tube between the surfaces our fundamental equation (9). We thus get, since there is no normal component perpendicular to the generating lines of the tube,

$$RdS - R'dS' = 0, \quad . \quad . \quad . \quad . \quad (12),$$

provided the tube does not cut through electrified matter between the two surfaces. Here R and R′ denote the resultant force at dS and $d'S$, which are supposed so small that the force may be considered uniform all over each of them. It appears then that *the product of the resultant force into the area of the normal section of a tube of force is constant for the same tube so long as it does not cut through electrified matter; or what amounts to the same, the resultant force at any point of a tube of force varies inversely as the normal section of the tube at that point.*

Important property of tubes of force $RdS = R'dS'$.

If we divide up any level surface into a series of small elements, such that the product RdS is constant for each element and equal to unity, and draw tubes of force through each small element, then the electric induction through any finite area of the surface is equal to the number of tubes of force which pass through that area; for if n be that number, we have, summing over the whole of the area—

$$\Sigma RdS = n \quad . \quad . \quad . \quad . \quad . \quad (13),$$

the left hand side of which is the electric induction through the finite area. It is clear, from the constancy of the product RdS for each tube of force, that if this is true for one level surface it will be true for every other cut by the tubes of force. It is evident that the proposition is true for any surface, whether a level surface or not, as may be seen by projecting the area considered by lines of force on a level surface, and applying to the cylinder thus formed the surface integral of electric induction, it being remarked as obvious that the same number of tubes of force pass through the area as through the projection. This enables us to state the proposition involved in equation (9) in the following manner:—

Charge measured by tubes of force.

The excess of the number of tubes of forces which leave a closed surface over the number which enter is equal to 4π times the algebraical sum of all the electricity within the surface.

(*N.B.*—The positive direction of a line of force is that direction in which a unit of + electricity would tend to move along it.) This proposition enables us to measure the charge of a body by means of the lines[1] of force. We have only to draw a surface inclosing the body, and very near to it, and count the lines of force entering and leaving the surface. If the number of the latter, diminished by the number of the former, be divided by 4π, the result is the charge on the body.

If we apply (13) to a portion of an equipotential surface so small that R may be considered uniform over the whole of it, we may write

$$R = \frac{n}{dS} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14);$$

Resultant force measured by lines of force.

or in words:—*The resultant force at any point is equal to the number of lines of force per unit of area of level surface at that point*, meaning thereby the number of lines of force which would pass through a unit of area of level surface if the force were uniform throughout, and equal to its value at the point considered.

We are now able to express by means of the lines of force the resultant force at any point of the field, and the charge in any element of space. The electrical language thus constructed was invented by Faraday, who continually used it in his electrical researches. In the hands of Sir William Thomson, and particularly of Professor Clerk Maxwell, this language has become capable of representing, not only qualitatively but also quantitatively, with mathematical accuracy, the state of the electric field. It has the additional advantages of being well fitted for the use of the practical electrician, and of lending itself very readily to graphical representation.

It will be convenient, before passing to electrical applications, to state here another general property of the potential which follows from our fundamental proposition.

Maximum or minimum potential impossible in free space.

The potential cannot have a maximum or minimum value at a point where there is no electricity.

For if a maximum value were possible, we could draw round the point a surface at every point of which the potential was decreasing outwards; consequently at every point of this surface the normal component of the resultant force in the outward direction would be positive, and a positive number of lines of force would leave the surface. But this is impossible, since, by our hypothesis, there is no electricity within. Similarly there could be no minimum value.

Case of space bounded by level surface.

From this it follows at once that *if the potential have the same value at every point of the boundary of a space in which there is no electrified body, then the potential is constant throughout that space, and equal to the value at the boundary.* For if the potential at any point within had any value greater or less than the value at the boundary, this would be a case of maximum or minimum potential at a point in free space, which we have seen to be impossible.

In order that there may be electrical equilibrium in a perfect conductor, it is necessary that the resultant electric force should be zero at every point of its substance. For if it were not so at any point the positive electricity there would move in the direction of the resultant force and the negative electricity in the opposite direction, which is inconsistent with our supposition of equilibrium. This condition must be satisfied at any point of the conductor, however near the surface. At the surface there must be no tangential component of resultant force, otherwise electricity would move along the surface. In other words, the resultant force at the surface must be normal; its magnitude is not otherwise restricted;[2] for by our hypothesis electricity cannot penetrate into the non-conducting medium.

These conditions are clearly sufficient. We may sum them up in the following single statement:—

Condition of electrical equilibrium.

If the electricity in any conductor is in equilibrium, the potential must have the same value at every point in its substance.

For if the potential be constant, its differential coefficients are zero, so that inside the conductor the resultant force vanishes. Also the surface of the conductor is a level surface, and therefore the resultant force is everywhere normal to it. This constant value of the potential we shall henceforth speak of as the *potential of the conductor.*

Electricity resides on the surface.

Since the potential is constant at every point in the substance of a charged conductor, we have at every point $\nabla^2 V = 0$, and hence by the equation of Poisson $\rho = 0$; that is, there is no electricity in the substance of the conductor. We thus get, as a theoretical conclusion from our hypothesis, the result already suggested by experiment, that electricity resides wholly on the surface of conductors.

If we apply the surface characteristic equation to any point of the surface of a conductor, we get

$$\sigma = -\frac{1}{4\pi}\frac{dV}{d\nu} = \frac{R}{4\pi}, \quad . \quad . \quad . \quad (15),$$

which gives the surface density in terms of the resultant force and reciprocally.

We may put this into the language of the lines of force by saying that *the charge on any portion of the surface of a conductor is equal to the number of lines of force issuing from it divided by 4π.*

Since the surface of a conductor in electric equilibrium

[1] Here we drop the distinction between line and tube of force. We shall hereafter suppose the lines of force to be always drawn so as to form unit tubes, and shall speak of these tubes as lines of force, thereby following the usual custom.

[2] Of course in practice there is an upper limit, at which disruptive discharge occurs.

is always a level surface, it follows, from what we have already proved about a space bounded by a surface of constant potential, that, *inside a hollow conductor the potential is constant, provided there be no electrified bodies within.* This is true, no matter how we electrify the conductor or what electrified bodies there may be *outside.* Hence, if we inclose any conductor A completely within another B, then electrify B and put A in metallic communication with it, A will not become charged either + or −; for, A being at the same potential as B, electricity will not tend to flow from the one to the other. This is in reality Biot's[1] experiment with the hemispheres, to which we have already alluded; only the point of view is slightly changed. The most striking experiment ever made in illustration of the present principle is that described by Faraday in his *Experimental Researches.* He constructed a hollow cube (12 feet in the edge) of conducting matter, and insulated it in the lecture-room of the Royal Institution. We quote in his own words the part of his description which bears on the present question:—

"1172. I put a delicate gold-leaf electrometer within the cube, and then charged the whole by an outside communication, very strongly for some time[2] together; but neither during the charge or after the discharge did the electrometer or air within show the least sign of electricity. . . . . I went into the cube and lived in it, and using all other tests of electrical states, I could not find the least influence upon them, though all the time the outside of the cube was powerfully charged, and large sparks and brushes were darting off from every point of its outer surface."

Indirect evidence for the law of inverse square.

The proposition that the potential is constant inside a hollow conductor containing no electrified bodies may be regarded as one of the most firmly established in the whole of experimental science. The experiments on which it rests are of extreme delicacy. It is of the greatest theoretical importance; for we can deduce from it the law of the inverse square. Taking the particular case of a spherical shell, uninfluenced by other bodies, on which of course the electrical distribution must from symmetry be uniform, it can be demonstrated mathematically that, if we assume the action between two elements of electricity to be a function of the distance between them, then that function must be the inverse square, in order that the potential may be constant throughout the interior. A demonstration of this proposition was given by Cavendish, who saw its importance; a more elaborate proof was afterwards given by Laplace; for a very elegant and simple demonstration we refer the mathematical reader to Clerk Maxwell's *Electricity*, vol. i. § 74. This must be regarded as by far the most satisfactory evidence for the law of the inverse square; for the delicacy of the tests involved infinitely surpasses that of the measurements made with the torsion balance; and now that we have instruments of greatly increased sensitiveness, like Thomson's quadrant electrometer, the experimental evidence might be still further strengthened.

General problem of electrical distribution.

In the problem to determine the distribution of electricity in a given system of conductors, the data are in most cases either the charge or the potential for each conductor. If the conductor is insulated it can neither give nor lose electricity, its charge is therefore given. If, on the other hand, it be connected with some inexhaustible source of electricity at a constant potential, its potential is given. Such a source the earth is assumed to be; and we shall henceforth take the potential of the earth as zero, and reckon the potential of all other bodies with reference to it. If all our electrical experiments were conducted in a space inclosed by a perfectly conducting envelope, the potential of this envelope would be the natural zero of our reckoning.

It will be useful to analyse more closely the distribution on a system of conductors, in order to see how far the above data really determine the solution of the electrical problem. For this purpose the following proposition is useful. If $e_1, e_2, \ldots . e_n$ be the charges at the points $1, 2, \ldots . n$ of any system, and V the potential at P, and if V′ be the potential at P due to $e_1', e_2', \ldots . e_n'$ at $1, 2, \ldots . n$, then the potential at P due to $e_1 + e_1'$, $e_2 + e_2', \ldots .$ at $1, 2, \ldots .$ is $V + V'$. This principle follows at once from the definition of the potential as a sum formed by the mere addition of parts due to all the single elements of the system.

Principle of electrical superposition.

Applied to a system of conductors in equilibrium, it may evidently be stated thus: If $E_1, E_2, \ldots . E_n$ and $V_1, V_2, \ldots . V_n$ be the respective charges and potentials for the conductors $1, 2, 3 \ldots . n$ in a state of equilibrium, $E_1', E_2', \ldots . E_n'$ and $V_1', V_2', \ldots . V_n'$ corresponding charges and potentials for another state of equilibrium, then $E_1 + E_1' \ldots . E_n + E_n'$, $V_1 + V_1' \ldots . V_n + V_n'$ will be corresponding charges and potentials for a third state of equilibrium.

Particular case of general problem.

Suppose that in the system of conductors $1, 2, 3, \ldots . . n$ the conductor 1 is kept at potential 1 and all the others at potential zero, then it can be shown that there is one and only one distribution of electricity fulfilling these conditions. Mathematically stated, the problem is to determine a function V, which shall satisfy the equation $\nabla^2 V = 0$ throughout the space unoccupied by conductors, and have the values $1, 0, 0, \ldots . . 0$ was respectively at each point of the surfaces of $1, 2, \ldots . . n$ respectively.

Consider the integral

$$I = \iiint \left\{ \overline{\frac{dV}{dx}}\Big|^2 + \overline{\frac{dV}{dy}}\Big|^2 + \overline{\frac{dV}{dz}}\Big|^2 \right\} dxdydz \; . \quad . \quad . \quad . \quad (16),$$

where the integration is extended all over the space unoccupied by conductors. If we consider all the values which this integral may have, consistent with the boundary conditions $V = 1$, $V = 0, \ldots . .$ &c. at the surfaces of $1, 2, \ldots . .$ &c., it is obvious that there must be a minimum value; for the integral is essentially positive. and cannot become less than zero.

$$\text{Now } \delta I = 2\iiint \left(\frac{dV}{dx}\frac{d\delta V}{dx} + \text{\&c.} \right) dxdydz$$

$$= 2\iiint \delta V \nabla^2 V dxdydz \quad . \quad . \quad . \quad . \quad . \quad (17)$$

by partial integration. The surface terms vanish, since $\delta V = 0$ at every surface. Hence $\nabla^2 V = 0$ is the condition for a maximum or minimum value of I, and since we know that a minimum value exists, there must be a solution of this equation. It can, moreover, be shown, by a method which we shall apply below to the more general problem, that there is only one solution of $\nabla^2 V = 0$ consistent with the given conditions, and this will of course be that which makes I a minimum. If our mathematical methods were powerful enough to determine V, we might proceed to find the surface density for each conductor by means of the formula $\sigma = -\frac{1}{4\pi}\frac{dV}{d\nu}$; then the charges on the conductors could be found by means of the integral $-\frac{1}{4\pi}\iint \frac{dV}{d\nu} dS$. In very few cases indeed could we actually find these charges; we have, however, demonstrated their existence and shown that our problem is definite.

Coefficients of capacity and induction.

Let these charges on $1, 2, \ldots n$ be called $q_{11}, q_{12} \cdots q_{1n}$. Corresponding to the data $0, 1, 0, \ldots . 0$ for the potentials of $1, 2, \ldots n$, we should get a series of charges $q_{21}, q_{22}, \ldots .$ q_{2n}, and so on; $q_{11}, q_{22}, q_{33} \ldots$ are called the coefficients of self-induction or capacity for the conductors $1, 2, 3, \ldots$; q_{12}, q_{13}, &c., are called the coefficients of induction of 1 on 2, 1 on 3, &c. It is obvious that these coefficients depend solely on the form and relative position of the conductors. It follows, from the principle of the superposition, that, if $1, 2, \ldots n$ be at the potentials $V_1, 0, 0 \ldots 0$, then the charges on them will be $q_{11}V_1, q_{12}V_1 \ldots . \; q_{1n}V$ We

[1] The experiment was first made by Cavendish. There is an account of it in his hitherto unpublished papers.

[2] Faraday was looking for what he called the absolute charge of matter; incidentally the experiment illustrates the point we are discussing.

may construct then a series of states of equilibrium represented thus :—

Potential,	V_1	0	0	......	0
Charge,	$q_{11}V_1$	$q_{12}V_1$	$q_{13}V_1$	...	$q_{1n}V_1$
Potential,	0	V_2	0	...	0
Charge	$q_{21}V_2$	$q_{22}V_2$	$q_{23}V_2$	......	$q_{2n}V_2$

and so on. Superposing all these, we get a system in equilibrium, in which the potentials are $V_1, V_2, \ldots V_n$, and the charges

$$\left.\begin{aligned} E_1 &= q_{11}V_1 + q_{21}V_2 + \ldots + q_{n1}V_n \\ E_2 &= q_{12}V_1 + q_{22}V_2 + \ldots + q_{n2}V_n \\ \&c. &= \&c. \end{aligned}\right\} \quad . \; . \; . \quad (18).$$

It appears therefore that the $2n$ quantities E_1, &c., V_1, &c., are connected by n linear equations; so that when n of them are given, the rest can be determined in terms of these in a definite manner.

Returning then to our general problem, we see that, when either the charge or the potential is given for each conductor, the electrical problem is determinate, and a solution is given by the linear equations of (18). The potential at any point of the field can be written down very easily. Suppose in fact v_1 to be the value at the point P of the function V which we determined in solving the case where the potentials 1, 0, 0, ... 0 are given for 1, 2, . . . n, v_2 the corresponding function for the case 0, 1, 0, . . . 0, and so on. Then the potential at P in the general case is obviously

$$V = V_1 v_1 + V_2 v_2 + \ldots + V_n v_n \quad . \; . \quad (19),$$

where $v_1, v_2, \ldots v_n$ are all known functions, and $V_1, V_2, \ldots V_n$ are all either given, or determined in terms of given quantities by the equations (18).

It is very easy to show that there is no other solution of the problem than the one we have found.

Suppose in fact that V' is a function different from V, which satisfies all the conditions of the problem. Consider the function $U = V - V'$, since V and V' both satisfy the equation $\nabla^2 V = 0$, we have $\nabla^2 U = 0$. Also at surfaces where V is given $U = 0$. At surfaces where V is not given, we have U = constant − constant = constant; and, since in this case the charge will be given, we shall have

$$\iint \frac{dV}{d\nu} dS = \iint \frac{dV'}{d\nu} dS \text{; and therefore} \iint \frac{dU}{d\nu} dS = 0.$$

Now we have

$$\iiint \left\{ \left(\frac{dV}{dx} - \frac{dV'}{dx}\right)^2 + \&c. \right\} dxdydz$$

$$\iiint \left\{ \left(\frac{dU}{dx}\right)^2 + \left(\frac{dU}{dy}\right)^2 + \left(\frac{dU}{dz}\right)^2 \right\} dxdydz$$

$$-\iint \frac{dU}{d\nu} U dS - \iiint U \nabla^2 U dxdydz.$$

The first term vanishes for all the surfaces,—for some because $U = 0$, for others because U is constant and $\iint \frac{dU}{d\nu} dS = 0$; and the second term vanishes because $\nabla^2 U = 0$.

Hence the integral on the left hand must vanish, and that too element by element, since every element is positive. Hence we must have

$$\frac{dV}{dx} = \frac{dV'}{dx}, \quad \frac{dV}{dy} = \frac{dV'}{dy}, \quad \frac{dV}{dz} = \frac{dV'}{dz}.$$

Hence V and V' can only differ by a constant. But such difference is precluded by the boundary conditions. Hence the functions are identical; in other words, there is but one solution to the problem we have proposed.

It is very easy to show, by methods of which we have already had an example, that the value of V thus found makes the integral

$$\frac{1}{8\pi} \iiint \left(\left(\frac{dV}{dx}\right)^2 + \left(\frac{dV}{dy}\right)^2 + \left(\frac{dV}{dz}\right)^2 \right) dxdydz$$

a minimum. Now, we shall show directly that this integral represents the potential energy of the system. It follows, therefore, that the distribution which we have found is in *stable* equilibrium.

If we solve the equations (18), we shall get

$$\left.\begin{aligned} V_1 &= p_{11}E_1 + p_{21}E_2 + \ldots p_{n1}E_n \\ V_2 &= p_{12}E_1 + p_{22}E_2 + \ldots p_{n2}E_n \\ &\&c. \end{aligned}\right\} \quad \ldots \quad (20).$$

A set of equations which we might obviously have arrived at by first principles. The physical meaning of the coefficients p_{11}, p_{12}, and p_{1n} is very obvious; they are the potentials, corresponding to a state of equilibrium, in which the charges on 1, 2, 3, ... n are 1, 0, 0, . . . 0, and so on. p_{11}, p_{12}, . . &c., are called coefficients of potential; and, *mutatis mutandis*, all the remarks already made about q_{11}, q_{12}, &c., apply to them. Many interesting and important theorems have been proved about these coefficients, for which we refer the reader to Maxwell (*Electricity*, vol. i. chap. 2), whose treatment of the subject we have in the main been following. One of these, of great importance, we shall prove here, because it leads us to state a very important general theorem, which we shall have occasion to use again.

Theorem of mutual potential energy. The mutual potential energy of two electrical systems, A and B, is the work done in removing the two systems to an infinite distance from each other, the internal arrangement of each system being supposed unaltered during the process. It is clear that we may suppose either that A is fixed and B moves off to infinity, or that B is fixed and A moves; the work done in both cases is, by Newton's third law of motion, the same. This is sometimes expressed by saying that *the potential of A on B is the same as that of B on A.*

In fact, the expression for the mutual potential energy is

$$\Sigma \frac{qq'}{D} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (21),$$

where q is any element of electricity belonging to A, and q' any element belonging to B, and D is the distance between them, the summation being extended so as to include every pair of elements. We may arrange (21) as follows :—

$$q_1' \Sigma_1 \frac{q}{D} + q_2' \Sigma_2 \frac{q}{D} + \&c.,$$

each group belonging to a point in B, or, as we may write it,

$$q_1' V_1 + q_2' V_2 + \&c., \text{ or } \Sigma q' V.$$

We may also arrange (21) in the form

$$q_1 \Sigma_1 \frac{q'}{D} + q_2 \Sigma_2 \frac{q'}{D} + \&c.,$$

each group belonging to a point in A. Hence we have the following equalities :—

$$\Sigma q' V = \Sigma \frac{qq'}{D} = \Sigma q V' \quad . \; . \; . \; . \; . \quad (22).$$

The first and last of these expressions are called respectively the potential of A on B, and the potential of B on A, and this equality explains the statement made above.

The two systems A and B may be different states of equilibrium of the same system, if we choose. In this case we may still farther modify the expression in (22), and write

$$V_1 \Sigma_1 q' + V_2 \Sigma_2 q' + \&c. = V_1' \Sigma_1 q + V_2' \Sigma_2 q + \&c. \quad \text{(See Gauss, } l.c.\text{)}$$

So that we may state the proposition thus :—If $E_1, E_2, \ldots E_n, V_1, V_2, \ldots V_n$, and $E_1', E_2', \ldots E_n', V_1', V_2', \ldots V_n'$ be the respective charges and potentials of the conductors in two different states of equilibrium, then we have

$$\Sigma E' V = \Sigma E V' \quad . \; . \; . \; . \quad (23).$$

If we take for the two states of the system

$$\frac{E}{V} \Big\| \frac{q_{11}}{1} \Big| \frac{q_{12}}{0} \Big| \frac{q_{13}}{0} \Big| \frac{\ldots}{\ldots} \Big| \frac{q_{1n}}{0} \Big|$$

$$\text{and} \quad \frac{E'}{V'} \Big\| \frac{q_{21}}{0} \Big| \frac{q_{22}}{1} \Big| \frac{q_{23}}{0} \Big| \frac{\ldots}{\ldots} \Big| \frac{q_{2n}}{0} \Big|$$

equation (23) becomes

$$q_{21} = q_{12} \quad . \; . \; . \quad (24),$$

or, in words, *the coefficient of induction of* 1 *on* 2 *is equal to that of* 2 *on* 1.

There is one more general theorem on electrical distribution which, from its great practical importance, deserves a place here. Suppose we take a hollow conductor of any form, place any electrical system inside it, and connect the conductor with the earth, then equilibrium will be established, in such a way that the potential of every portion of the conductor is zero. Now, the potential being zero at all infinitely distant points, we may regard the outside space as inclosed by a surface of zero potential; hence the potential at every point in this space must be the same, and there can be no electrical action anywhere outside.

Again, removing the internal system, let us place any system outside the conductor, and, besides, charge it to any desired extent, keeping it insulated this time. Then the outer and inner surfaces of the conductor will be level surfaces; and, since there is no electricity inside the inner surface, the potential in the interior will be constant. Hence the external system, in a state of equilibrium, exerts no action whatever within. Now we may evidently, without mutual disturbance, superpose such an internal and external system as we have described, and still get a system in equilibrium. It is, moreover, clear that we can in this way satisfy the most general conditions that can be assigned. Hence, since we know that there can be only one solution of the problem of electrical equilibrium, the synthetical one thus obtained represents the actual state of affairs. When, therefore, a hollow conductor with any external and internal systems is in equilibrium, *the equilibrium of the internal is independent of that of the external system.*

Moreover, if we draw any surface *in the substance of the hollow conductor*, no lines of force cross it in one direction or the other; therefore the whole amount of electricity within must be zero; in other words, *the charge on the internal surface of the conductor is equal and opposite to the algebraical sum of the charges on all the bodies within.*

Electrical screens. These propositions contain the principle of what are called electrical screens, *i.e.* sheets of metal used to defend electrical instruments, &c., from external influences. On the practical efficiency of gratings in this way, see Maxwell (§ 203); on the application to the theory of lightning conductors, see a paper by him in the reports of the British Association for 1876.

If we take the simple case where there is no external system, but only a charge on the hollow conductor, we get a complete explanation of Faraday's ice-pail experiment.

Potential energy of a system of conductors. The potential energy of a system of charged conductors is the work required to bring them from a neutral state to the charges and potentials which they have at any time. The state of zero potential energy here contemplated is of course that in which there is an equal amount of + and − electricity everywhere in the system, or, as we might put it, the state in which there is no electrical separation. Now if Q denote the potential energy of the system, we have with the notation of (21)

$$Q = \Sigma \frac{qq'}{D} \quad . \; . \; . \; . \quad . \; . \quad (25),$$

the summation including every pair of elements in the system. If the system be in equilibrium, then, reasoning as above, it is obvious that ΣEV is just twice $\Sigma \frac{qq'}{D}$, inasmuch as each pair of elements will come in twice. Hence we get

$$Q = \tfrac{1}{2} \Sigma EV \quad . \; . \; . \; . \; . \; . \; . \quad (26).$$

This is an expression of the greatest importance. We can give it various forms; by means of (18) and (20) we get

$$Q = \tfrac{1}{2} \sum_{r=1}^{r=n} \sum_{s=1}^{s=n} q_{rs} V_r V_s = \tfrac{1}{2} \sum_{r=1}^{r=n} \sum_{s=1}^{s=n} p_{rs} E_r E_s \quad . \; . \quad (27).$$

So that Q is a homogeneous quadratic function of the potentials or of the charges. If, therefore, we increase the potentials of *all* the conductors, or the charges of *all* the conductors in any ratio, we increase thereby the potential energy in the duplicate of that ratio.

We can by a transformation, which is a particular case of a theorem of Green's, obtain a very remarkable volume integral for the potential energy of an electrical system.

Green's theorem. Let V denote the potential at any point in the field. Consider the integral

$$\frac{1}{8\pi} \iiint \left(\left(\frac{dV}{dx}\right)^2 + \left(\frac{dV}{dy}\right)^2 + \left(\frac{dV}{dz}\right)^2 \right) dxdydz,$$

where the integration is to be extended throughout the whole of the space unoccupied by conductors We have by partial integration

$$\iiint \left(\frac{dV}{dx}\right)^2 dxdydz = \iint V \frac{dV}{dx} dydz - \iiint V \frac{d^2V}{dx^2} dxdydz,$$

and two similar equations. Hence

$$\frac{1}{8\pi} \iiint \left(\left(\frac{dV}{dx}\right)^2 + \left(\frac{dV}{dy}\right)^2 + \left(\frac{dV}{dz}\right)^2 \right) dxdydz$$

$$= -\frac{1}{8\pi} \iint V \frac{dV}{d\nu} dS - \frac{1}{8\pi} \iiint V \nabla^2 V dxdydz,$$

where the surface integration extends over the surface of all the conductors, and it is to be noticed that $d\nu$ is drawn *from the conductor* into the insulating medium. If ρ and σ be volume and surface densities,

$$\sigma = -\frac{1}{4\pi} \frac{dV}{d\nu}, \text{ and } \rho = -\frac{1}{4\pi} \nabla^2 V.$$

Thus we get

$$\frac{1}{8\pi} \iiint \left(\left(\frac{dV}{dx}\right)^2 + \left(\frac{dV}{dy}\right)^2 + \left(\frac{dV}{dz}\right)^2 \right) dxdydz$$

$$= \tfrac{1}{2} \iint V \sigma dS + \tfrac{1}{2} \iiint V \rho dxdydz \quad . \; . \quad (28).$$

This result includes a more general case than our present one; for it shows that the potential energy of an electrical system is given by the integral on the left hand side in all cases, whether there is equilibrium or not. It is not even restricted to the case of perfect conductors and perfect non-conductors, for a slight modification of our preliminary statements would include that case as well. At present, however, we have $\rho = 0$ everywhere, and V constant at the surface and in the substance of each conductor, so that the right hand side is simply the expression $\tfrac{1}{2}\Sigma EV$ which we have already found for the potential energy; we may therefore write

$$Q = \frac{1}{8\pi} \iiint \left(\left(\frac{dV}{dx}\right)^2 + \left(\frac{dV}{dy}\right)^2 + \left(\frac{dV}{dz}\right)^2 \right) dxdydz$$

$$= \frac{1}{8\pi} \iiint R^2 dv \quad . \; . \; . \; . \; . \; . \quad (29),$$

R being the resultant force at any point of the field, and dv the element of volume. It is clear that we may if we like extend the integration over the *whole* field, since in the substance of any conductor $R = 0$.

Force tending to produce any change of configuration. When we know the potential energy of an electrical system it is very easy to find the force which resists or tends to produce any change of configuration. Two particular cases are of common occurrence and of considerable interest. First, let the charges on all the conductors be kept constant. Let the variable which is altered by the supposed change of configuration be ϕ, and let Φ be the corresponding force[1] tending to increase ϕ. Then, since no energy is supplied from without, if we suppose the displacement made infinitely slowly, so that no kinetic energy is generated, we have

[1] Or generalized force component, *i.e.*, the amount of work per unit of ϕ done in increasing ϕ.

$$\Phi\delta\phi+\delta Q=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30),$$

or

$$\Phi=-\frac{dQ}{d\phi} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31).$$

Referring to the second of the expressions in (27), we see that this may be written

$$\Phi=-\tfrac{1}{2}\sum_{r=1}^{r=n}\sum_{s=1}^{s=n}E_rE_s\frac{dp_{rs}}{d\phi}\,.$$

From this it is evident that in similarly electrified states of the same system the force tending to produce a given displacement varies as the square of the electrification. It is important to remark that in the present case the system tends to move so that its potential energy is *decreased.*

Secondly, let us suppose that the potentials of the different conductors are kept constant during any displacement, energy being supplied from without.

We shall suppose the change made in two steps. First, we shall suppose the given displacement to take place while the charges remain constant. On this supposition the force exerted will, to the first order of small quantities, be the same as that exerted when we suppose the potential not to vary; hence

$$\Phi\delta\phi+\tfrac{1}{2}\Sigma E\delta V=0\,.$$

Next, supply energy from without so that the potentials become again V_1, V_2, &c., . . . and the charges $E_1+\delta E_1$, $E_2+\delta E_2$, &c. The final result will be the same, to first order of small quantities, as if the two changes had been made simultaneously. Now, applying the theorem of mutual potential energy to the two states of our system,

$$\frac{E}{V}\left\|\frac{E_1}{V_1+\delta V_1}\right|\frac{E_2}{V_2+\delta V_2}\right| \;.\;.\;.\; \text{and} \;\; \frac{E}{V}\left\|\frac{E_1+\delta E_1}{V_1}\right|\frac{E_2+\delta E_2}{V_2}\right| \;.\;.\;.\;,$$

we have $\qquad \Sigma(E_1+\delta E_1)(V_1+\delta V_1)=\Sigma(EV),$

hence $\qquad \Sigma E\delta V=-\Sigma V\delta E \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32);$

therefore $\quad \Phi=-\tfrac{1}{2}\Sigma E\dfrac{dV}{d\phi}=\tfrac{1}{2}\Sigma V\dfrac{dE}{d\phi}=\dfrac{dQ}{d\phi}(V \text{ const.}) \quad . \quad . \quad (33).$

By (27) this may be written

$$\Phi=\tfrac{1}{2}\sum_{r=1}^{r=n}\sum_{s=1}^{s=n}V_rV_s\frac{dq_{rs}}{d\phi}\,.$$

The energy supplied from without is

$$\tfrac{1}{2}\{\Sigma(E+\delta E)V-\Sigma E(V+\delta V)\}$$

$$=\tfrac{1}{2}\Sigma\delta EV-\tfrac{1}{2}\Sigma E\delta V=-\Sigma E\delta V=2\Phi\delta\phi=2\delta Q, \text{ by (32).}$$

In other words, when the potentials of a system are kept constant by supply of energy from without, the system tends to move so as to increase the potential energy of electrical separation, and the amount of energy supplied from without is double this increase. If we suspend side by side two balls, each connected with the positive pole of a battery, the other pole of which is connected with the ground, the balls will tend to separate, and in separating they will gain with reference to gravity a certain amount δQ of potential energy; the charges on the balls will also increase to an extent representing an increase of electrical potential energy δQ, and the batteries will be drawn upon for an amount of $2\delta Q$.

Cases where problem has been solved.

The problem of electrical equilibrium has been completely solved in very few cases. We proceed to give a short sketch of what has been done in this way, which may indicate to the reader what is known on this head.

Ellipsoid.

We can deduce the distribution and potential in the case of an ellipsoid from known propositions about the attractions of ellipsoidal shells of gravitating matter.

Consider an ellipsoidal shell, the axes of whose bounding surfaces are (a, b, c) $(a+da, b+db, c+dc)$, where $\dfrac{da}{a}=\dfrac{db}{b}=\dfrac{dc}{c}=\mu$. The potential of such a shell at any internal point is constant, and the equipotential surfaces for external space are ellipsoids confocal with (a, b, c). (See Thomson and Tait, §§ 519 *sqq.*) Hence if we distribute electricity on an ellipsoid (a, b, c) such that its density at every point is proportional to the thickness of the shell formed by the similar ellipsoids (a,b,c) $(a+da,b+db,c+dc)$, the distribution will be in equilibrium. Thus if $\sigma=A\theta\rho$, where θ is the thickness at any point and ρ the volume density of the shell; then the quantity of electricity on any element dS is A times the mass of the corresponding element of the shell; and if Q be the whole quantity of electricity on the ellipsoid, Q = A times the whole mass of the shell. The mass of the shell is $\frac{4}{3}\pi\rho d(abc)=4\pi\mu abc\rho$, therefore $Q=A4\pi\mu abc\rho$. Also $\theta=\mu p$ where p is the perpendicular from the centre of the ellipsoid on the tangent plane. Whence we get

$$\sigma=\frac{Qp}{4\pi abc} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34);$$

that is, the density at any point varies directly as the distance of the tangent plane at that point from the centre.

Returning again to our ellipsoidal shell, we know that the resultant force at any external point P due to this shell is to that due to a "confocal shell" passing through the point in the ratio of the masses. Let the volume density in the two be ρ, and let the perpendicular on the tangent plane at P to the confocal $(\sqrt{a^2+\lambda}, \sqrt{b^2+\lambda}, \sqrt{c^2+\lambda})$ through P be ϖ. Then the thickness of the shell at P is $\mu\varpi$, and the force at P due to the shell $4\pi\rho\mu\varpi$. Hence the force due to the original shell is

$$-\frac{dV}{d\nu}=4\pi\rho\mu\varpi\frac{abc}{\sqrt{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)}} \quad . \quad . \quad . \quad (\alpha),$$

$d\nu$ being an element of the normal at P. Now if x,y,z be the coordinates of P, we have, by differentiation of

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}+\frac{z^2}{c^2+\lambda}=1\,,$$

$$\frac{2xdx}{a^2+\lambda}+\frac{2ydy}{b^2+\lambda}+\frac{2zdz}{c^2+\lambda}=\left\{\frac{x^2}{(a^2+\lambda)^2}+\frac{y^2}{(b^2+\lambda)^2}+\frac{z^2}{(c^2+\lambda)^2}\right\}d\lambda.$$

Suppose we take dx, dy, dz in the direction of the normal, then $dx=d\nu\dfrac{\varpi x}{a^2+\lambda}$, &c., and the last equation reduces to

$$d\lambda=2\varpi d\nu\,.$$

Hence from (α) we get

$$-dV=\frac{2\pi\rho\mu abcd\lambda}{\sqrt{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)}}\,.$$

Integrating this from λ to ∞, and remembering that the potential vanishes at an infinite distance, we get

$$V=2\pi\rho\mu abc\int_\lambda^\infty\frac{d\lambda}{\sqrt{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)}} \quad . \quad . \quad . \quad . \quad (\beta).$$

We pass from this to the electrical case by putting for $4\pi\rho\mu abc$, which is the mass of the shell, Q, which represents the quantity of electricity on the ellipsoid. We thus get

$$V=\frac{Q}{2}\int_\lambda^\infty\frac{d\lambda}{\sqrt{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)}} \quad . \quad . \quad . \quad (35),[1]$$

which gives the potential due to a charge Q on an isolated ellipsoid abc at any point on the confocal $(\sqrt{a^2+\lambda}, \sqrt{b^2+\lambda}, \sqrt{c^2+\lambda})$. It is obvious that, of the three confocals at P, that is meant which belongs to the same family as (a, b, c), *e.g.*, if (a,b,c) be an ellipsoid, as opposed to a hyperboloid of one or two sheets, then $(\sqrt{a^2+\lambda}, \sqrt{b^2+\lambda}, \sqrt{c^2+\lambda})$ must be an ellipsoid.

If we put $\lambda=0$, we get the value of the potential V_0 at the surface. Now $\dfrac{Q}{V_0}$ is what we have defined above as the capacity of the ellipsoid; we get therefore in the reciprocal of the integral

$$\frac{1}{2}\int_0^\infty\frac{d\lambda}{\sqrt{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)}} \quad . \quad . \quad . \quad (36),$$

an expression for the capacity of an isolated ellipsoid.

In the particular case of an ellipsoid of revolution, the above integral, which is in general an elliptic integral, can be found in finite terms. In the case of a planetary ellipsoid, $a=b>c$; and we find for the capacity

$$\frac{\sqrt{a^2-c^2}}{\frac{1}{2}\pi-\epsilon} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37),$$

where ϵ is the least angle whose tangent is $\dfrac{c}{\sqrt{a^2-c^2}}$.

If we make $c=0$, then $\epsilon=0$; and the planetary ellipsoid reduces to a circular disc, the capacity for which is therefore $\dfrac{2a}{\pi}$, that is, $\dfrac{1}{1{\cdot}571}$ that of a sphere of the same radius

[1] This demonstration was suggested by that given by Thomson (*Reprint of Papers*, p. 10) to establish a slightly different formula.

(for the capacity of a sphere is obviously equal to its radius). Cavendish had arrived by experiment at the value $\dfrac{1}{1{\cdot}57}$ (see Thomson's *Reprint*, p. 180), a very remarkable result for his time. It is very easy, by taking the limit of the right hand side of (34), to find the expression for the density at a distance r from the centre of the disc; it is

$$\sigma=\frac{Q}{4\pi a\sqrt{a^2-r^2}}=\frac{V}{2\pi^2\sqrt{a^2-r^2}} \quad . \qquad (38).$$

In the case of an ovary ellipsoid, $a=b<c$; and the capacity is

$$\frac{2\sqrt{c^2-a^2}}{\log\left(\dfrac{c+\sqrt{c^2-a^2}}{c-\sqrt{c^2-a^2}}\right)} \quad . \quad . \quad . \quad . \quad (39);$$

from which several limiting cases may be deduced.

Formula (34), applied to a very elongated ovary ellipsoid, shows us that the density at the pointed ends is very great compared with that at the equator. The ratio of the densities in fact increases indefinitely with the ratio of the longest to the shortest dimension. We have in such an infinitely elongated ellipsoid an excellent type of a pointed conductor.

Points and edges.

The effect of a point or an edge on a conductor may be very easily shown by drawing a series of level surfaces, the first of which is the surface of the conductor itself, which has, say, an edge on it. The consecutive surfaces have sharpness of curvature corresponding to the edge, which gets less and less as we recede from the conductor. The level surfaces at an infinite distance are spheres. Tracing, then, any tube of force from an infinite distance, where the sections of all are equal, inwards towards the discontinuity, we see that the section becomes narrower as the curvature of the level surfaces sharpens, and at a mathematical edge the section is infinitely small, and therefore the force is infinitely great. At a mathematical point this is doubly true. At such places the force tending to drive the electricity into the insulating medium becomes infinite. In practice the medium gives way, and disruptive discharge of some kind occurs.

Sphere with given force.

We can find the distribution on a spherical conductor influenced by given forces, such for instance as would arise from *rigidly* electrified bodies in the neighbourhood.

The method of procedure would be as follows:—Let U be the potential of the rigidly electrified system alone at any point of the sphere. Then the problem is to determine a function V, which shall satisfy the equation ∇^2V at every point of space, and have the value $C-U$ at the surface of the sphere, where C is a constant to be determined by the conditions of the problem. Expand $C-U$ in series of surface harmonics, and let the result be

$$C-U=\gamma_0+\gamma_1+\gamma_2+ \;.\;.\;.\; \text{\&c.} \;.\;.\;.\; (\alpha).$$

Then the value of V is

$$V=\gamma_0+\gamma_1\frac{r}{a}+\gamma_2\left.\frac{r}{a}\right|^2+ \;.\;.\;.\; \text{inside the sphere} \;.\;.\; (\beta),$$

and

$$V'=\gamma_0\frac{a}{r}+\gamma_1\left.\frac{a}{r}\right|^2+\gamma_2\left.\frac{a}{r}\right|^3+ \;.\;.\;.\;.\; \text{outside} \;.\;. \quad (\gamma).$$

For these evidently satisfy Laplace's equation, have the given value (α) at the surface of the sphere, and are finite and continuous everywhere. From (β) and (γ), by means of the surface characteristic equation, we can deduce an expression for the density at any point of the sphere, and for the whole charge. If the latter is given we have a condition to determine C; if, on the other hand, the value of the potential of the sphere were given, then this would be the value of C.

The case of two mutually influencing spheres was treated by Poisson in the famous memoir which really began the mathematical theory of electricity. We regret that we cannot afford space for more than a mere sketch of his methods.

Consider the potentials due to the distributions on each sphere. Let a and b be the radii of the two spheres, r and r' the distances of any point P from their respective centres, and μ and μ' the cosines of the angles r and r' make with the line joining the centres of the spheres. Since the distributions are evidently symmetrical about the central line, we can obviously expand the potentials due to each distribution in zonal harmonics relative to the corresponding sphere. Hence, if $4\pi a\phi\left(\mu,\dfrac{r}{a}\right)$ denote potential due to sphere a at any point inside it, we have

$$4\pi a\phi\left(\mu,\frac{r}{a}\right)=A_0+A_1Q_1\frac{r}{a}+A_2Q_2\left.\frac{r}{a}\right|^2+ \;.\;.\;.\; (\alpha).$$

The potential at any external point is

$$A_0\frac{a}{r}+A_1Q_1\left.\frac{a}{r}\right|^2+A_2Q_2\left.\frac{a}{r}\right|^3+ \;.\;.\;.\;.\;.\; (\beta),$$

which may be written $4\pi\dfrac{a^2}{r}\phi\left(\mu,\dfrac{a}{r}\right)$.

Similarly we have for the other sphere

$$4\pi b\Phi\left(\mu',\frac{r'}{b}\right)=B_0+B_1Q_1'\frac{r'}{b}+B_2Q_2'\left.\frac{r'}{b}\right|^2+ \;.\;.\;.\;.\; (\gamma)$$

for the potential at any internal, and $4\pi\dfrac{b^2}{r'}\Phi\left(\mu',\dfrac{b'}{r'}\right)$ for the potential at any external point.

The whole potential, then, will be given by

$$V=4\pi\frac{a^2}{r}\phi\left(\mu,\frac{a}{r}\right)+4\pi\frac{b^2}{r'}\Phi\left(\mu',\frac{b}{r'}\right)$$

at any point external to both spheres.

Also

$$V=4\pi a\phi\left(\mu,\frac{r}{a}\right)+4\pi\frac{b^2}{r'}\Phi\left(\mu',\frac{b}{r'}\right) \text{ inside } a; \text{ and}$$

$$V=4\pi\frac{a^2}{r}\phi\left(\mu,\frac{a}{r}\right)+4\pi b\Phi\left(\mu',\frac{r'}{b}\right) \text{ inside } b.$$

Now, the conditions of the problem require that the values of V in the two last cases shall be constant. Our functions are, therefore, to be determined by the equations

$$\left.\begin{aligned}a\phi\left(\mu,\frac{r}{a}\right)+\frac{b^2}{r}\Phi\left(\mu',\frac{b}{r'}\right)&=h\\ \frac{a^2}{r}\phi\left(\mu,\frac{a}{r}\right)+b\Phi\left(\mu',\frac{r'}{b}\right)&=g\end{aligned}\right\} \;.\;.\;.\;.\; (\delta),$$

which are to be satisfied with obvious restrictions on r and r' in each case. Reverting, however, to the expressions (α), (β), (γ), &c., we see that we need not solve the problem in the general form thus suggested; for it will be sufficient if we determine the constants A_0, A_1, &c., B_0, B_1, &c. Now, if we make $\mu=1$, $\mu'=1$,—that is, consider only points on the central line,—then $Q_1=1$, $Q_2=1$, &c., $Q_1'=1$, $Q_2'=1$, &c.[1] A_0, A_1 &c. B_0, B_1, &c., are the coefficients of $\left.\dfrac{a}{r}\right|$, $\left.\dfrac{a}{r}\right|^2$, &c., and $\left.\dfrac{b}{r'}\right|$, $\left.\dfrac{b}{r'}\right|^2$, &c., in the expressions for the potentials inside the spheres a and b. Hence, if $f\left(\dfrac{r}{a}\right)$ and $F\left(\dfrac{r'}{b}\right)$ denote the values of $\phi\left(\mu,\dfrac{r}{a}\right)$, $\Phi\left(\mu',\dfrac{r'}{b}\right)$, when $\mu=1$ and $\mu'=1$, we need only solve the equations

$$\left.\begin{aligned}af\left(\frac{r}{a}\right)+\frac{b^2}{c-r}F\left(\frac{b}{c-r}\right)&=h\\ \frac{a^2}{c-r'}f\left(\frac{a}{c-r'}\right)+bF\left(\frac{r'}{b}\right)&=g\end{aligned}\right\} \;.\;.\; (\epsilon),$$

where we have replaced r and r' by their values $c-r'$ and $c-r$, c being the distance between the centres of a and b. Poisson then eliminates the function F, by choosing a new variable ξ, such that $r'=\dfrac{b^2}{c-\xi}$, and remarks that we may give to ξ any value between $+a$ and $-a$, and therefore we may write r for ξ; we thus have the same variable in both the equations, and $F\left(\dfrac{b}{c-r}\right)$ which occurs in both may be eliminated. The result is

$$af\left(\frac{r}{a}\right)+\frac{a^2b}{c^2-b^2-cr}f\left(\frac{ac-ar}{c^2-b^2-cr}\right)=h-\frac{gb}{c-r} \;.\;.\;.\; (\zeta).$$

This is the functional equation on which depends the solution of the problem of two mutually influencing spheres.

Poisson treats very fully the case of two spheres in contact; for which case, taking $a=1$, the above equation becomes

$$f(r)-\frac{b}{b+(1+b)(1-r)}f\left(\frac{1+b-r}{b+(1+b)(1-r)}\right)=h-\frac{gb}{1+b-r} \;.\;.\; (\eta).$$

[1] We are, of course, assuming aquaintance with the properties of spherical harmonics.