Def. The three lines thus proved to exist for every possible homogeneous stress are called its axes. The planes of their pairs are called its normal planes; the mutual forces between parts of the body separated by these planes, or the forces on portions of the bounding surface parallel to them, are called the principal tensions.

Cor. 1. The Principal Tensions of the stress are the roots of the determinant cubic referred to in the demonstration.

Cor. 2. If a stress be specified by the notation P(X), &c., as explained above, its normal planes are the principal planes of the surface of the second degree whose equation is

$$P(X)X^2+Q(Y)Y^2+R(Z)Z^2+2T(X)YZ+2T(Y)ZX$$

and its principal tensions are equal to the reciprocals of the squares of the lengths of the semi-principal-axes of the same surface (quantities which are negative of course for the principal axis or axes which do not cut the surface when the surface is a hyperboloid of one or of two sheets).

Cor. 3. The ellipsoid whose equation, referred to the rectangular axes of a stress, is

$$(1-2eF)X^2+(1-2eG)Y^2+(1-2eH)Z^2=1,$$

where F, G, H denote the principal tensions, and e any infinitely small quantity, *represents the stress*, in the following manner:—

From any point P in the surface of the ellipsoid draw a line in the tangent plane half-way to the point where this plane is cut by a perpendicular to it through the centre; and from the end of the first-mentioned line draw a radial line to meet the surface of a sphere of unit radius concentric with the ellipsoid. The tension at this point of the surface of a sphere of the solid is in the line from it to the point P; and its amount per unit of surface is equal to the length of that infinitely small line, divided by e.

Cor. 4. Any stress is fully specified by six quantities, viz., its three principal tensions (F, G, H), and three angles (θ, ϕ, ψ) or three numerical quantities equivalent to the nine direction cosines specifying its axes.

Chapter IV.—*On the Distribution of Displacement in a Strain.*

Prop. In every homogeneous strain any part of the solid bounded by an ellipsoid remains bounded by an ellipsoid.

For all particles of the solid in a plane remain in a plane, and two parallel planes remain parallel. Consequently every system of conjugate diametral planes of an ellipsoid of the solid retain the property of conjugate diametral planes with reference to the altered curve surface containing the same particles. This altered surface is therefore an ellipsoid.

Prop. There is a single system (and only a single system, except in the cases of symmetry) of three rectangular planes for every homogeneous strain, which remain at right angles to one another in the altered solid.

Def. 1. These three planes are called the normal planes of the strain, or simply the strain-normals. Their lines of intersection are called the axes of the strain. The elongations of the solid per unit of length along these axes or perpendicular to these planes are called the Principal Elongations of the strain.

Remark. The preceding propositions and definitions are not limited to infinitely small strains, but are applicable to whatever extent the body may be strained.

Prop. If a body, while experiencing an infinitely small strain, be held with one point fixed and the normal planes of the strain parallel to three fixed rectangular planes through the point O, a sphere of the solid of unit radius having this point for its centre becomes, when strained, an ellipsoid, whose equation, referred to the strain-normals through O, is

$$(1-2x)X^2+(1-2y)Y^2+(1-2z)Z^2=1,$$

if x, y, z denote the elongations of the solid per unit of length, in the directions respectively perpendicular to these three planes; and the position, on the surface of this ellipsoid, attained by any particular point of the solid, is such that if a line be drawn in the tangent plane, half-way to the point of intersection of this plane with a perpendicular from the centre, a radial line drawn through its extremity cuts the primitive spherical surface in the primitive position of that point.

Cor. 1. For every stress, there is a certain infinitely small strain, and conversely, for every infinitely small strain, there is a certain stress, so related that if, while the strain is being acquired, the centre and the strain-normals through it are unmoved, the absolute displacements of particles belonging to a spherical surface of the solid represent, in intensity (according to a definite convention as to units for the representation of force by lines) and in direction, the force (reckoned as to intensity, in amount per unit of area) experienced by the enclosed sphere of the solid, at the different parts of its surface, when subjected to the stress.

Cor. 2. Any strain is fully specified by six quantities, viz., its three principal elongations, and three angles (θ, ϕ, ψ), or nine direction cosines, equivalent to three independent quantities specifying its axes.

Def. 2. A stress and an infinitely small strain related in the manner defined in Cor. 1, are said to be of the same type. The ellipsoid by means of which the distribution of force over the surface of a sphere of unit radius is represented in one case, and by means of which the displacements of particles from the spherical surface are shown in the other, may be called the geometrical type of either.

Cor. Any stress- or strain-type is fully specified by *five* quantities, viz., two ratios between its principal strains or elongations and three quantities specifying the angular position of its axes.

Chapter V.—*Conditions of Perfect Concurrence between Stresses and Strains.*

Def. 1. Two stresses are said to be coincident in direction, or to be perfectly concurrent, when they only differ in absolute magnitude. The same relative designations are applied to two strains differing from one another only in absolute magnitude.

Cor. If two stresses or two strains differ by one being reverse to the other, they may be said to be negatively coincident in direction, or to be directly opposed or directly contrary to one another.

Def. 2. When a homogeneous stress is such that the normal component of the mutual force between the parts of the body on the two sides of any plane whatever through it is proportional to the augmentation of distance between the same plane and another parallel to it and initially at unity of distance, due to a certain strain experienced by the same body, the stress and the strain are said to be perfectly concurrent; also to be coincident in direction. The body is said to be yielding directly to a stress applied to it, when it is acquiring a strain thus related to the stress; and in the same circumstances, the stress is said to be working directly on the body, or to be acting in the same direction as the strain.

Cor. 1. Perfectly concurrent stresses and strains are of the same type.

Cor. 2. If a strain is of the same type as the stress, its reverse will be said to be negatively of the same type, or to be directly opposed to the strain. A body is said to be working directly against a stress applied to it when it is acquiring a strain directly opposed to the stress; and in the same circumstances, the matter round the body is said to be yielding directly to the reactive stress of the body upon it.

Chapter VI.—*Orthogonal Stresses and Strains.*

Def. 1. A stress is said to act right across a strain, or to act orthogonally to a strain, or to be orthogonal to a strain, if work is neither done upon nor by the body in virtue of the action of the stress upon it while it is acquiring the strain.

Def. 2. Two stresses are said to be orthogonal when either coincides in direction with a strain orthogonal to the other.

Def. 3. Two strains are said to be orthogonal when either coincides in direction with a stress orthogonal to the other.

Examples.—(1) A uniform cubical compression, and any strain involving no alteration of volume, are orthogonal to one another.

(2) A simple extension or contraction in parallel lines unaccompanied by any transverse extension or contraction, that is, "a simple longitudinal strain," is orthogonal to any similar strain in lines at right angles to those parallels.

(3) A simple longitudinal strain is orthogonal to a "simple tangential strain"[1] in which the sliding is parallel to its direction or at right angles to it.

(4) Two infinitely small simple tangential strains in the same plane,[2] with their directions of sliding mutually inclined at an angle of 45°, are orthogonal to one another.

(5) An infinitely small simple tangential strain is orthogonal to every infinitely small simple tangential strain, in a plane either parallel to its plane of sliding or perpendicular to its line of sliding.

Chapter VII.—*Composition and Resolution of Stresses and of Strains.*

Any number of simultaneously applied homogeneous stresses are equivalent to a single homogeneous stress which is called their resultant. Any number of superimposed homogeneous strains are equivalent to a single homogeneous resultant strain. Infinitely small strains may be independently superimposed; and in what follows it will be uniformly understood that the strains spoken of are infinitely small, unless the contrary is stated.

Examples.—(1) A strain consisting simply of elongation in one set of parallel lines, and a strain consisting of equal contraction in a direction at right angles to it, applied together, constitute a single strain, of the kind which that described in Example (3) of the preceding chapter is when infinitely small, and is called a plane distortion, or a simple distortion. It is also sometimes called a simple tangential strain, and when so considered, its plane of sliding may be regarded as either of the planes bisecting the angles between planes normal to the lines of the component longitudinal strains.

(2) Any two simple distortions in one plane may be reduced to a single simple distortion in the same plane.

(3) Two simple distortions not in the same plane have for their resultant a strain which is a distortion unaccompanied by change of volume, and which may be called a compound distortion.

(4) Three equal longitudinal elongations or condensations in three directions

[1] That is, a homogeneous strain in which all the particles in one plane remain fixed, and other particles are displaced parallel to this plane.

[2] "The plane of a simple tangential strain," or the plane of distortion in a simple tangential strain, is a plane perpendicular to that of the particles supposed to be held fixed, and parallel to the lines of displacement of the others.

at right angles to one another are equivalent to a single dilatation or condensation equal in all directions. The single stress equivalent to three equal tensions or pressures in directions at right angles to one another is a negative or positive pressure equal in all directions.

(5) If a certain stress or infinitely small strain be defined (Chapter III. Cor. 3, or Chapter IV.) by the ellipsoid

$$(1+A)X^2+(1+B)Y^2+(1+C)Z^2+DYZ+EZX+FXY=1,$$

and another stress or infinitely small strain by the ellipsoid

$$(1+A')X^2+(1+B')Y^2+(1+C')Z^2+D'YZ+E'ZX+F'XY=1,$$

where A, B, C, D, E, F, &c., are all infinitely small, their resultant stress or strain is that represented by the ellipsoid

$$(1+A+A')X^2+(1+B+B')Y^2+(1+C+C')Z^2+(D+D')YZ+(E+E')ZX+(F+F')XY=1.$$

Chapter VIII.—*Specification of Strains and Stresses by their Components according to chosen Types.*

Prop. Six stresses or six strains of six distinct arbitrarily chosen types may be determined to fulfil the condition of having a given stress or a given strain for their resultant, provided those six types are so chosen that a strain belonging to any one of them cannot be the resultant of any strains whatever belonging to the others.

For, just six independent parameters being required to express any stress or strain whatever, the resultant of any set of stresses or strains may be made identical with a given stress or strain by fulfilling six equations among the parameters which they involve; and therefore the magnitudes of six stresses or strains belonging to the six arbitarily chosen types may be determined, if their resultant be assumed to be identical with the given stress or strain.

Cor. Any stress or strain may be numerically specified in terms of numbers expressing the amounts of six stresses or strains of six arbitrarily chosen types which have it for their resultant.

Types arbitrarily chosen for this purpose will be called types of reference. The specifying elements of a stress or strain will be called its components according to types of reference. The specifying elements of a strain may also be called its coordinates, with reference to the chosen types.

Examples.—(1) Six strains in each of which one of the six edges of a tetrahedron of the solid is elongated while the others remain unchanged, may be used as types of reference for the specification of any kind of strain or stress. The ellipsoid representing any one of those six types will have its two circular sections parallel to the faces of the tetrahedron which do not contain the stretched side.

(2) Six strains consisting, any one of them, of an infinitely small alteration either of one of the three edges, or of one of the three angles between the faces, of a parallelepiped of the solid, while the other five angles and edges remain unchanged, may be taken as types of reference, for the specification of either stresses or strains. In some cases, as for instance in expressing the probable elastic properties of a crystal of Iceland spar, it might possibly be convenient to use an oblique parallelepiped for such a system of types of reference; but more frequently it will be convenient to adopt a system of types related to the deformations of a cube of the solid.

Chapter IX.—*Orthogonal Types of Reference.*

Def. A normal system of types of reference is one in which the strains or stresses of the different types are all six mutually orthogonal (fifteen conditions). A normal system of types of reference may also be called an orthogonal system. The elements specifying, with reference to such a system, any stress or strain, will be called orthogonal components or orthogonal coordinates.

Examples.—(1) The six types described in Example (2) of Chapter VIII. are clearly orthogonal, if the parallelepiped referred to is rectangular. Three of these are simple longitudinal extensions, parallel to the three sets of rectangular edges of the parallelepiped. The remaining three are plane distortions parallel to the faces, their axes bisecting the angles between the edges. They constitute the system of types of reference uniformly used hitherto by writers on the theory of elasticity.

(2) The six strains in which a spherical portion of the solid is changed into ellipsoids having the following equations—

$$\begin{aligned}(1+A)X^2+Y^2+Z^2&=1\\ X^2+(1+B)Y^2+Z^2&=1\\ X^2+Y^2+(1+C)Z^2&=1\\ X^2+Y^2+Z^2+DYZ&=1\\ X^2+Y^2+Z^2+EZX&=1\\ X^2+Y^2+Z^2+FXY&=1,\end{aligned}$$

are of the same kind as those considered in the preceding example, and therefore constitute a normal system of types of reference. The resultant of the strains specified, according to those equations, by the elements A, B, C, D, E, F, is a strain in which the sphere becomes an ellipsoid whose equation—see above, Chapter VII. Ex. (5)—is

$$(1+A)X^2+(1+B)Y^2+(1+C)Z^2+DYZ+EZX+FXY=1$$

(3)[1] A compression equal in all directions (I.), three simple distortions having their planes at right angles to one another and their axes[2] bisecting the angles between the lines of intersection of these planes (II.) (III.) (IV.), any simple or compound distortion consisting of a combination of longitudinal strains parallel to those lines of intersections (V.), and the distortion (VI.), constituted from the same elements which is orthogonal to the last, afford a system of six mutually orthogonal types which will be used as types of reference below in expressing the elasticity of cubically isotropic solids. (Compare Chapter X. Example 7 below.)

[1] This example, as well as (7) of Chapter X. (5) of XI., and the example of Chapter XII., are intended to prepare for the application of the theory of Principal Elasticities to cubically and spherically isotropic bodies, in Part II. Chapter XV.

[2] The "axes of a simple distortion" are the lines of its two component longitudinal strains.

Chapter X.—*On the Measurement of Strains and Stresses*

Def. Strains of any types are said to be to one another in the same ratios as stresses of the same types respectively, when any particular plane of the solid acquires, relatively to another plane parallel to it, motions in virtue of those strains which are to one another in the same ratios as the normal components of the forces between the parts of the solid on the two sides of either plane due to the respective stresses.

Def. The magnitude of a stress and of a strain of the same type are quantities which, multiplied one by the other, give the work done on unity of volume of a body acted on by the stress while acquiring the strain.

Cor. 1. If x, y, z, ξ, η, ζ denote orthogonal components of a certain strain, and if P, Q, R, S, T, U denote components, of the same type respectively, of a stress applied to a body while acquiring that strain, the work done upon it per unit of its volume will be

$$Px+Qy+Rz+S\xi+T\eta+U\zeta.$$

Cor. 2. The condition that two strains or stresses specified by (x, y, z, ξ, η, ζ) and (x', y', z', ξ', η', ζ'), in terms of a normal system of types of reference, may be orthogonal to one another is

$$xx'+yy'+zz'+\xi\xi'+\eta\eta'+\zeta\zeta'=0.$$

Cor. 3. The magnitude of the resultant of two, three, four, five, or six mutually orthogonal strains or stresses is equal to the square root of the sum of their squares. For if P, Q, &c., denote several orthogonal stresses, and F the magnitude of their resultant; and x, y, &c., a set of proportional strains of the same types respectively, and r the magnitude of the single equivalent strain, the resultant stress and strain will be of one type, and therefore the work done by the resultant stress will be Fr. But the amounts done by the several components will be Px, Qy, &c., and therefore

$$Fr=Px+Qy+\&c.$$

Now we have, to express the proportionality of the stresses and strains,

$$\frac{P}{x}=\frac{Q}{y}=\&c.=\frac{F}{r}.$$

Each member must be equal to

$$\frac{P^2+Q^2+\&c.}{Px+Qy+\&c.}$$

and also equal to

$$\frac{Px+Qy+\&c.}{x^2+y^2+\&c.}$$

Hence $\frac{F}{r}=\frac{P^2+Q^2+\&c.}{Fr}$, which gives $F^2=P^2+Q^2+\&c.$

and $\frac{F}{r}=\frac{Fr}{x^2+y^2+\&c.}$, which gives $r^2=x^2+y^2+\&c.$

Cor. 4. A definite stress of some particular type chosen arbitrarily may be called unity; and then the numerical reckoning of all strains and stresses becomes perfectly definite.

Def. A uniform pressure or tension in parallel lines, amounting in intensity to the unit of force per unit of area normal to it, will be called a stress of unit magnitude, and will be reckoned as positive when it is tension, and negative when pressure.

Examples.—(1) Hence the magnitude of a simple longitudinal strain, in which lines of the body parallel to a certain direction experience elongation to an extent bearing the ratio κ to their original dimensions, must be called κ.

(2) The magnitude of the single stress equivalent to three simple pressures in directions at right angles to one another each unity is $-\sqrt{3}$; a uniform compression in all directions of unity per unit of surface is a negative stress equal to $\sqrt{3}$ in absolute value.

(3) A uniform dilatation in all directions, in which lineal dimensions are augmented in the ratio $1:1+x$, is a strain equal in magnitude to $x\sqrt{3}$; or a uniform "cubic expansion" E is a strain equal to $\frac{E}{\sqrt{3}}$.

(4) A stress compounded of unit pressure in one direction and an equal tension in a direction at right angles to it, or which is the same thing, a stress compounded of two balancing couples of unit tangential tensions in planes at angles of 45° to the direction of those forces, and at right angles to one another amounts in magnitude to $\sqrt{2}$.

(5) A strain compounded of a simple longitudinal extension x, and a simple longitudinal condensation of equal absolute value, in a direction perpendicular to it, is a strain of magnitude $x\sqrt{2}$; or, which is the same thing (if $\sigma=2x$), a simple distortion such that the relative motion of two planes at unit distances parallel to either of the planes bisecting the angles between the two planes mentioned above is a motion σ parallel to themselves, is a strain amounting in magnitude to $\frac{\sigma}{\sqrt{2}}$.

(6) If a strain be such that a sphere of unit radius in the body becomes an ellipsoid whose equation is

$$(1-A)X^2+(1-B)Y^2+(1-C)Z^2-DYZ-EZX-FXY=1,$$

the values of the component strains corresponding, as explained in Example (2) of Chap. IX., to the different coefficients respectively, are

$$\tfrac{1}{2}A,\ \tfrac{1}{2}B,\ \tfrac{1}{2}C,\ \frac{D}{2\sqrt{2}},\ \frac{E}{2\sqrt{2}},\ \frac{F}{2\sqrt{2}}.$$

For the components corresponding to A, B, C are simple longitudinal strains, in which diameters of the sphere along the axes of coordinates become elongated from 2 to 2 + A, 2 + B, 2 + C respectively; D is a distortion in which diameters in the plane YOZ, bisecting the angles YOZ and Y′OZ, become respectively elongated and contracted from 2 to 2 + ½D, and from 2 to 2 − ½D; and so for the others. Hence, if we take x, y, z, ξ, η, ζ to denote the magnitudes of six component strains, according to the orthogonal system of types described in Examples (1) and (2) of Chap. IX., the resultant strain equivalent to them will be one in which a sphere of radius 1 in the solid becomes an ellipsoid whose equation is

$$(1-2x)X^2+(1-2y)Y^2+(1-2z)Z^2-2\sqrt{2}(\xi YZ+\eta ZX+\zeta XY)=1,$$

and its magnitude will be

$$\sqrt{(x^2+y^2+z^2+\xi^2+\eta^2+\zeta^2)}.$$

(7) The specifications, according to the system of reference used in the preceding Example, of the unit strains of the six orthogonal types defined in Example (3) of Chap. IX. are respectively as follows:—

	x	y	z	ξ	η	ζ
(I.)	$\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}$	0	0	0
(II.)	0	0	0	1	0	0
(III.)	0	0	0	0	1	0
(IV.)	0	0	0	0	0	1
(V.)	l	m	n	0	0	0
(VI.)	l'	m'	n'	0	0	0

where l, m, n, l', m', n' denote quantities fulfilling the following conditions:—

$$\begin{aligned} l^2+m^2+n^2&=1,\\ l+m+n&=0,\\ ll'+mm'+nn'&=0,\\ l'^2+m'^2+n'^2&=1,\\ l'+m'+n'&=0. \end{aligned}$$

(8) If $(1-2eP)X^2+(1-2eQ)Y^2+(1-2eR)Z^2-2e\sqrt{2}(SYZ+TZX+UXY)=1$ be the equation of the ellipsoid representing a certain stress, the amount of work done by this stress, if applied to a body while acquiring the strain represented by the equation in the preceding example, will be

$$Px+Qy+Rz+S\xi+T\eta+U\zeta.$$

Cor. Hence, if variables X, Y, Z be transformed to any other set (X′, Y′, Z′) fulfilling the condition of being the coordinates of the same point, referred to another system of rectangular axes, the coefficients x, y, z, &c., $x_{,}, y_{,}, z_{,}$, &c., in two homogeneous quadratic functions of three variables,

$$(1-2x)X^2+(1-2y)Y^2+(1-2z)Z^2-2\sqrt{2}(\xi YZ+\eta ZX+\zeta XY)$$

and

$$(1-2x_{,})X^2+(1-2y_{,})Y^2+(1-2z_{,})Z^2-2\sqrt{2}(\xi_{,}YZ+\eta_{,}ZX+\zeta_{,}XY),$$

and the corresponding coefficients x', y', z', &c., $x'_{,}, y'_{,}, z'_{,}$, &c., in these functions transformed to x', y', z', will be so related that

$$x'x'_{,}+y'y'_{,}+z'z'_{,}+\xi'\xi'_{,}+\eta'\eta'_{,}+\zeta'\zeta'_{,}=xx_{,}+yy_{,}+zz_{,}+\xi\xi_{,}+\eta\eta_{,}+\zeta\zeta_{,};$$

or the function $xx_{,}+yy_{,}+zz_{,}+\xi\xi_{,}+\eta\eta_{,}+\zeta\zeta_{,}$ of the coefficients is an "invariant" for linear transformations fulfilling the conditions of transformation from one to another set of rectangular axes. Since $x+y+z$ and $x_{,}+y_{,}+z_{,}$ are clearly invariants also, it follows that $AA_{,}+BB_{,}+CC_{,}+2DD_{,}+2EE_{,}+2FF_{,}$ is an invariant function of the coefficients of the two quadratics

$$AX^2+BY^2+CZ^2+2DYZ+2EZX+2FXY$$

and

$$A_{,}X^2+B_{,}Y^2+C_{,}Z^2+2D_{,}YZ+2E_{,}ZX+2F_{,}XY,$$

which it is easily proved to be by direct transformation.

This is the simplest form of the algebraic theorem of invariance with which we are concerned.

CHAPTER XI.—*On Imperfect Concurrences of two Stress or Strain Types.*

Def. The concurrence of any stresses or strains of two stated types is the proportion which the work done when a body of unit volume experiences a stress of either type, while acquiring a strain of the other, bears to the product of the numbers measuring the stress and strain respectively.

Cor. 1. In orthogonal resolution of a stress or strain, its component of any stated type is equal to its own amount multiplied by its concurrence with that type; or the stress or strain of a stated type which, along with another or others orthogonal to it, have a given stress or strain for their resultant, is equal to the amount of the given stress or strain reduced in the ratio of its concurrence with that stated type.

Cor. 2. The concurrence of two coincident stresses or strains is unity; or a perfect concurrence is numerically equal to unity.

Cor. 3. The concurrence of two orthogonal stresses and strains is zero.

Cor. 4. The concurrence of two directly opposite stresses or strains is -1.

Cor. 5. If $x, y, z, \xi, \eta, \zeta$, are orthogonal components of any strain or stress r, its concurrences with the types of reference are respectively

$$\frac{x}{r}, \frac{y}{r}, \frac{z}{r}, \frac{\xi}{r}, \frac{\eta}{r}, \frac{\zeta}{r},$$

where

$$r=(x^2+y^2+z^2+\xi^2+\eta^2+\zeta^2)^{\frac{1}{2}}.$$

Cor. 6. The mutual concurrence of two stresses or strains is

$$ll'+mm'+nn'+\lambda\lambda'+\mu\mu'+\nu\nu',$$

if $l, m, n, \lambda, \mu, \nu$ denote the concurrences of one of them with six orthogonal types of reference, and $l', m', n', \lambda', \mu', \nu'$ those of the other.

Cor. 7. The most convenient *specification of a type* for strains or stresses, being in general a statement of the components, according to the types of reference, of a unit strain or stress of the type to be specified, becomes a statement of its concurrences with the types of reference when these are orthogonal.

Examples.—(1) The mutual concurrence of two simple longitudinal strains or stresses, inclined to one another at an angle θ, is $\cos^2\theta$.

(2) The mutual concurrence of two simple distortions in the same plane, whose axes are inclined at an angle θ to one another, is $\cos^2\theta-\sin^2\theta$, or $2\sin(45°-\theta)\cos(45°-\theta)$.

Hence the components of a simple distortion δ along two rectangular axes in its plane, and two others bisecting the angle between these taken as axes of component simple distortions, are

$$\delta(\cos^2\theta-\sin^2\theta) \text{ and } \delta.2\sin\theta\cos\theta$$

respectively, if θ be the angle between the axis of elongation in the given distortion and in the first component type.

(3) The mutual concurrence of a simple longitudinal strain and a simple distortion is

$$\sqrt{2}.\cos\alpha\cos\beta,$$

if α and β be the angles at which the direction of the longitudinal strain is inclined to the line bisecting the angles between the axes of the distortion; it is also equal to

$$\frac{1}{\sqrt{2}}(\cos^2\phi-\cos^2\psi),$$

if ϕ and ψ denote the angles at which the direction of the longitudinal strain is inclined to the axis of the distortion.

(4) The mutual concurrence of a simple longitudinal strain and of a uniform dilatation is $\frac{1}{\sqrt{3}}$.

(5) The specifying elements exhibited in Example (7) of the preceding Chapter are the concurrences of the new system of orthogonal types described in Example (3) of Chap. IX. with the ordinary system, Examples (1) and (2), Chap. IX.

CHAPTER XII.—*On the Transformation of Types of Reference for Stresses or Strains.*

To transform the specification $(x, y, z, \xi, \eta, \zeta)$ of a stress or strain with reference to one system of types into $(x_1, x_2, x_3, x_4, x_5, x_6)$ with reference to another system of types. Let $(a_1, b_1, c_1, e_1, f_1, g_1)$ be the components, according to the original system, of a unit strain of the first type of the new system; let $(a_2, b_2, c_2, e_2, f_2, g_2)$ be the corresponding specification of the second type of the new system; and so on. Then we have, for the required formulæ of transformation—

$$\begin{aligned} x&=a_1x_1+a_2x_2+a_3x_3+a_4x_4+a_5x_5+a_6x_6,\\ y&=b_1x_1+b_2x_2+b_3x_3+b_4x_4+b_5x_5+b_6x_6,\\ &\ldots\ldots\ldots\ldots\\ &\ldots\ldots\ldots\ldots\\ \zeta&=g_1x_1+g_2x_2+g_3x_3+g_4x_4+g_5x_5+g_6x_6. \end{aligned}$$

Example.—The transforming equations to pass from a specification $(x, y, z, \xi, \eta, \zeta)$ in terms of the system of reference used in Examples (6) and (7), Chapter X., to a specification $(\sigma, \xi, \eta, \zeta, \varpi, \omega)$ in terms of the new system described in Example (3) of Chapter IX., and specified in Example (7) of Chapter X., are as follows:—

$$\begin{aligned} x&=\frac{1}{\sqrt{3}}\sigma+l\varpi+l'\omega,\\ y&=\frac{1}{\sqrt{3}}\sigma+m\varpi+m'\omega\\ z&=\frac{1}{\sqrt{3}}\sigma+n\varpi+n'\omega,\\ \xi&=\xi,\ \eta=\eta,\ \zeta=\zeta; \end{aligned}$$

where, as before stated, l, m, n, l', m', n' are by quantities fulfilling the conditions

$$\begin{aligned} l^2+m^2+n^2&=1,\\ l+m+n&=0,\\ l'^2+m'^2+n'^2&=1\\ l'+m'+n'&=0.\\ ll'+mm'+nn'&=0. \end{aligned}$$

PART II.—ON THE DYNAMICAL RELATIONS BETWEEN STRESSES AND STRAINS EXPERIENCED BY AN ELASTIC SOLID.

CHAPTER XIII.—*Interpretation of the Differential Equation of Energy.*

In a paper on the Thermo-elastic Properties of Matter, published in the first number of the *Quarterly Mathematical Journal*, April 1855, and republished in the *Philosophical Magazine*, 1877, second half year, it was proved, from general principles in the theory of the Transformation of Energy, that the amount of work (w) required to reduce an elastic solid, kept at a constant temperature, from one stated condition of internal strain to another depends solely on these two conditions, and not at all on the cycle of varied states through which the body may have been made to pass in effecting the change, provided always there has been no failure in

the elasticity under any of the strains it has experienced. Thus for a homogeneous solid homogeneously strained, it appears that w is a function of six independent variables $x, y, z, \xi, \eta, \zeta$, by which the condition of the solid as to strain is specified. Hence to strain the body to the infinitely small extent expressed by the variation from $(x, y, z, \xi, \eta, \zeta)$ to $(x+dx, y+dy, z+dz, \xi+d\xi, \eta+d\eta, \zeta+d\zeta)$, the work required to be done upon it is

$$\frac{dw}{dx}dx+\frac{dw}{dy}dy+\frac{dw}{dz}dz+\frac{dw}{d\xi}d\xi+\frac{dw}{d\eta}d\eta+\frac{dw}{d\zeta}d\zeta.$$

The stress which must be applied to its surface to keep the body in equilibrium in the state $(x, y, z, \xi, \eta, \zeta)$ must therefore be such that it would do this amount of work if the body, under its action, were to acquire the arbitrary strain $dx, dy, dz, d\xi, d\eta, d\zeta$; that is, it must be the resultant of six stresses:—one orthogonal to the five strains $dy, dz, d\xi, d\eta, d\zeta$, and of such a magnitude as to do the work $\frac{dw}{dx}dx$ when the body acquires the strain dx; a second orthogonal to $dx, dz, d\xi, d\eta, d\zeta$, and of such a magnitude as to do the work $\frac{dw}{dy}dy$ when the body acquires the strain dy; and so on. If a, b, c, f, g, h denote the respective concurrences of these six stresses, with the types of reference used in the specification $(x, y, z, \xi, \eta, \zeta)$ of the strains, the amounts of the six stresses which fulfil those conditions will (Chapter XI.) be given by the equations

$$P=\frac{1}{a}\frac{dw}{dx},\quad Q=\frac{1}{b}\frac{d}{dy},\quad R=\frac{1}{c}\frac{dw}{dz},$$

$$S=\frac{1}{f}\frac{dw}{d\xi},\quad T=\frac{1}{g}\frac{dw}{d\eta},\quad U=\frac{1}{h}\frac{dw}{d\zeta};$$

and the types of these component stresses are determined by being orthogonal to the fives of the six strain-types, wanting the first, the second, &c., respectively.

Cor. If the types of reference used in expressing the strain of the body constitute an orthogonal system, the types of the component stresses will coincide with them, and each of the concurrences will be unity. Hence the equations of equilibrium of an elastic solid referred to six orthogonal types are simply

$$P=\frac{dw}{dx},\quad Q=\frac{dw}{dy},\quad R=\frac{dw}{dz}$$

$$S=\frac{dw}{d\xi},\quad T=\frac{dw}{d\eta},\quad U=\frac{w}{d\zeta}.$$

CHAPTER XIV.—*Reduction of the Potential Function, and of the Equations of Equilibrium, of an Elastic Solid to their simplest Forms.*

If the condition of the body from which the work denoted by w is reckoned be that of equilibrium under no stress from without, and if $x, y, z, \xi, \eta, \zeta$ be chosen each zero for this condition, we shall have, by Maclaurin's theorem,

$$w=H_2(x, y, z, \xi, \eta, \zeta)+H_3(x, y, z, \xi, \eta, \zeta)+\text{\&c.},$$

where H_2, H_3, &c., denote homogeneous functions of the second order, third order, &c., respectively. Hence $\frac{dw}{dx}, \frac{dw}{dy}$, &c., will each be a linear function of the strain coordinates, together with functions of higher orders derived from H_3, &c. But experience shows (section 37 above) that, within the elastic limits, the stresses are very nearly if not quite proportional to the strains they are capable of producing; and therefore H_3, &c., may be neglected, and we have simply

$$w=H_2(x, y, z, \xi, \eta, \zeta).$$

Now in general there will be twenty-one terms, with independent coefficients, in this function; but by a choice of types of reference, that is, by a linear transformation of the independent variables, we may, in an infinite variety of ways, reduce it to the form

$$w=\tfrac{1}{2}(Ax^2+By^2+Cz^2+F\xi^2+G\eta^2+H\zeta^2).$$

The equations of equilibrium then become

$$P=\frac{A}{a}x,\quad Q=\frac{B}{b}y,\quad R=\frac{C}{c}z,$$

$$S=\frac{F}{f}\xi,\quad T=\frac{G}{g}\eta,\quad U=\frac{H}{h}\xi,$$

the simplest possible form under which they can be presented. The interpretation can be expressed as follows.

Prop. An infinite number of systems of six types of strains or stresses exist in any given elastic solid such that, if a strain of any one of those types be impressed on the body, the elastic reaction is balanced by a stress orthogonal to the five others of the same system.

CHAPTER XV.—*On the Six Principal Strains of an Elastic Solid.*

To reduce the twenty-one coefficients of the quadratic terms in the expression for the potential energy to six by a linear transformation, we have only fifteen equations to satisfy; while we have thirty disposable transforming coefficients, there being five independent elements to specify a type, and six types to be changed. Any further condition expressible by just fifteen independent equations may be satisfied, and makes the transformation determinate. Now the condition that six strains may be mutually orthogonal is expressible by just as many equations as there are different pairs of six things, that is, fifteen. The well-known algebraic theory of the linear transformation of quadratic functions shows for the case of six variables—(1) that the six coefficients in the reduced form are the roots of a "determinant" of the sixth degree necessarily real; (2) that this multiplicity of roots leads determinately to one, and only one system of six types fulfilling the prescribed conditions, unless two or more of the roots are equal to one another, when there will be an infinite number of solutions and definite degrees of isotropy among them; and (3) that there is no equality between any of the six roots of the determinant in general, when there are twenty-one independent coefficients in the given quadratic.

Prop. Hence a single system of six mutually orthogonal types may be determined for any homogeneous elastic solid, so that its potential energy when homogeneously strained in any way is expressed by the sum of the products of the squares of the components of the strain, according to those types, respectively multiplied by six determinate coefficients.

Def. The six strain-types thus determined are called the Six Principal Strain-types of the body.

The concurrences of the stress-components used in interpreting the differential equation of energy with the types of the strain-coordinates in terms of which the potential of elasticity is expressed, being perfect when these constitute an orthogonal system, each of the quantities denoted above by a, b, c, f, g, h, is unity when the six principal strain-types are chosen for the coordinates. The equations of equilibrium of an elastic solid may therefore be expressed as follows:—

$$P=Ax,\quad Q=By,\quad R=Cz,$$

$$S=F\xi,\quad T=G\eta,\quad U=H\zeta,$$

where $x, y, z, \xi, \eta, \zeta$ denote strains belonging to the six Principal Types, and P, Q, R, S, T, U the components according to the same types, of the stress required to hold the body in equilibrium when in the condition of having those strains. The amount of work that must be spent upon it per unit of its volume, to bring it to this state from an unconstrained condition, is given by the equation

$$w=\tfrac{1}{2}(Ax^2+By^2+Cz^2+F\xi^2+G\eta^2+H\zeta^2).$$

Def. The coefficients A, B, C, F, G, H are called the six Principal Elasticities of the body.

The equations of equilibrium express the following propositions:—

Prop. If a body be strained according to any one of its six Principal Types, the stress required to hold it so is directly concurrent with the strain.

Examples.—(1) If a solid be cubically isotropic in its elastic properties, as crystals of the cubical class probably are, any portion of it will, when subject to a uniform positive or negative normal pressure all round its surface, experience a uniform condensation or dilation in all directions. Hence a uniform condensation is one of its six principal strains. Three plane distortions with axes bisecting the angles between the edges of the cube of symmetry are clearly also principal strains, and since the three corresponding principal elasticities are equal to one another, any strain whatever compounded of these three is a principal strain. Lastly, a plane distortion whose axes coincide with any two edges of the cube, being clearly a principal distortion, and the principal elasticities corresponding to the three distortions of this kind being equal to one another, any distortion compounded of them is also a principal distortion.

Hence the system of orthogonal types treated of in Examples (3) Chap. IX., and (7) Chap. X., or any system in which, for (II.), (III.), and (IV.), any three orthogonal strains compounded of them are substituted, constitutes a system of six Principal Strains in a solid cubically isotropic. There are only three distinct Principal Elasticities for such a body, and these are—(A) its modulus of compressibility, (B) its rigidity against diagonal distortion in any of its principal planes (three equal elasticities), and (C) its rigidity against rectangular distortions of a cube of symmetry (two equal elasticities).

(2) In a perfectly isotropic solid the rigidity against all distortions is equal. Hence the rigidity (B) against diagonal distortion must be equal to the rigidity (C) against rectangular distortion, in a cube; and it is easily seen that if this condition is fulfilled for one set of three rectangular planes for which a substance is isotropic, the isotropy must be complete. The conditions of perfect or spherical isotropy are therefore expressed in terms of the conditions referred to in the preceding example, with the farther condition B=C.

A uniform condensation in all directions, and any system whatever of five orthogonal distortions, constitute a system of six Principal Strains in a spherically isotropic solid. Its Principal Elasticities are simply its Modulus of Compressibility and its Rigidity.

Prop. Unless some of the six Principal Elasticities be equal to one another, the stress required to keep the body strained otherwise than according to one or other of six distinct types is oblique to the strain.

Prop. The stress required to maintain a given amount of strain is a maximum or a maximum-minimum, or a minimum, if it is of one of the six Principal Types.

Cor. If A be the greatest and H the least of the six quantities A, B, C, F, G, H, the principal type to which the first corresponds is that of a strain *requiring a greater stress to maintain it* than any other strain of equal amount; and the principal type to which the last corresponds is that of a strain which *is maintained by a less stress* than any other strain of equal amount in the same body. The stresses corresponding to the four other principal strain-types have each the maximum-minimum property in a determinate way.

Prop. If a body be strained in the direction of which the concurrences with the principal strain-types are $l, m, n, \lambda, \mu, \nu$, and to an amount equal to r, the stress required to maintain it in this state will be equal to Ωr, where

$$\Omega=(A^2l^2+B^2m^2+C^2n^2+F^2\lambda^2+G^2\mu^2+H^2\nu^2)^{\frac{1}{2}},$$

and will be of a type of which the concurrences with the principal types are respectively

$$\frac{Al}{\Omega},\ \frac{Bm}{\Omega},\ \frac{Cn}{\Omega},\ \frac{F\lambda}{\Omega},\ \frac{G\mu}{\Omega},\ \frac{H\nu}{\Omega}.$$

Prop. A homogeneous elastic solid, crystalline or non-crystalline, subject to magnetic force or free from magnetic force, has neither any right-handed or left-handed, nor any dipolar, properties dependent on elastic forces simply proportional to strains.

Cor. The elastic forces concerned in the luminiferous vibrations of a solid or fluid medium possessing the right- or left-handed property, whether axial or rotatory, such as quartz crystal, or tartaric acid, or solution of sugar, either depend on the heterogeneousness or on the magnitude of the strains experienced.

Hence as they *do not depend on the magnitude of the strain*, they *do depend on its heterogeneousness through the portion of a medium containing a wave.*

Cor. There cannot possibly be any characteristic of elastic forces simply proportional to the strains in a homogeneous body, corresponding to certain peculiarities of crystalline form which have been observed,—for instance corresponding to the plagiedral faces discovered by Sir John Herschel to indicate the optical character, whether right-handed or left-handed, in different specimens of quartz crystal, or corresponding to the distinguishing characteristics of the crystals of the right-handed and left-handed tartaric acids obtained by M. Pasteur from racemic acid, or corresponding to the dipolar characteristics of form said to have been discovered in electric crystals.

CHAPTER XVI.—*Application of Conclusions to Natural Crystals.*

It is easy to demonstrate that a body, homogeneous when regarded on a large scale, may be constructed to have twenty-one arbitrarily prescribed values for the coefficients in the expression for its potential energy in terms of any prescribed system of strain coordinates. This proposition was first enunciated in the paper on the Thermo-elastic Properties of Solids, published April 1855, in the *Quarterly Mathematical Journal* alluded to above. We may infer the following.

Prop. A solid may be constructed to have arbitrarily prescribed values for its six Principal Elasticities and an arbitrary orthogonal system of six strain-types, specified by fifteen independent elements, for its principal strains: for instance, five arbitrarily chosen systems of three rectangular axes, for the normal axes of five of the Principal Types; those of the sixth consequently in general distinct from all the others, and determinate; and the six times two ratios between the three stresses or strains of each type, also determinate. The fifteen equations expressing (Chap. VI.) the mutual orthogonality of the six types determine the twelve ratios for the six types, and the three quantities specifying the axes of the sixth type in the particular case here suggested: or generally the fifteen equations determine fifteen out of the thirty quantities (viz. twelve ratios and eighteen angular coordinates) specifying six Principal Types.

Cor. There is no reason for believing that natural crystals do not exist for which there are six unequal Principal Elasticities, and six distinct strain-types for which the three normal axes constitute six distinct sets of three principal rectangular axes of elasticity.

It is easy to give arbitrary illustrative examples regarding Principal Elasticities: also, to investigate the principal strain-types and the equations of elastic force referred to them or to other natural types, for a body possessing the kind of symmetry as to elastic forces that is possessed by a crystal of Iceland spar, or by a crystal of the "tesseral class," or of the included "cubical class." Such illustrations and developments, though proper for a students' text book of the subject, are unnecessary here.

For applications of the Mathematical Theory of Elasticity to the question of the earth's rigidity and elasticity as a whole, and to the equilibrium of elastic solids in general, which are beyond the scope of the present article, the reader is referred to Thomson and Tait's *Natural Philosophy*, §§ 588, 740, 832, 849, and Appendix C.

CHAPTER XVII.—*Plane Waves in a Homogeneous Æolotropic Solid.*

A plane wave in a homogeneous elastic solid is a motion in which every line of particles in a plane parallel to one fixed plane experiences simply a motion of translation—but a motion differing from the motions of particles in planes parallel to the same. Let OX, OY, OZ be three fixed rectangular axes; OX perpendicular to the wave front (as any of the parallel planes of moving particles referred to in the definition is called), and OY, OZ *in* the wave front. Let $x+u$, $y+v$, $z+w$ be the coordinates at time t of a particle which, if the solid were free from strain, would be at (x, y, z). The definition of wave motion amounts simply to this, that u, v, w are functions of x and t.

The *strain* of the solid (Chap. VII. above) is the resultant of a simple longitudinal strain in the direction OX, equal to $\frac{du}{dx}$, and two differential slips $\frac{dv}{dx}, \frac{dw}{dx}$, parallel to OY and OZ, constituting simple distortions of which the numerical magnitudes (Chap. X.) are

$$\frac{dv}{dx}\sqrt{2}, \quad \text{and} \quad \frac{dw}{dx}\sqrt{2}.$$

Put then

$$\frac{du}{dx}=\xi,\quad \frac{dv}{dx}\sqrt{2}=\eta,\quad \frac{dw}{dx}\sqrt{2}=\zeta \quad . \ . \ . \ . \ (1);$$

and let W denote the work per unit of bulk required to produce the strain represented by this notation. We have (Chap. XV.)

$$W=\tfrac{1}{2}(A\xi^2+B\eta^2+C\zeta^2+2D\eta\zeta+2E\zeta\xi+2F\xi\eta) \quad . \ . \ . \ (2),$$

where A, B, C, D, E, F denote moduluses of elasticity of the solid. Let p, q, r denote the three components of the traction per unit area of the wave front. We have (Chap. XV.)

$$\left.\begin{aligned} p&=A\xi+F\eta+E\zeta\\ q\sqrt{\tfrac{1}{2}}&=F\xi+B\eta+D\zeta\\ r\sqrt{\tfrac{1}{2}}&=E\xi+D\eta+C\zeta \end{aligned}\right\} \quad . \ . \ . \ . \ . \ . \ . \ (3),$$

Now let ξ, η, ζ be taken such that

$$\left.\begin{aligned} A\xi+F\eta+E\zeta&=M\xi\\ F\xi+B\eta+D\zeta&=M\eta\\ E\xi+D\eta+C\zeta&=M\zeta \end{aligned}\right\} \quad . \ . \ . \ . \ . \ . \ (4)$$

the determinantal cubic gives three real positive values for M, and with M equal to any one of these values, (4) determine the ratios $\xi:\eta:\zeta$. Hence when the solid is strained in any one of the three ways thus determined we have

$$p=M\frac{du}{dx},\quad q=M\frac{dv}{dx},\quad r=M\frac{dw}{dx} \quad . \ . \ . \ . \ (5).$$

The three components of the whole force due to the tractions on the sides of an infinitely small parallelepiped δx, δy, δz of the solid are clearly

$$\frac{dp}{dx}\delta x.\delta y\delta z,\quad \frac{dq}{dx}\delta x.\delta y\delta z,\ \text{and}\ \frac{dr}{dx}\delta x.\delta y\delta z \quad . \ . \ . \ . \ . \ (6),$$

and therefore, if ρ be its density, and consequently $\rho\delta x\ \delta y\ \delta z$ its mean, the equations of its motion are

$$\rho\frac{d^2u}{dt^2}=\frac{dp}{dx},\quad \rho\frac{d^2v}{dt^2}=\frac{dq}{dx},\quad \rho\frac{d^2w}{dt^2}=\frac{dr}{dx} \quad . \ (7).$$

These, putting for p, q, r their values by (5), become

$$\rho\frac{d^2u}{dt^2}=M\frac{d^2u}{dx^2},\quad \rho\frac{d^2v}{dt^2}=M\frac{d^2v}{dx^2},\quad \rho\frac{d^2w}{dt^2}=M\frac{d^2w}{dx^2} \quad (8).$$

And by (4) and (1) we have

$$\left.\begin{aligned} Au+(Fv+Ew)\sqrt{2}&=Mu\\ Fu+(Bv+Dw)\sqrt{2}&=Mv\sqrt{2}\\ Eu+(Dv+Cw)\sqrt{2}&=Mw\sqrt{2} \end{aligned}\right\} \quad . \ . \ . \ . \ . \ (9).$$

Let M_1, M_2, M_3 be the three roots of the determinantal cubic, and b_1, c_1; b_2, c_2; b_3, c_3, the corresponding values of the ratios $\frac{v}{u}, \frac{w}{u}$ determined by (9). The complete solution of (8), subject to (9), is

$$\left.\begin{aligned} u&=u_1+u_2+u_3\\ v&=b_1u_1+b_2u_2+b_3u_3\\ w&=c_1u_1+c_2u_2+c_3u_3\\ \text{where}\quad u_1&=f_1\left(x+t\sqrt{\tfrac{M_1}{\rho}}\right)+F_1\left(x-t\sqrt{\tfrac{M_1}{\rho}}\right)\\ u_2&=f_2\left(x+t\sqrt{\tfrac{M_2}{\rho}}\right)+F_2\left(x-t\sqrt{\tfrac{M_2}{\rho}}\right)\\ u_3&=f_3\left(x+t\sqrt{\tfrac{M_3}{\rho}}\right)+F_3\left(x-t\sqrt{\tfrac{M_3}{\rho}}\right) \end{aligned}\right\} \quad (10),$$

f_1, F_1, f_2, F_2, f_3, F_3 denoting arbitary functions. Hence we conclude that there are three different wave-velocities,

$$\sqrt{\frac{M_1}{\rho}},\ \sqrt{\frac{M_2}{\rho}},\ \sqrt{\frac{M_3}{\rho}}$$

and three different modes of waves, determined by equations (9).

Waves in an Isotropic Solid.—If the solid be isotropic, we have

$$\left.\begin{aligned} &B=C\\ &D=E=F=0\\ &M_1=A.\ M_2=M_3=B \end{aligned}\right\} \quad . \ . \ . \ . \ (11)$$

Hence, instead of three different waves with different velocities, we have just two,—a wave (like that of sound in air or other elastic fluid), in which the motions are perpendicular to the wave front, and the other (like the waves of light in an isotropic medium) in which the motions are parallel to the wave front.

Waves in an Incompressible Solid (Æolotropic or Isotropic).—If the solid be incompressible, we have **$A=\infty$. and u must be zero.**

Hence $$W=B\eta^2+C\zeta^2+2D\eta\zeta$$

and by a determinantal quadratic, instead of cubic, we find two wave-velocities and two wave-modes, in each of which the motion is parallel to the wave front. In the case of isotropy the two wave velocities are equal.

It is to be noticed that M_1, M_2, M_3 in the preceding investigation are not generally true "principal moduluses," but special moduluses corresponding to the particular plane chosen for the wave front. In the particular case of isotropy, however, the equal moduluses M_2, M_3 of (11) are principal moduluses, being each equal to *the* modulus of rigidity, but M_1 is a mixed modulus of compressibility and rigidity—not a principal modulus. In the case of incompressibility, the two moduluses found from the determinantal quadratic by the process indicated above are not principal moduluses generally, because the distortions by the differential motions of planes of particles parallel to the wave front must generally give rise to tangential stresses orthogonal to them, which do not influence the wave motion. (W. TH.)

ELATERIUM, a drug consisting of a sediment deposited by the juice of the fruit of *Ecbalium Elaterium*, the squirting cucumber (see vol. vi. p. 688.) To prepare it, the fruit is sliced lengthwise and slightly pressed; the greenish and slightly turbid juice thus obtained is strained and set aside; and the deposit of elaterium formed after a few hours is collected on a linen filter, rapidly drained, and dried on porous tiles at a gentle heat. Elaterium is met with in commerce in light, thin, friable, flat or slightly incurved opaque cakes, of a greyish-green colour, bitter taste, and tea-like smell. The best kind is the English, prepared at Hitchin, Market Deeping, Mitcham, and elsewhere; the Maltese is generally very inferior. Elaterium is an exceedingly powerful hydragogue and drastic purgative, and not unfrequently produces vomiting. Its active principal is *elaterin*, a crystallizable body of the formula $C_{20}H_{28}O_5$.

ELBA, the Αἰθαλία of the Greeks, and Ilva of the Romans, is an island in the Mediterranean Sea, forming part of the Italian province of Livorno, and lying about 6 miles from the mainland of Italy, from which it is separated by the channel of Piombino, and about 34 miles E. of Corsica. It has a very irregular coast outline, is 18 miles long and $2\frac{1}{4}$ to $10\frac{1}{2}$ miles broad, and has a total area of nearly 90 square miles. It is throughout mountainous, and the highest point, Monte Capanne, is 2925 feet above sea-level. The western portion of the island is granitic, the eastern consists mainly of the sandstone locally known as *verrucano*, which in some places passes into a talc slate. In the vicinity of Porto Ferrajo the hills are cretaceous. The climate is mild, and, except at some spots on the coast, healthy. Springs are numerous, and the soil is not infertile; but agriculture and cattle-rearing are neglected, and there are no manufactures. Wine, wheat, aloes, dyer's lichen, and olives and other fruits are produced. The sardine and tunny fisheries, and the manufacture of sea-salt are of some importance; but the principal industry is mining. The iron mines are mostly in the vicinity of Rio Inferiore, and yield abundance of ore, chiefly hematite, of excellent quality. On account of the lack of fuel the ore is not smelted on the island, but is shipped direct to Follonica on the neighbouring coast of Italy, and to the ports of France and England. Marble, alabaster, sulphur, and ores of tin, lead, and silver are among the other mineral products. The principal places in Elba are the chief town Porto Ferrajo, with about 5000 inhabitants, the residence of Napoleon from May 4, 1814, to February 26, 1815, Rio Ferrajo, San Pietro, Porto Longone, and the village of Capoliveri. The population of the island in 1871 was 21,755.

The Argonauts, in quest of Circe, are said to have loaded at Portus Argous (Αργῷος λιμήν), now Porto Ferrajo, in Elba. The island was early famous for the richness of its mines, alluded to by Virgil (*Æn.* x. 173). It was attacked by Phayllus with a Syracusan fleet, 453 B.C., and subsequently by Apelles, who is stated to have subjugated it. In the 10th century it became a possession of the Pisans, from whom it was taken by the Genoese in 1290. It fell eventually into Spanish hands, came in 1736 under the jurisdiction of Naples, and in 1801 was ceded to the king of Etruria by the treaty of Lunéville. It was united to France in 1803, made over to Napoleon by the Treaty of Paris in 1814, restored to Tuscany in the following year, and in 1860 annexed to Italy.

ELBE, the Albis of the Romans and the Labe of the Bohemians, a large river of Germany, with a total length of 705 miles, and a drainage area of about 55,000 square miles. It rises in Bohemia not far from the frontiers of Silesia, on the southern side of the Riesengebirge or Giants' Mountains, in 50° 46′ N. lat. and 15° 32′ E. long. Of the numerous small streams (Seifen or Flessen, as they are named in the district) whose confluent waters compose the infant river, the most important are the Weisswasser, or White Water, and the Elbseifen; the former rises to the S.W. of the Schneekuppe in the White Meadow, and the latter in a stone fountain in the Elb Meadow. Augmented successively by the Adler, the Iser, the Moldau, and the Eger, it cuts its way through the Mittelgebirge of Bohemia, traverses the sandstone mountains of Saxon Switzerland, and with a general N.W. direction continues to meander through Saxony, Anhalt, and Hanover, until at length it falls into the German Ocean about 53° 5′ N. lat. and 8° 50′ E. long. The principal towns on its banks are Leitmeritz, Pirna, Dresden, Meissen, Torgau, Wittenberg, Magdeburg, Wittenberge, Harburg, Hamburg, and Altona. A short distance above Hamburg the stream divides into a number of branches, but they all reunite before reaching the ocean. At its source the Elbe is about 4600 feet above the level of the sea; after the first 40 miles of its course it is still 658 feet; but at Dresden it is only 279, and at Arneburg in Brandenburg only 176. At Königgratz the width is about 100 feet, at the mouth of the Moldau about 300, at Dresden 960, and at Magdeburg over 1000. The tide is perceptible as far up as Geesthacht. Of the fifty and more tributaries belonging to the system the most important are the Moldau, the Eger, the Mulde, and the Saale,—the Moldau having a course of 267 miles, the Eger of 235, the Mulde of 185, and the Saale of 220. Though the channel in some places, and especially in the estuary, is encumbered with sandbanks and shallows, the Elbe is of great importance as a means of communication, steamboats being able to ascend the main stream as far as Melnick, and to reach Prague by means of the Moldau. Some idea of the extent of its traffic may be obtained from the statement that in 1870 at Schandau 489 passenger-steamers and 2658 vessels and barges of various kinds passed up the stream, and 489 passenger steamers, 2865 ships, and 1505 rafts down the stream. By one line of canal it communicates with Lübeck, by another with Bremen, and by others with the great network of Mecklenburg and Brandenburg; and several new lines are projected, by which a direct way will be opened up to Hanover, Leipsic, and various other important cities.—For details see Dr Th. H. Schunke's "Die Schifffahrts-Kanäle im Deutschen Reiche," in Petermann's *Mittheil.*, 1877.