covers. If now we remove the proof-plane in the direction of the normal, keeping it, as nearly as possible, parallel to the surface, the electricity will not leave it; but we shall carry safely away all that it had when in contact with the surface of the body. We may return to the consideration of the proof-plane when we come to study mathematically the laws of electrical distribution.

In the experiment with which we are now concerned, Coulomb used a very delicate balance (a force of $\frac{1}{1000}$ of a milligramme was sufficient to keep the wire twisted through 90°). When the proof-plane was applied to any point of the external surface of the wooden cylinder, and then introduced into the torsion balance, it repelled the electrified ball of the balance with great force. When it was carefully introduced into one of the holes, made to touch the bottom, and then carefully withdrawn so as not to touch the edge of the hole, it produced no appreciable effect on the balance.

Coulomb varied this experiment as follows. He insulated and electrified a hollow sphere of metal (fig. 7), and by carefully introducing a proof-plane through a small opening tested the electrical condition of the interior surface. He found no sensible trace of electricity inside, except very near the edge of the small opening. Hence we conclude that if the sphere had been closed entirely there would have been no electrification inside. Many experiments have been made to illustrate the proposition that electricity resides entirely on the surface of conductors.

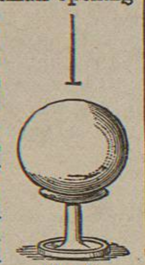

Fig. 7.

Franklin's experiment.

Franklin put a long chain inside a metal teapot, which he insulated and electrified. When he seized the chain by a hook at the end of a glass rod and pulled it out of the teapot he observed that a pair of pith balls, suspended side by side from the teapot, collapsed more and more as the chain was drawn out, and he concluded that the electrification of the teapot, being now spread over a greater surface, had become weaker.

Biot's experiment.

The following experiment of Biot's has become classical. A spherical conductor A (fig. 8) is supported on an insulating stem D. B and C are two hollow hemispheres fastened to insulating handles E and F. When these are fitted together they form a sphere somewhat larger than A, with a small hole in it through which the stem D can pass. If we electrify A very strongly, so that when put in the electric cage it powerfully deflects the electrometer, and then close B and C over A, and make either B or C touch it, then separate B and C, and test A, B, and C in the cage, we shall find that all the electricity has gone from A and spread itself over B and C.

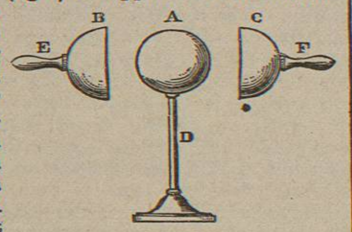

Fig. 8.

The following is an ingenious experiment of Faraday's, involving the same principle. AB (fig. 9) is a wire ring supported on an insulating stand; C is a conical muslin bag fitted to the ring with two strings fastened to the vertex of the cone, giving the experimenter the means of quickly turning the bag inside out. If the bag be electrified in the first position in the figure and tested with the proof-plane and electric cage, it will be found that the outside only is electrified. If we turn the bag inside out and test it, we shall find as before that what is now the outside, and was formerly the inside, is alone electrified. The electricity has thus passed through the bag so as to be on the outside in both cases.

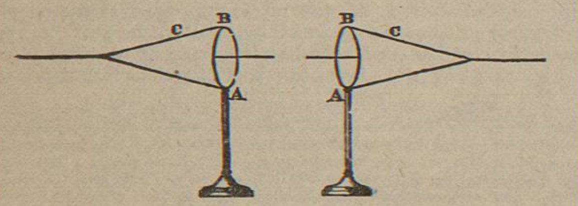

Fig. 9

Before leaving for a time tne question of the distribution of electricity on conductors, it may be well to warn the student to accept with due reserve the proposition that electricity resides entirely on the surface of conductors, and to remind him that such a proposition is from the nature of the case incapable of direct experimental proof, for we cannot operate in the substance of a mass of metal. Some of the experiments we have quoted bear more directly on the question than others. Coulomb's experiment, for instance, shows, strictly speaking, merely that electricity avoids cavities and re-entrant angles. Again, in Faraday's experiment with the linen bag, we have not clearly defined what we mean by the outside of the body. The proposition has on the whole been suggested rather than proved. Its meaning will become clearer as we go more and more into the theory of distribution,[1] and we shall meet with it by-and-by as one of the first propositions in the mathematical theory.

Laws of Electric Force.

Before proceeding to give an account of Coulomb's quantitative experiments on electrical distribution, we shall describe shortly the methods by which he arrived at the laws of electric force, and did for electricity what Newton did for astronomy, *i.e.*, laid the foundation for a mathematical theory of the subject based on the hypothesis of *action at a distance.*

Experimental determination of the elementary law of force

In this research Coulomb used the form of balance which we described above. The clamp holding the fixed ball of the balance is so adjusted that the centre of the ball falls in a horizontal line through zero of the graduation on the glass cylinder and the prolongation of the suspending wire; the torsion button is turned till its arm is at zero; the disc, button and all, is then turned till the disc on the arm and the centre of the movable ball are in a line with the zero of the lower graduation. The fixed ball, which had been removed to allow of the last adjustment, being replaced, and the movable ball having come to rest in contact with it, both are electrified by means of a small metal ball carried on an insulating stem of shellac. The balls repel each other, and the movable ball takes up a certain position of equilibrium; the corresponding angle is read off. The torsion button is then turned through an angle which is noted, so as to bring the balls nearer together. The new position of the beam is again read off; this may be repeated a third time. We are then in possession of data from which we can draw conclusions as to the law of electrical force at different distances.

Let us assume that the force between two elements of positive electricity (supposed collected at two points, technically speaking, "regarded as physical points") varies inversely as the square of the distance between them. It will be shown in the mathematical theory that two spheres *uniformly*[2] electrified, as we shall at present

[1] One additional caution may be useful, viz., not to confound this proposition with another of fundamental importance, of which we can give direct experimental proof of the most conclusive nature "that there is no electrical action inside a hollow conductor containing no charged bodies."

[2] This condition is not absolutely satisfied in any experiment; it is approximately satisfied in Coulomb's experiment.

assume the two balls in the balance to be, repel each other, as if their electricity were collected at their centres.

Let ϵ be the angle of equilibrium in any case, τ the angle of torsion. O (fig. 10) is the centre of the beam, F and M the centres of the fixed and movable ball (we suppose OF=OM); OK is perpendicular to FM. Then $FM^2 \propto \sin^2\frac{\epsilon}{2}$. Hence moment of the force on M about

$$O \propto \frac{\cos\frac{\epsilon}{2}}{\sin^2\frac{\epsilon}{2}},$$ and the torsonial couple $\propto \tau + \epsilon$.

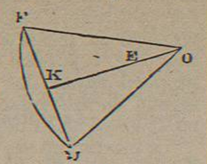

Fig. 10.

Hence in the three cases the value of $(\tau+\epsilon)\sin\frac{\epsilon}{2}\tan\frac{\epsilon}{2} = A$ (say) must be the same, if the law of the inverse square agree with the experiments.

Coulomb made many experiments of the kind we have described. The following is the result which he has given of one such :—

	Observed.	Calculated.	Difference.
τ	ϵ	ϵ	
0	36° 0′	36° 0′	...
126°	18 0	18 6	6′
567	8 30	9 4	34

The third column is obtained from the two preceding. A is calculated by putting $\tau = 0$ and $\epsilon = 36°$ in the formula

$$(\tau + \epsilon)\sin\frac{\epsilon}{2}\tan\frac{\epsilon}{2} = A.$$

Then using this value of A and the observed value of τ, the formula is employed to find ϵ in the two second cases. The agreement between the observed and calculated values of ϵ is the test of the truth of the law we have assumed. The agreement in the second line is as good as can be expected when possible errors of experiment are considered. It will be seen, moreover, that the calculated is in excess of the observed value, which is what we ought to expect, owing to the loss of electricity which goes on during the time consumed in the experiment. That there is such a loss may be proved experimentally by simply leaving the movable ball to itself after any of the three operations; it will be seen to move slowly towards the fixed ball. We shall return hereafter to this loss of electricity, with regard to the exact nature of which authorities are not quite agreed.[1] In the third line the agreement is less good, but here the proximity of the balls renders the supposition of uniformity no longer even approximately allowable. The mutual repulsion tends to drive the electricity on each ball farther from the other ball, and thus the action between the balls is as if the electricity on each were collected at points beyond the centre, so that the observed repulsion must be less than that calculated on the supposition of uniformity of distribution.

Coulomb also made experiments with the torsion balance to test whether the law of the inverse square applies to the attraction as well as to the repulsion of electrified bodies. His experiments confirmed the law; but the difficulty of operating is much greater in this case than in the former. He therefore adopted another method of experimenting. A small conducting disc was fixed normally on the end of a small shellac needle, which was hung up, so as to be horizontal, on a fibre of raw silk attached to a horizontal scale. An insulated conducting globe was set up with its centre in the same vertical plane as the scale, and in the same horizontal plane as the centre of the small disc. The globe and disc were oppositely electrified, and the period of oscillation of the needle was found by observing the duration of 15 swings. The time of oscillation follows the pendulum law, and varies inversely as the square root of the force acting on the needle, hence the duration of 15 oscillations will vary inversely as the square root of the force, *i.e.* directly as the distance between the centres of the globe and disc, if the law of the inverse square hold. Coulomb's experiment gave the following results :—

Distance of centres of globe and disc.	Duration of 15 oscillations.	Ratio of distance to duration.
9	20	2·22
18	41	2·28
24	60	2·50

The numbers in the third column ought to be all equal. The deviation from equality are not greater than can fairly be explained by loss of electricity and errors of observation.

Coulomb also investigated, both by means of the torsion balance and by the method of oscillations, the relation between electric force and quantity.

He electrified the two balls of the torsion balance by simultaneous contact with another ball, and observed the angle of equilibrium; he then halved the quantity on the fixed ball by touching it with an equal neutral ball, and reduced the torsion till the angle of equilibrium, and, in consequence, the distance between the balls was the same as before; he found the torsional couple in the second case to be somewhat less than half what it was in the first. He therefore concluded that the force between two elements of electricity varies as the product of the quantities.

Coulomb's experiments were repeated, and his results confirmed by Riess,[2] and by Marié-Davy.[3] Experiments which, when properly interpreted, lead to the same results, were made by Snow Harris,[4] and by Egen.[5]

We have then arrived at this general law of electric force :—

Statement of law.

If two quantities q, q' of electricity be supposed collected at two points, whose distance is d, the force between them acts in the straight line joining the points and $\propto \frac{qq'}{r^2}$

So far, this law might be merely an approximation to the truth. Later on, however, it will be seen to be logically deducible from experiments which in delicacy infinitely surpass those just described. The law of Coulomb is in fact established as certainly as the law of gravitation itself.[6]

Definition of absolute electrostatic unit.

By means of the law now given the *unit of electrical quantity* can be defined in a satisfactory and practical manner. This unit we now state to be *that quantity of positive electricity which, when collected into a point, repels with unit of force an equal quantity similarly collected into a point at unit distance from the former.*

If we take centimetre, gramme, and second as our units of length, mass, and time, the unit force will be that force which in a second generates in a gramme of matter a velocity of a centimetre per second.

[1] This is only one of the many experimental difficulties which beset the use of the torsion balance, one of the most difficult of all instruments to use successfully. To appreciate the skill and sagacity of Coulomb in this and other matters, the student must read more detailed accounts (Riess and Mascart, or *Mémoires de l'Acad.*, about 1785) of his labours than we can give here. He will be richly repaid for his trouble. Nothing is better calculated to rouse the failing enthusiasm of the tyro in experimental electricity than a perusal of the works of Coulomb, unless it be to read the *Experimental Researches* of Faraday.

[2] *Reibungselectricität*, Bd. i. p. 94. [3] Mascart, i. p. 67.

[4] *Phil. Trans.*, 1834 and 1836. In connection with which we call the attention of the student to the classical paper of Sir W. Thomson, *Reprint of Papers on Electrostatics and Magnetism*, p. 15 *sqq*.

[5] Riess, Bd. i. p. 94.

[6] We suppose, of course, that we are dealing always with one and the same dielectric throughout.

The law of electric force between two quantities q and q' now becomes

$$\text{Force} = \frac{qq'}{d^2}.$$

The unit of quantity which we have just defined is called the electrostatic unit, in contradistinction to the electromagnetic unit which we shall define hereafter.

Since the dimension of unit of force is $[LMT^{-2}]$, where L,M,T symbolize units of length, mass, and time, we have for the dimension of unit of electrical quantity [Q]

$$[Q] = [LF^{\frac{1}{2}}] = [L^{\frac{3}{2}}M^{\frac{1}{2}}T^{-1}].$$

Quantitative Results concerning Distribution.

It has already been indicated that electricity in equilibrium resides on the surface of conducting bodies. We must now review shortly the experimental method by which this surface distribution has been more closely investigated. We shall state here some of the general principles arrived at, and one or two of the results, reserving others for quotation when we come to the mathematical theory of electrical distribution.

The most important experiments are due to Coulomb. He used the proof-plane and the torsion balance. Riess, who afterwards made similar experiments, used methods similar to those of Coulomb.

Allusion has already been made to the use of the proof-plane, and it has been stated that when applied to any part of the surface of an electrified body, it brings away just as much electricity as originally occupied the part of the surface which it covers. If, therefore, we electrify the movable ball of the torsion balance in the same sense as the body we are to examine, and note the repulsion caused by the proof-plane when introduced in place of the fixed ball after having touched in succession two parts of the surface of the body, we can, from the indications of the balance, calculate the ratio of the quantities of electricity on the plane in the two cases, and hence the ratio of the electrical densities at the two points of the surface. We suppose, of course, that the proof-plane is small enough to allow us to assume that the electrical density is sensibly uniform over the small area covered by it. In some of his experiments Riess used a small sphere (about two lines in diameter) instead of the small disc of the proof-plane as Coulomb used it. The sphere in such cases ought to be very small, and even then, except in the case of plane surfaces, its use is objectionable, unless the object be merely to determine, by twice touching the same point of the same conductor, the ratio of the whole charges on the conductor at two different times. The fundamental requisite is that the testing body shall, when applied, alter the *form* of the testing body as little as possible,[1] and this requisite is best satisfied by a small disc, and the better the smaller the disc is. The theoretically correct procedure would be to have a small portion of the actual surface of the body movable. If we could remove such a piece so as to break contact with all neighbouring portions simultaneously, then we should, by testing the electrification of this in the balance, get a perfect measure of the mean electric surface density on the removed portion. We shall see that Coulomb did employ a method like this.

There are various ways of using the torsion balance in researches on distribution. We may either electrify the movable ball independently (as above described), or we may electrify it each time by contact with the proof-plane when it is inserted into the balance. It must be noticed that the repulsion of the movable ball is in the first case proportional to the charge on the proof-plane, but in the second to the square of the charge, so that the indications must be reduced differently.

In measuring we may either bring the movable ball to a fixed position, in which case the whole torsion required to keep it in this position is proportional to the charge on the proof-plane (or to its square, if the second of the above modes of operation be adopted), or we may simply observe the angle of equilibrium and calculate the quantity from that. It is supposed, for simplicity of explanation in all that follows, that the former of the two alternatives is adopted, and that the movable ball is always independently charged.

The gradual loss of electricity experienced more or less by every insulated conductor has already been alluded to. This loss forms one of the greatest difficulties to be encountered in such experiments as we are now describing. If we apply the proof-plane to a part of a conductor and take the balance reading, giving a torsion τ_1 say, and repeat the observation, after time t, we shall get a different torsion τ owing to the loss of electricity in the interval. This loss, partly if not mainly due to the insulating supports, depends on a great many circumstances, some of which are entirely beyond even the observation of the experimenter. We may admit, however, what experiment confirms within certain small limits, that the rate of loss of electricity is proportional to the charge, and we shall call $\frac{\tau_1 - \tau_2}{t}$ (the loss per unit of time on hypothesis of uniformity) the coefficient of dissipation (δ). This coefficient, although, as we have implied, tolerably constant for one experiment, will vary very much from experiment to experiment, and from day to day; it depends above all on the weather.

Supposing we have determined this coefficient by such an observation as the above, then we can calculate the torsion τ', which we should have observed had we touched the body at any interval t' after the first experiment; for we have, provided t' be small,

$$\tau' = \tau_1 - \delta t' = \tau_2 + \delta(t - t').$$

In particular, if $t' = \frac{1}{2}t$, we have

$$\tau' = \tfrac{1}{2}(\tau_1 + \tau_2).$$

Coulomb used this principle in comparing the electric densities at two points A and A′ of the same conductor. He touched the two points a number of times in succession, first A, then A′, then A again, and so on, observing the corresponding torsions τ_1, τ_1', τ_2, τ_2', &c., the intervals between the operations being very nearly equal. He thus got for the ratio of the densities at A and A′ the values $\frac{\tau_1 + \tau_2}{2\tau_1}$, $\frac{2\tau_2}{\tau_1' + \tau_2'}$, $\frac{\tau_2 + \tau_3}{2\tau_2'}$, &c. These values ought to be all equal: the mean of them was taken as the best result.

In certain cases, where the rapidity of the electric dissipation was too great to allow the above method to be applied, Riess used the method of paired proof-planes. For a description of this, and for some elaborate calculations on the subject of electrical dissipation, the reader is referred to Riess's work.

The cage method is well adapted for experiments on distribution. The proof-plane, proof-sphere, or paired proof-planes may all be used in conjunction with it. If the cage be fairly well insualted, and a tolerably delicate Thomson's electrometer be used, so that the cage may

[1] It is evident from what we have advanced here that the use of the proof-plane to determine the electric density at points of a surface where the curvature is very great, *e.g.*, at edges or conical points is inadmissible. If we attempt to determine the electrical density at the vertex of a cone by applying a proof-sphere there, as Riess did, we shall very evidently get a result much under the mark, owing to the blunting of the point when the sphere is *in situ*. We should, on the other hand, for an opposite reason, get too large a result by applying a proof-plane edgewise to a point of a surface where the curvature is continuous.

be made large, and the surface density on its outside therefore small, there will be little loss of the external charge; and the method has this advantage, that dissipation from the proof-plane inside the cage does not affect the result of the measurement in hand, it being indifferent, *qua* effect on the electrometer, whether the electricity inside the cage be on the proof-plane, in the air, or elsewhere, provided merely it be inside. The state of the cage as to electrified air, &c., is easily tested by the electrometer at any time.

Coulomb's Results.—If we electrify a sphere, and test the electrical density at two points of its surface. experiment will show, as would be expected from the symmetry of the body, that the density at the two points is the same. If we test the electric density at any point of a sphere, and then halve its charge by division with an equal neutral sphere, and test the electric density again, we shall find it half what it was before. The electric density at any point is therefore proportional to the whole charge on the sphere, or to the *mean density*, meaning by that the whole charge divided by the whole surface of the sphere.

If, instead of a sphere, we operate with an ellipsoid generated by the revolution of an ellipse about its major axis, we shall find that the electric density is not uniform as in the case of the sphere, but greater at the sharp ends of the major axis than at the equator, and the ratio of the densities increases indefinitely as we make the ellipsoid sharper and sharper. This leads us to state a principle of great importance in the theory of electrical distribution, viz., that the electrical density is very great at any pointed part of a conductor.

If we determine the ratio of the densities at two points of an ellipsoid,[1] diminish the charge, and redetermine the same ratio, we shall find that, although the actual densities are diminished, the ratio remains the same; and if we determine the density at any point of the ellipsoid, and then halve its charge by touching it with an equal and similar ellipsoid (they must be placed with their axes in the same straight line, and made to touch at the poles),[2] and redetermine the density at the same point as before, we shall find that the density in the second case is half that in the first. We have in fact, in general, the important proposition that—

The density at any point of a conductor is proportional to the whole charge on the conductor, or, what is the same, to the mean density.

The following case given by Coulomb is interesting; it shows the tendency of electricity towards the projecting parts, ends, or points of bodies. The conductor was a cylinder with hemispherical ends,—the length of the cylinder being 30 inches, its diameter 2 inches. Coulomb gives the following results:—

Distance from end.	Density.
5 in.	1·00
2	1·25
1	1·80
0	2·30

The density at the end is thus more than twice that at the middle.

Other results, taken from Coulomb's unpublished papers, may be found in Biot,[3] Mascart, or Riess. His results for a circular disc we shall quote further on.

Riess made a series of experiments on cubes, cones, &c.; but as these are not of theoretical interest, the calculation in such cases being beyond the powers of analysis at present, and as the use of the proof-plane or sphere with bodies where edges and points occur is not free from objection, we content ourselves with referring to Riess's work for an account of the results.

Coulomb made a series of experiments on bodies of different forms, which he built up out of spheres of different sizes, or out of spheres and cylinders. These are of very great interest, partly on account of the close agreement of some of the results with the deductions subsequently made by Poisson from the mathematical theory, and partly on account of the clearness with which they convey to the mind the general principles of electric distribution. His method in most cases was to build up the conductor and electrify it with all the different parts in contact, and then after separating the parts widely, to determine the *mean density* or the whole amount of electricity on each part by the proof-plane or otherwise.

For spheres in contact he found the following results,—S, Q, σ; S′, Q′, σ' denoting the surface, quantity of electricity, and mean surface density for the two spheres respectively.

$\frac{S'}{S}$	$\frac{Q'}{Q}$	$\frac{\sigma}{\sigma'}$
3·36	3·8	1·09
14·80	11·1	1·33
62·00	37·6	1·65

From this it appears that although the whole amount of electricity on the large sphere is greater than that on the small, yet the mean density for the smaller sphere is greater than for the larger. The above result also affords an experimental illustration of the action of the earth in discharging a conductor connected with it. Comparing the conductor to the small sphere and the earth to the large sphere of 62 times the superficial area of the small one, if we start with charge Q on small sphere and then put the two in contact, the charge on the small sphere will be reduced to $\frac{1}{38\cdot6}$ Q, so that the mean density is diminished in the ratio 1 : 38·6. This ratio increases indefinitely as the ratio $\frac{S'}{S}$ increases. These results are in satisfactory agreement with Poisson's calculations. Coulomb was led by his observations to assign 2 as the limit of the ratio of the mean densities when the ratio of the diameters of the spheres is infinitely great; the mathematical theory gives $\frac{\pi^2}{6}$ or 1·65.

Coulomb also determined the density at the apex or smaller end of the body formed by two unequal spheres in contact. The following are his results, the mean density of the larger sphere being unity:—

Ratio of radii.	Density at apex.	
	Observed.	Calculated.
1	1·27	1·32
2	1·55	1·83
4	2·35	2·48
8	3·18	3·09
8	4·00	4·21

When two equal spheres are placed in contact the distribution will of course be the same in each; Coulomb found that, from the point of contact up to a point on the surface of either sphere distant from it by about 20°, no trace of electricity could be observed; at 30°, 60°, 90°,

[1] We suppose in all these experiments that we are dealing with a single body, sufficiently distant not only from all electrified bodies but from all neutral conductors to be undisturbed by them. This condition is essential.

[2] It would not do to make the pole of one touch the equator of the other, or to place them otherwise unsymmetrically.

[3] *Traité de Physique.*

180° respectively, the electric density had the relative values ·20, ·77, ·96, 1·00. When the spheres are unequal the distribution is no longer alike on each. On the small sphere it is less uniform, and the density at the point of the small sphere diametrically opposite the point of contact is greater than anywhere else on the body. The distribution on the larger sphere is more uniform than on the smaller, and the more unequal the spheres are the more uniform is the distribution on the larger, and the smaller the unelectrified part in the neighbourhood of the point of contact.

The following results of Coulomb are useful illustrations of distribution on elongated and pointed bodies :—

Three equal spheres (2 in. diameter) in contact, with their centres in the same straight line : the mean densities were 1·34, 1·00, 1·34 on the spheres 1, 2, and 3 respectively.

Six equal spheres as before: mean densities on 1, 2, and 3 = 1·56, 1·05, 1·00.

Twelve equal spheres : mean densities on 1, 2, and 6 = 1·70, 1·14, 1·00.

Twenty-four equal spheres : mean densities on 1, 2, and 12 = 1·75, 1·07, 1·00.

Large (8 in. diameter) sphere with four small (2 in.) spheres applied to it, all the centres in line: the mean density on large sphere being 1, that on the small one next it was ·60 that on the extreme small one 2·08.

Large sphere 1, and twenty-four (2 to 25) small ones: mean densities on 1, 2, 13, 24, 25 = 1·00, ·60, 1·28, 1·46, 2·17.

MATHEMATICAL THEORY OF ELECTRICAL EQUILIBRIUM.

We take as the basis of our theory the assumptions already laid down under the head Provisional Theory, and in addition the precise elementary law of electrical action established by Coulomb. We shall also suppose that we have only perfect conductors and perfect non-conductors to deal with, the medium being in all cases the same, viz., air. When we have to deal with electrified non-conductors we shall suppose the electrification to be rigid, *i.e.* incapable of disturbance by any electric force we have to consider.

In our mathematical outline we have in view the requirements of the physical more than the mathematical student, and shall pass over many points of great interest and importance to the latter, for full treatment of which we must refer him to original sources, such as the classical papers of Green, the papers of Sir William Thomson, and the works of Gauss. Of peculiar interest mathematically is the elegant and powerful memoir of the last—*Allgemeine Lehrsätze in Beziehung auf die im verkehrten Verhältnisse des Quadrats der Entfernung wirkenden Anziehungs- und Abstossungskräfte*, in which will be found detailed discussions of the continuity of the integrals used in the potential theory, &c. The works of Green and Thomson are too well known in this country to require farther remark.

When, in what follows, we speak of the *electric field*, we mean simply a portion of space which we are considering with reference to its electrical properties ; it will be found conducive to clearness to regard that space as *bounded*. In general the natural boundary would be the walls of the experimenting room ; but, for mathematical purposes, we shall, unless the contrary is stated, suppose our field to be bounded by a sphere of radius so great that the action at a point on its circumference due to an electrified body in the field is infinitely small.

The *resultant force* at a point in the electric field is the force which would be exerted on a unit of + electricity placed there without disturbing the electrical distribution elsewhere. It is plain that the resultant force has a definite magnitude and direction at every point in the field, and consequently is in modern mathematical language a *vector*. A curve drawn in the field such that its tangent at every point is in the direction of the resultant force at that point is called a *line of force*. We can draw such a line through every point of space, and if we suspend at any point a small conducting needle, it is obvious, from what we have already laid down about induction, that it will take up a position very nearly parallel to the line of force ; so that if we start from any point and carry the centre of the needle always in the direction in which the needle points we should trace out a line of force.

The *potential* at any point is the work done by a unit of + electricity in passing from that point to the infinitely distant boundary of the electric field, the electric distribution being supposed undisturbed. It is usual to call the infinitely distant boundary a place of zero potential. Zero is to be understood in the sense of "point or position from which we reckon."[1]

Consider two points P, Q, infinitely near each other in the field, and draw a curve from P passing through Q to ∞. Then, if F be the component parallel to PQ of the resultant force at P, we have by our definition

$$F.PQ = V_P - V_Q\,;$$

or in differential notation

$$Fds = -dV,$$

hence

$$F = -\frac{dV}{ds} \quad . \quad . \quad . \quad . \quad . \quad (1),$$

and

$$V = \int_P^\infty Fds = \int_P^\infty (Xdx + Ydy + Zdz) \quad . \quad . \quad . \quad (2),$$

where V denotes the potential at P, and X,Y,Z the components parallel to the co-ordinate axes of the resultant electric force. We clearly have as particular cases of (1)

$$X = -\frac{dV}{dx} \qquad Y = -\frac{dV}{dy} \qquad Z = -\frac{dV}{dz} \quad . \quad . \quad (3).$$

We may remark that, in all cases which we shall consider at present, the work done in passing from any point to any other point is the same whatever the intermediate path of our exploring unit. Hence V as above defined is a single valued function, and the formulæ (3) gives the components of resultant force when V is known.

The work done by a unit of + electricity in passing by any path from P to Q is called the *electromotive force* from P to Q; it is obviously equal to the difference of the potentials at the two points. Thus

$$V_P - V_Q = \int_P^Q (Xdx + Ydy + Zdz) \quad . \quad . \quad . \quad (4).$$

is the electromotive force from P to Q.

Expression of V in terms of definite integral.

Suppose we concentrate m units of electricity at any point P, and require the potential due to this at a point Q, distant D from P. Applying (2), and, since any path to ∞ may be chosen, taking the integral along the production of PQ to ∞, we get

$$V = \int_D^\infty \frac{m}{r^2}\,dr = \frac{m}{D} \quad . \quad . \quad . \quad . \quad (5).$$

If we have any number of discrete points with charges m_1, m_2, m_3, ... at distances D_1, D_2, D_3, ... from Q, since the work done by the exploring unit under the action of the whole is got by adding up the work done under the action of each part separately, we clearly have

$$V = \frac{m_1}{D_1} + \frac{m_2}{D_2} + \&c. = \Sigma\,\frac{m}{D} \quad . \quad . \quad . \quad . \quad (6).$$

From this we may pass to the case of a continuous volume distribution. If ρ be the volume density at the point $\xi\eta\zeta$, and V the potential at xyz, we have

$$V = \iiint \frac{\rho\, d\xi\, d\eta\, d\zeta}{D} \quad . \quad . \quad . \quad . \quad . \quad . \quad (7),$$

where D denotes $+\sqrt{(\xi - x)^2 + (\eta - y)^2 + (\zeta - z)^2}$, and the integral is to be extended over every part of the field where there is any charge,—or, which is the same thing, over the whole field, on the understanding that $\rho = 0$ where there is no charge.

If, as will generally be the case, the electricity is distributed on a surface in such a way that on an element dS of surface there is a quantity σdS of electricity, where σ is a finite surface density, then

$$V = \iint \frac{\sigma dS}{D} \quad . \quad . \quad . \quad . \quad . \quad (8),$$

where D has the same meaning as before, and the integral is extended all over the electrified surface or surfaces.

[1] It may be well here to warn the reader that measurement of potential is relative, just as much as measurement of distance is, and to caution him against the fallacious idea of absolute zero of potential.

We may make here the important remark that, so long as ρ or σ is not infinite, the integrals in (7) and (8) are finite and continuous. This depends on the fact, which we cannot stop to prove, that the part of the potential at P, contributed by an infinitely small portion of electricity surrounding P, is infinitely small.

In practice, therefore, the electric potential is always continuous; for although we may in theory speak of discrete points and electrified lines where finite electrification is condensed into infinitely small space, yet no such cases ever occur in nature. It may also be shown for any electrical system of finite extent, that, as the distance of P from O, any fixed point at a finite distance from the system is increased indefinitely, the potential at P approaches more and more nearly the value $\frac{M}{D}$, where M is the algebraical sum of all the electricity in the system, and D the distance of P from O. Hence at any point infinitely distant from O, V = 0.

We next proceed to prove the following proposition, which will form the basis of the subsequent theory :—

Surface integral of electric induction.

The surface integral of electric induction taken all over the surface inclosing any space is equal to 4π times the algebraical sum of all the electricity in that space.

By the electric induction across any element of the surface (taken so small that the resultant force at every point of it may be regarded as uniform) is meant the product of the area of the element into the component of the resultant force in the direction of the normal to the element which is drawn outwards with respect to the inclosed space. Thus dS being an element of surface, ϵ the angle between the positive direction of the resultant force R and the outward normal ν, and E the sum of all the electricity in the inclosed space, the proposition in symbols is—

$$\iint R\cos\epsilon\, dS = 4\pi E \quad . \quad . \quad . \quad . \quad (9).$$

We shall prove it in the manner most naturally suggested by the theory of electrical elements acting at a distance, by first showing that it is true for a single element e either outside or inside the surface. Let us suppose e to be at a point P, fig. 11, within S, which for greater generality we may suppose to be a re-entrant surface. Draw a small cone of vertical solid angle $d\omega$ at P, and let it cut the surface in the elements QR, Q′R′, Q″R″; let the outward normals to these be QM, Q′M′, Q″M″. The elements of the surface integral contributed by QR, Q′R′, Q″R″ are obviously $\frac{QRe\cos\epsilon}{PR^2}$, &c.; but $QR = \frac{d\omega PR^2}{\cos\epsilon}$,

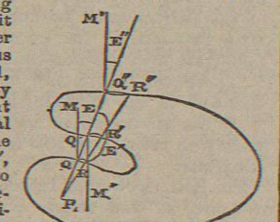

Fig. 11.

$Q'R' = -\frac{d\omega PR'^2}{\cos\epsilon'}$, and $Q''R'' = \frac{d\omega PR''^2}{\cos\epsilon''}$; hence the three elements of the integral become $+ed\omega, -ed\omega, +ed\omega$; and the sum is $ed\omega$. Adding now the contributions from all the little cones which fill up the solid angle of 4π about P, we get

$$\iint R\cos\epsilon\, dS = e\iint d\omega = 4\pi e.$$

Had the point P been outside, the numbers of emergences and entrances would have been equal, the contribution of each cone zero, and on the whole

$$\iint R\cos\epsilon\, dS = 0.$$

Combining these results, we see that the proposition is true for a single element. Hence, by summation for all the elements, we can at once extend it to any electrical system ; for all the elements external to S give zero, and all the internal elements give $4\pi\Sigma e = 4\pi E$.

Let us apply the above proposition to the space enclosed by the infinitely small parallelepiped whose centre is at xyz, and the co-ordinates of whose angles are $x \pm \frac{1}{2}dx$, $y \pm \frac{1}{2}dy$, $z \pm \frac{1}{2}dz$. The contributions to the surface integral from the two faces perpendicular to the x-axis are $-\left(\frac{dV}{dx} + \frac{dx}{2}\frac{d^2V}{dx^2}\right)dydz$ and $\left(\frac{dV}{dx} - \frac{dx}{2}\,\frac{d^2V}{dx^2}\right)dydz$.

Adding these and the four parts from the remaining sides, and equating to $4\pi\rho dxdydz$, which is the 4πE in this case, we have

$$\frac{d^2V}{dx^2} + \frac{d^2V}{dy^2} + \frac{d^2V}{dz^2} + 4\pi\rho = 0\,,$$

or, as it is usually abbreviated,

$$\nabla^2 V + 4\pi\rho = \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10).$$

Equation (10), originally found by Laplace for the case $\rho = 0$, and extended by Poisson, has been called the characteristic equation of the potential. It may be applied at any point where ρ is finite and the electric force continuous. It might be shown by examining the integrals representing X, Y, Z, and $\frac{dV}{dx}$, &c., that the electric force is continuous wherever there is finite volume density. Equation (10) may be looked on either as an equation to determine the potential when ρ is given, or as an equation to determine ρ when V is given. We shall have occasion to use it in both ways.

Conditions at an electrified surface.

The characteristic equation cannot be applied in the form (10) when the resultant force is discontinuous. This will be found to be the case at a surface on which electricity is distributed with finite surface density. Let us consider the values of the resultant force at two points, P and Q, infinitely near each other, but on opposite sides of such a surface. Resolve the resultant force tangentially and normally to the surface. If we consider the part of the force which arises from an infinitely small circular disc, whose radius, though infinitely small, is yet infinitely great compared with the distance between P and Q, we see that infinitely little is contributed to the tangential component at P or Q by this disc, while it can be readily shown that the part of the normal component arising therefrom is $2\pi\sigma$, directed *from* the disc *in each* case, when σ is the surface density. Hence, since the part of the resultant force arising from all the rest of the electrified system obviously is not discontinuous between P and Q, the tangential component is continuous when we pass through an electrified surface, but the normal component is suddenly altered by $4\pi\sigma$.

For a thorough investigation of these points the reader is referred to Gauss or Green. We can arrive very readily at the amount of the discontinuity of the normal force by applying (9) to the cylinder formed by carrying an infinitely short generating line round the element dS, so that one end of the cylinder is on one side of dS and the other on the other, the lateral dimensions being infinitely small, but still infinitely greater than the longitudinal. The only part of the integral which is of the order of dS is the part arising from the two ends ; hence if N, N′ be the value of the normal components on the two sides of S, we clearly get

$$(N - N')\, dS = 4\pi\sigma dS\,, \text{ or } N - N' = 4\pi\sigma.$$

If V_1, V_2 denote the potentials on the two sides of S, and ν_1, ν_2 the normals to dS, drawn towards these sides respectively, then we may obviously write our equation

$$\frac{dV_1}{d\nu_1} + \frac{dV_2}{d\nu_2} + 4\pi\sigma = 0 \quad . \quad . \quad . \quad . \quad . \quad (11).$$

Surface characteristic equation.

Written in this form the equation has been called the surface characteristic equation of the potential. It may be looked upon as a surface condition, which must be fulfilled by the values of V on the two sides of an electrified surface on which the surface density σ is given, and where, in consequence, there is discontinuity in the first differential coefficients of V ; or it may be looked on as an equation to determine σ when V_1 and V_2 are given.

Level surface.

We have seen that we can draw through every point of the electric field a line of force. A surface constructed so that the potential at every point of it has the same value is called an *equipotential or level surface*. We can obviously draw such a surface passing through every point of the field. It is clear, too, that the line of force at every point must be perpendicular to the level surface passing through that point. For since no work is done on a unit of + electricity in passing from one point of a level surface to a neighbouring point, there can be no component of the resultant force tangential to the surface; in other words, the direction of the resultant force is perpendicular to the surface. This is expressed otherwise by saying that the lines of force are orthogonal trajectories to the level surface.

If we take a small portion of a level surface, and draw through every point of the boundary a line of force, we shall thus generate a tubular surface which will cut orthogonally every level surface which it meets. Such a surface is called a *tube of force*.

Let a tube of force cut two level surfaces in the elements dS and dS′. Apply to the space contained by the part of the