated to redness and plunged in water. A length of 260 centimetres cut from this, suspended, and elongated by successive tractions.

[8] Another length of 260 centimetres cut from the same, and similarily treated.

[9] One piece, successively elongated by simple tractions till it broke.

79. The following tables show the effects of differences of temperature on the Young's Modulus, rigidity-modulus, and modulus of compressibility of various substances:—

Substance.	Density.	Young's Modulus in million grms. per square centimetre.		
		15°	100°	200°
Lead	11·232	173	163	...
Gold	18·035	558	531	548
Silver	10·304	715	727	637
Palladium	11·225	979	...	...
Copper	8·936	1052	938	786
Platinum	21·083	1552	1418	1296
Steel, drawn, English	7·622	1728	2129	1928
Cast steel	7·919	1956	1901	1792
Iron, Berry	7·757	2079	2188	1770

The above results are from Wertheim's "Mémoires" on Elasticity, *Ann. de Chim. et Phys.*, tom xii. (1844).

The change in the rigidity-modulus produced by change of temperature was investigated by Kohlrausch. He found that it is expressed by the formula $n = n_0 (1 - \alpha t - \beta t^2)$, where n_0 denotes the value of the rigidity-modulus at 0° C., n its value at temperature t, and α, β coefficients the values of which for iron, copper, and brass are as follows:—

	α	β
Iron	0·000447	0·00000052
Copper	0·000520	0·00000028
Brass	0·000428	0·00000136

Moduluses of Compressibility of Water, Alcohol, and Ether at Different Temperatures.[1]

Temp. Cent.	Modulus of compressibility in grammes per square centimetre.			Authority.
	Water.	Alcohol.	Ether.	
0°	$20·6 \times 10^6$	$12·4 \times 10^6$	$9·5 \times 10^6$	For water, Grassi, *Ann. de Chim.*, tome xxxi. (1851).
1·5	$20·2 \times 10^6$	...	...	
4·1	$20·7 \times 10^6$	...	...	
10·8	$21·5 \times 10^6$	...	...	
13·4	$21·6 \times 10^6$	...	...	
14·0	......	...	$8·07 \times 10^6$	For ether and alcohol, Amaury and Descamp, *Comptes Rendus*, tome xvii. p. 1564 (1869).
15·0	......	$11·4 \times 10^6$	..	
18·0	$22·4 \times 10^6$	...	...	
25·0	$22·6 \times 10^6$	...	...	
34·0	$22·8 \times 10^6$	...	...	
43·0	$23·3 \times 10^6$	...	...	
53·0	$23·5 \times 10^6$	...	...	

80. *Tempering soft iron by long-continued stress.*—Preliminary experiments by Mr J. T. Bottomley towards the investigation promised in section 5 above have discovered a very remarkable property of soft iron wire respecting its ultimate tensile strength. Eight different specimens, tested by the gradual application of more and more weight within ten minutes of time in each case until the wire broke, bore from 43½ to 46 ℔ (average 45·2) just before breaking, with elongations of from 17 per cent to 22 per cent. Another specimen left with 43 ℔ hanging on it for 24 hours, and then tested by the gradual addition of weights during 25 minutes till it broke, bore 49¼ ℔ before breaking, with elongation of 15 per cent. Another left for 3 days 11 hours 40 minutes with 43 ℔ hanging on it, and then tested by the gradual addition of weights during 34 minutes till it broke, bore 51½ ℔ just before breaking, with elongation of 14·4 per cent. Another specimen of the same wire was set up with 40 ℔ hanging on it on the 5th of July 1877, on the 6th of July 3 ℔ were added, on the 9th 1½ ℔ more, and on the 10th ¾ ℔ more, making in all on this date 45¼ ℔. Thenceforward day by day, with occasional intervals of two days or three days, the weight was increased first by half a pound at a time, and latterly by a quarter of a pound at a time, until on the 3d of September the wire broke with 57¼ ℔ (elongation not recorded). This gradual addition of weight therefore had increased the tensile strength of the metal by 26·7 per cent.!

81. *Experiments made for this article.*—There are many subjects in the theory of elasticity regarding which information to be obtained by experiment only is greatly wanted. Several of these have been pointed out above (section 21), and while this article was being put in type, experiments were made in the physical laboratory of the university of Glasgow with a view of answering some of the questions proposed. Mr Donald M'Farlane, besides making the experiments referred to in sections 3 and 21, investigated the effects of applying different amounts of pull to a steel pianoforte wire which had been twisted to nearly its limits of elasticity, and which was kept twisted by means of a couple. The results proved a deviation from Hooke's law by showing a diminution of the torsional rigidity, of about 1·6 per cent., produced by hanging a weight of 112 ℔ on the wire. Of this 1·2 per cent. is accounted for by elongation and by shrinkage of the diameter, leaving ·4 per cent. of diminution of the rigidity-modulus.

It was also found that when the wire was twisted far beyond its limits of elasticity, and then freed from torsional stress, a weight hung on it caused it to untwist slightly. When the weight was removed and reapplied again and again, the lower end of the wire always turned in the same direction as the permanent twist when the weight was removed, and in the opposite direction when it was applied. This result shows the development of æolotropic quality in the substance of the wire, according to which a small cube cut from any part of it far out from the axis, with two sides of the cube parallel to the length, and the other two pairs of sides making angles of 45° with the length, would show different compressibilities in the directions perpendicular to the last-mentioned pairs of sides.

Another very interesting result, discovered in the course of these experiments, was that when a length of five metres of the steel wire, with a weight of 39 ℔ hung upon it, was twisted to the extent of 95 turns, it became gradually elongated to the extent of $\frac{1}{1000}$ of the length of the wire; when farther twisted it began to shorten till, when 25 turns had been given (in all 120 turns), the weight had risen from its lowest position through nearly $\frac{1}{5000}$ of the length of the wire, so that the previous elongation had been diminished by about ¼ of its amount.

Experiments were also made by Mr Andrew Gray and Mr Thomas Gray for the purpose of determining the effects of various amounts of permanent twist in altering the rigidity-modulus and the Young's modulus of wires of copper, iron, and steel. A copper wire, of 3·15 metres in length and ·154 centimetre diameter, No. 17 B.W.G., which had a rigidity-modulus of 442 million grammes per square centimetre to begin with, was found to have 420 after 10 turns, showing a diminution in the modulus of $\frac{1}{20}$ of its own amount. The diminution went on rapidly until 100 turns of permanent twist had been given, when the modulus was as low as 385. The diminution of the modulus continued with further twist, but very slowly, up to 1225 turns, when the modulus was found to be 371, showing a diminution to the extent of $\frac{1}{6}$ of its original value! There was little farther change until 1400 turns had been given, when the modulus began to increase. At 1525 turns its value was 373, and at 1625 it was 377. Twenty turns more broke the wire before the torsional elasticity had been again determined.

A piece of iron wire of nearly the same length, about three metres, but of smaller diameter (·087 centimetre), showed continued diminution of torsional rigidity as far as 1350 turns of permanent twist, when the diminution had amounted to 14 per cent. of the primitive value, 36 turns more broke the wire before another determination of torsional rigidity had been made.

The steel pianoforte wire also showed a diminution of torsional rigidity with permanent twist, and (as did the copper wire) showed first a diminution and then a slight augmentation. The amount of the diminution in the steel wire was enormously greater than the surprisingly great amount which had been discovered in the copper wire, and the ultimate augmentation was considerably greater in the steel than what it had been in the copper before rupture. Thus after 473 turns of permanent twist the torsional modulus had diminished from 751 million grammes per square centimetre to 414! 95 more turns of permanent twist augmented the rigidity from 414 to 430, and when farther twisted the wire broke before another observation had been made. The vibrator used in these experiments was a cylinder of lead weighing 56 ℔, which was kept hanging on the wire while it was being twisted, and in fact during the whole of about 100 hours from the beginning of the experiment till the wire broke, except on two occasions for a few minutes, while the top fastening which had given way was being resoldered. The period of vibration was augmented from 39·375 seconds to 51·9 seconds by the twist. The wire took the twist very irregularly, some parts not beginning to show much signs of permanent twist till near the end of the experiment

In two specimens of copper wire of the same length and gauge as those described above, the Young's modulus was found to be increased 10 per cent. by 100 turns of permanent twist.

Five metres of the steel pianoforte wire, bearing a weight of 39 ℔, was in one of Mr M'Farlane's experiments twisted 120 turns, and then allowed to untwist, and 38¼ turns came out, leaving the wire in equilibrium with 81¾ turns of permanent twist. Its Young's modulus was then found not to differ as much as ½ per cent. from the value it had before the wire was twisted.

MATHEMATICAL THEORY OF ELASTICITY.[1]

Part I.—On Stresses and Strains.[2]

Chapter I.—*Initial Definitions and Explanations.*

Def. A stress is an equilibrating application of force to a body.

Cor. The stress on any part of a body in equilibrium will thus signify the force which it experiences from the matter touching that part all round, whether entirely homogeneous with itself, or only so across a portion of its bounding surface.

Def. A strain is any definite alteration of form or dimensions experienced by a solid.

Examples.—Equal and opposite forces acting at the two ends of a wire or rod of any substance constitute a stress upon it. A body pressed equally all round—for instance, any mass touched by air on all sides—experiences a stress. A stone in a building experiences stress if it is pressed upon by other stones, or by any parts of the structure, in contact with it. Any part of a continuous solid mass, simply resting on a fixed base, experiences stress from the surrounding parts in consequence of their weight. The different parts of a ship in a heavy sea experience stresses from which they are exempt when the water is smooth.

If a rod of any substance become either longer or shorter, it is said to experience a strain. If a body be uniformly condensed in all directions it experiences a strain. If a stone, a beam, or a mass of metal in a building, or in a piece of framework, becomes condensed or dilated in any direction, or bent, or twisted, or distorted in any way, it is said to experience a strain, to become strained, or often in common language, simply "to strain." A ship is said to "strain" if in launching, or when working in a heavy sea, the different parts of it experience relative motions.

Chapter II.—*Homogeneous Stresses and Homogeneous Strains.*

Def. A stress is said to be homogeneous throughout a body when equal and similar portions of the body, with corresponding lines parallel, experience equal and parallel pressures or tensions on corresponding elements of their surfaces.

Cor. When a body is subjected to any homogeneous stress, the mutual tension or pressure between the parts of it on two sides of any plane amounts to the same per unit of surface as that between the parts on the two sides of any parallel plane; and the former tension or pressure is parallel to the latter.

A strain is said to be homogeneous throughout a body, or the body is said to be homogeneously strained, when equal and similar portions, with corresponding lines parallel, experience equal and similar alterations of dimensions

Cor. All the particles of the body in parallel planes remain in parallel planes, when the body is homogeneously strained in any way.

Examples.—A long uniform rod, if pulled out, or a pillar loaded with a weight, will experience a uniform strain, except near its ends. There will be a sensible heterogeneousness of the strain, because of the end attachments, or other circumstances preventing the ends from expanding laterally to the same extent as the middle does.

A piece of cloth held in a plane, and distorted so that a warp and woof, instead of being perpendicular to one another, become two sets of parallels cutting one another obliquely, experiences a homogeneous strain. The strain is heterogeneous as to intensity, from the axis to the surface of a cylindrical wire under torsion, and heterogeneous as to direction in different positions in a circle round the axis.

Chapter III.—*On the Distribution of Force in a Stress.*

Theorem.—In every homogeneous stress there is a system of three rectangular planes, each of which is perpendicular to the direction of the mutual force between the parts of the body on its two sides.

For let P(X), P(Y), P(Z) denote the components, parallel to X, Y, Z, any three rectangular lines of reference, of the force experienced per unit of surface at any portion of the solid bounded by a plane parallel to (Y, Z); Q(X), Q(Y), Q(Z), the corresponding components of the force experienced by any surface of the solid parallel to (Z, X); and R(X), R(Y), R(Z), those of the force at a surface parallel to (X, Y). Now, by considering the equilibrium of a cube of the solid with faces parallel to the planes of reference (fig. 15), we see that the couple of forces Q(Z) on its two faces perpendicular to Y is balanced by the couple of forces R(Y) on the faces perpendicular to Z. Hence we must have

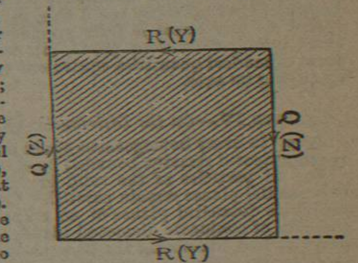

Fig. 15

$$Q(Z) = R(Y).$$

Similarly it is seen that

$$R(X) = P(Z)$$

and

$$P(Y) = Q(X).$$

For the sake of brevity, these pairs of equal quantities (being tangential forces respectively perpendicular to X, Y, Z) may be denoted by T(X), T(Y), T(Z).

Consider a tetrahedral portion of the body (surrounded it may be with continuous solid) contained within three planes A, B, C, through a point O parallel to the planes of the pairs of lines of reference, and a third plane K cutting these at angles α, β, γ respectively; so that as regards the areas of the different sides we shall have

$$A = K \cos\alpha, \quad B = K \cos\beta, \quad C = K \cos\gamma.$$

The forces actually experienced by the sides A, B, C have nothing to balance them except the force actually experienced by K. Hence those three forces must have a single resultant, and the force on K must be equal and opposite to it. If, therefore, the force on K per unit of surface be denoted by F, and its direction cosines by l, m, n, we have

$$F.K.l = P(X)A + T(Z)B + T(Y)C,$$
$$F.K.m = T(Z)A + Q(Y)B + T(X)C,$$
$$F.K.n = T(Y)A + T(X)B + R(Z)C;$$

and, by the relations between the cases stated above, we deduce

$$Fl = P(X)\cos\alpha + T(Z)\cos\beta + T(Y)\cos\gamma$$
$$Fm = T(Z)\cos\alpha + Q(Y)\cos\beta + T(X)\cos\gamma$$
$$Fn = T(Y)\cos\alpha + T(X)\cos\beta + R(Z)\cos\gamma$$

Hence the problem of finding (α, β, γ), so that the force $F(l, m, n)$ may be perpendicular to it, will be solved by substituting $\cos\alpha$, $\cos\beta$, $\cos\gamma$ for l, m, n in these equations. By the elimination of $\cos\alpha$, $\cos\beta$, $\cos\gamma$ from the three equations thus obtained, we have the well-known cubic determinantal equation, of which the roots, necessarily real, lead, when no two of them are equal, to one and only one system of three rectangular axes having the stated property.

[1] The modulus seems to be a minimum near the temperature of maximum density.

[1] The substance of Chap. I.-XVI. of this part of the present article was read before the Royal Society by Prof. Wm. Thomson, M.A., F.R.S., April 24, 1856, and published in the *Transactions*. Chap. XVII., containing the mathematical theory of Waves in an æolotropic or isotropic elastic solid, is new.

[2] These terms were first definitively introduced into the Theory of Elasticity by Rankine, and have been found very valuable in writing on the subject. It will be seen that there is a slight deviation from Rankine's definition of the word "stress." It is here applied to the direct action experienced by a body from the matter around it, and not, as proposed by him, to the elastic reaction of the body equal and opposite to that action.