He finds a solution,

$$f(r)=\frac{bh}{(1+b)(1-r)}\int_0^1\frac{\left(t^{-\frac{1}{1+b}}-1\right)e^{\frac{br}{(1+b)(1-r)}}}{1-t}\,dt\,. \quad . \quad (\theta).$$

It is then easy to find F(r), and write down the general expressions for the potential. Poisson goes on to show that the density at the point of contact of the spheres is zero. He finds, for the mean density on the two spheres 1 and b respectively,

$$A=\frac{bh}{1+b}\int_0^1\frac{t^{-\frac{1}{1+b}}-1}{1-t}\,dt\,,$$

this being, in fact, the value of $f(0)$,

$$\text{and } B=\frac{h}{b(1+b)}\int_0^1\frac{e^{-\frac{1}{1+b}}-1}{1-t}\,dt\,.$$

He shows that the calculation of the ratio β of A to B may be reduced to the calculation of the first of these integrals only. For the difference $4\pi b^2B-4\pi A$ between the charges on 1 and b he finds the elegant expression

$$\frac{4\pi^2bh}{1+b}\cos\frac{\pi}{1+b}\,,$$

from which it follows that the whole charge is always greater on the sphere of greater radius. He then calculates the value of β for various values of b, and its limit for $b=0$, and next the ratio of the densities at the two points diametrically opposite the point of contact, and finds for the mean density on each of two equal spheres in contact $A=h\log 2$. He also calculates for this last case the ratio of the greatest to the mean density. In the case of two unequal spheres, the ratio of the greatest density on the smaller to the mean density on the larger is found for various values of b. He then passes on to investigate the densities for various values of μ.

Plana and Roche.

All these results are compared with the measurements of Coulomb, and found in satisfactory accordance with them. In his first memoir, Poisson considers the case where the distance between the spheres is great compared with the radii; and in a subsequent memoir he considers the case of two spheres at any distance.

Plana (*Sur la distribution de l'électricité à la surface des deux Sphères*, Turin, 1845) extended the calculations of Poisson, using much the same methods. He also calculated approximately the mean densities in the case of several spheres in contact, and arrived at results which agreed satisfactorily with the experiments of Coulomb. For a table of his results, see the end of the first volume of Riess's *Reibungselectricität*. An account of the work of Roche, who also followed in the footsteps of Poisson, will be found in Mascart, t. i. p. 290 *sqq*.

Synthetical method of Green.

The researches of Green led him to a very valuable synthetical method, by means of which we can construct an infinite number of cases where we can find the electrical distribution. Suppose that we take *any distribution whatever* of electricity, for which we know the potential at any point, and consequently the level surfaces. Take any level surface, or parts of level surfaces, inclosing the whole of the electricity, and suppose these level surfaces to become actual conducting sheets of metal. Suppose the electrical distribution inside to be rigid, and connect the sheets of metal with the earth, so as to reduce them to potential zero. The sheets will become charged in such a way that the whole potential at every point in them and external to them is zero. Let now U be the potential at any external point due to inside distribution, and V that due to the charge on the sheets, then we have everywhere on or outside the sheets, $U+V=0$, or $V=-U$. Now U is constant at every point of each sheet; hence V is so also. Hence the distribution to which V is due is an equilibrium distribution *per se*. Removing now our internal distribution, and changing the sign of that on the sheets, we have a distribution of electricity in equilibrium on a set of conductors of known form, the potential of which at any external point is $V=U$, where U is known. Also the potential V is clearly constant inside every conductor. Hence, applying the characteristic surface equation, we get for the density at any point of any of our conductors the expression

$$\sigma=-\frac{1}{4\pi}\frac{dU}{d\nu}\,.$$

We might make this a little more general, and state our result thus:—*If we distribute on a level surface or surfaces of any electrical system, completely inclosing that system, electricity with surface density at every point* $\sigma=-\frac{k}{4\pi}\frac{dU}{d\nu}$, *this distribution will of itself be in equilibrium, and the potential at any external point will be* kU.

We have given a physical demonstration of this important theorem. The mathematical reader will easily see the application to this case of the general reasoning about the solution of $\nabla^2V=0$, of which we have already given examples. For a simple but interesting case of this general theorem, see Thomson and Tait's *Natural Philosophy*, vol. i. § 508.

Methods of Sir W. Thomson. Electric images.

To Sir William Thomson we owe the elegant and powerful methods of "Electric Images" and "Electric Inversion." By means of these he arrived, by the use of simple geometrical reasoning, at results which before had required the higher analysis. We shall endeavour to illustrate these by two simple examples. We do not follow the methods of the author (for which, see his papers), but take advantage of what we have already laid down.

Let A be any point outside a sphere (fig. 12) of radius a, and centre C. Let $AC=f$, and take B in CA such that $CB\cdot CA=a^2$, or $CB=\frac{a^2}{f}$; then it is easily proved that, if P be any point on the sphere,

$$\frac{BP}{AP}=\frac{a}{f}\,.$$

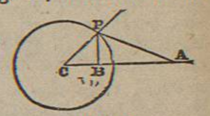

Fig. 12.

Hence if E be any quantity of electricity, we have

$$\frac{E}{AP}-\frac{\frac{a}{f}E}{BP}=0\,.$$

Therefore, if we place a quantity E of electricity at A, and a quantity $-\frac{a}{f}$E at B, the sphere will be a level surface of these two, that, namely, for which the potential is zero. Another level surface of the system is evidently an infinitely small sphere surrounding A. Hence it follows, from the theorem of Green which we have just discussed, that a distribution of electricity on the sphere, the density of which is given by $\sigma=\frac{R}{4\pi}$ (where R is the resultant force due to E and $-\frac{a}{f}$E at any point of the sphere), together with a quantity E at A, gives a system in equilibrium, the potential due to which at any point outside the sphere is the same as that of E at A, and $-\frac{a}{f}$E at B.

It appears, therefore, that the action of the electricity induced on the uninsulated sphere by the electrified point A is equivalent at all external points to the action of $-\frac{a}{f}$E at B. The electrified point B is called by Sir William Thomson the electrical image of A in the sphere. It is obvious that the whole charge on the sphere is $-\frac{a}{f}$E, and we can very easily find the density at any point.

In fact, resolving along CP, which we know to be the direction of resultant force, the forces due to A and B, we get

$$R=\frac{E}{AP^2}\cos CPA-\frac{\frac{a}{f}E}{BP^2}\cos CPB$$

$$=\frac{E}{AP^2}\left(\frac{a^2+AP^2-f^2}{2aAP}\right)-\frac{fE}{aAP^2}\left(\frac{f^2+AP^2-a^2}{2fAP}\right)=-\frac{(f^2-a^2)E}{aAP^3}\,.$$

$$\therefore\ \sigma=-\frac{(f^2-a^2)E}{4\pi aAP^3}\ .\ .\ .\ .\ .\ .\ .\ .\ (40).$$

We might have any number of external points and find the image of each. We should thus get a system which might be called the image of the external system. The distribution induced in an uninsulated sphere by such an external system could easily be found by adding up the effect of each external element found by means of its image. Similar methods might also be applied to an internal system. The solution can be generalized without difficulty to the case where either the charge or potential of the sphere is given.

Suppose the charge Q given; superpose on the distribution found above a uniform distribution of amount $Q+\frac{a}{f}E$. This will produce a constant potential $\frac{Q}{a}+\frac{E}{f}$ all over the sphere, and therefore will not disturb the equilibrium. We have thus got the required distribution of the given charge Q under the influence of A. The density of any point is given by

$$\sigma=\frac{Q}{4\pi a^2}+\frac{E}{4\pi af}-\frac{(f^2-a^2)E}{4\pi aAP}\ .\ .\ .\ .\ (41).$$

So far the method of images is simply a synthetical method for obtaining distributions on a sphere. But Sir William Thomson has shown us how to convert it into an instrument for transforming any electrical problem into a variety of others.

f P be any point (fig. 13), O a fixed point, and P′ be taken in OP such that $OP.OP'=a^2$, then P′ is called the inverse of P with respect to O, which is called the origin of inversion, or simply the origin; a is the radius of inversion. We may[1] thus invert any locus of points into another locus of points, which we may call the inverse of the former.

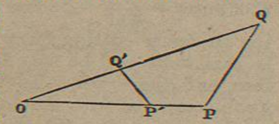

Fig. 13.

Let P, Q and P′, Q′ be any two points and their inverses. Let us suppose that there is a charge E at Q, and a charge E′ at Q′, which is the image of E in a sphere with radius a and centre O; so that $E'=\frac{a}{OQ}E$. Let V and V′ be the respective potentials of E and E′ at P and P′. Then we have obviously

$$\frac{V'}{V}=\frac{a}{r'}=\frac{r}{a}\,,$$

where $OP=r$, $OP'=r'$. It is very easy to show that, if ds, dS, dv, σ, ρ, be elements of length, surface, and volume, and surface and volume densities, and the same symbols with dashes the inverses of these, then we have

$$\left.\begin{aligned}&\frac{ds'}{ds}=\frac{a^2}{r^2}=\frac{r'^2}{a^2}\,;\ \frac{dS'}{dS}=\frac{a^4}{r^4}\ \&c.\\ \text{and }&\frac{\sigma'}{\sigma}=\frac{r^3}{a^3}=\frac{a^3}{r'^3}\,;\ \frac{\rho'}{\rho}=\frac{r^5}{a^5}=\frac{a^5}{r'^5}\\ \text{also }&\frac{E'}{E}=\frac{r}{a}=\frac{a}{r'}\,;\ \frac{V'}{V}=\frac{r}{a}=\frac{a}{r'}\end{aligned}\right\}\quad (42)\,.$$

By means of these equations it is easy to invert any electrical system. Take, for example, the case of any conductor in electrical equilibrium; then, since its potential is everywhere constant, it inverts into a surface distribution, the potential at any point of which distant r' from the origin is by (42) $\frac{a}{r'}$ C, where C is the constant potential of the conductor. The surface density at any point of the system is found from that of the corresponding point on the conductor by the equation

[1] For the general properties of curves and their inverses, the reader may consult Salmon's *Solid Geometry*. He will have no difficulty in proving for himself such as we shall require here.

Again, if we consider the system thus found, it is obvious that, if we place a quantity $-a$C of electricity at the origin, this will make the potential at every point of the system zero, and we have a solution of the case of an uninsulated conductor, whose surface is the inverse of that of the given conductor, under the influence of an electrified point.

As an example of the use of this method, let us invert the uniform distribution on a sphere with respect to an origin on its circumference, the radius of inversion being the diameter of the sphere. The sphere inverts into an infinite plane, touching at the other end A of the diameter through the origin. Let C be the potential on the sphere so that $\sigma=\frac{C}{2\pi d}$, where d is the diameter. Hence the density at any point P on an infinite plane influenced by a quantity $-Cd$ of electricity placed at a point O distant d from it is given by

$$\sigma'=\frac{d^2C}{2\pi r'^3}\,.$$

Again, inverting points inside the sphere, for which the potential is constant, we get the potential due to the distribution on the infinite plane, at points on the other side from the inducing point, the result being

$$V'=\frac{dC}{r'}\,,$$

which is the same as that due to dC at O. Hence the potential at a point on the same side as O is that due to a quantity dC placed at O′, where $O'A=OA$. O′ is in fact the image of O. If we write Q for $-Cd$, then we get

$$\left.\begin{aligned}\sigma'&=\frac{Qd}{2\pi r'^3}\\ V'&=\frac{-Q}{r'}\end{aligned}\right\}\ .\ .\ .\ .\ .\ (43).$$

These results might of course have been deduced as particular cases of a sphere and point.

Many beautiful applications of these methods will be found in the *Reprint* of Sir William Thomson's papers and in Maxwell's *Electricity and Magnetism*. Two of these are of especial importance. Adopting the method of successive influences given by Murphy (*Electricity*, 1833, p. 93), and conjoining with it the method of images, Sir William Thomson treated the problem of two spheres. For his results, see *Reprint*, pp. 86–97. At the end of that paper two valuable tables are given—I. "Showing the quantities of electricity on two equal spherical conductors of radius r, and the mutual force between them, when charged to potentials u and v respectively;" II. "Giving the potentials and force when the charges D and E are given." The ratio of u to v in the first case and of D to E in the second is also given, for which at a given distance there is neither attraction nor repulsion. An interesting experiment on this curious phenomenon is described in Riess, Bd. i. § 186. For an application of dipolar co-ordinates to the problem of two spheres, see Maxwell.

Spheric bowl.

Thomson also applied his methods to determine the distribution on spherical bowls of different apertures. See *Reprint*, p. 178 *sqq*. His numerical results on p. 186 are extremely interesting, as affording a picture of the effect of gradually closing a conductor, and are of great value in giving the experimenter an idea as to what aperture he may allow himself in a vessel which he desires should be for practical purposes electrically closed.

It would lead us too far to discuss here the analytical method of conjugate functions, and the allied geometrical method of inversion in two dimensions. A full account of these, with important applications, will be found in Maxwell, vol. i. § 182 *sqq*.

We shall conclude our applications with a brief notice of a few of the ordinary electrostatical instruments, referring the reader for an account of some others to the article ELECTROMETER.

If two plates be placed parallel to each other, and one of them raised to potential V, while the other is connected with the earth, then there will be certain charges E and F on the two plates. If p and r be the coefficients of self-induction for A and B, and q the coefficient of mutual induction, then in the present case

$$E = pV, \quad F = qV,$$

and the energy of the distribution is obviously

$$Q = \tfrac{1}{2}EV = \tfrac{1}{2}pV^2,$$

so that the work done by completely discharging the condenser $\propto V^2$. If we suppose the plates very large compared with the distance between them, then we may treat the case, for all points not very near the edge, as if the plates were infinite.

In this case the lines of force are straight, and the number of lines of force which leave any area on A is equal to that of those which enter the opposite area on B. Hence the surface densities on the plates are equal and opposite in sign. Also we clearly have

$$\sigma = \frac{R}{4\pi} = \frac{V}{4\pi d} \quad . \quad . \quad . \quad . \quad . \quad . \quad (44).$$

For the number of lines of force which cross any unit of area parallel to the plates is constant, and therefore the resultant force is constant at every point between the plates.

Principle of accumulators. It appears, therefore, from (44) that if we make the distance between our plates very small, the density on the inner surface will be very great, and the whole charge on A very great. An apparatus of this kind for collecting large quantities of electricity at a moderate potential is called an accumulator or condenser. One of the first instruments of this kind was Franklin's pane, which consisted of two sheets of tinfoil pasted opposite each other on the two sides of a pane of glass. There is of course a practical limit to the increase of capacity in such arrangements, because a spark will pass when the insulating medium is too thin. The greater dielectric strength of glass makes it more convenient than air for an insulating medium, and we shall see by-and-by that it has other advantages as well. When the plate A is of finite size there will in general be a distribution of electricity on the back comparable with the charge which A would hold at potential V if B were absent. When the distance between the plates is small, by far the greater portion of the capacity is due to the presence of B. Advantage of this principle has been taken

Condensing electroscope. in the condensing electroscope of Volta, which is an ordinary gold-leaf apparatus, except that the knob is replaced by a circular disc on which is placed another disc fitted with an insulating handle; the discs are covered with a thin coat of varnish which serves as an insulating medium. If we connect with either disc, say the lower, a source of electricity of feeble potential V, and connect the upper disc at the same time with the earth, then a large quantity of electricity at potential V collects on the lower disc. Now remove all connections, and lift away the upper disc. The capacity of the lower disc is thereby enormously diminished. Therefore, since the charge is unaltered, its potential must rise correspondingly; and the gold leaves may diverge very vigorously, although a simple connection with the lower disc alone would scarcely have moved them. This instrument is of great use in all cases where we have an unlimited supply of electricity at feeble potential. Sir William Thomson has devised an accumulator of measurable capacity, called the Guard Ring Accumulator, which is a modification of the arrangement we are discussing.

AB (fig. 14) is a flat cylindrical metal box, the upper end of which is truly plane, and has a circular aperture, into which fits, without touching, a plane disc C, which is supported on the bottom of the box by insulating supports, so that its upper surface is in the same plane with the lid of the box. DE is a metal disc which can be moved by a screw through measured distances, always remaining parallel to AB. When desired, C can be put in communication with AB. It may then be regarded as forming part of an infinite plate, so that if AB be at potential V, and DE at potential zero, then the surface density on C will be equal to $\frac{V}{4\pi d}$, where d is the distance between the plates; and if A be the area of C the whole amount of electricity on C is $\frac{AV}{4\pi d}$. If now we break the connection between C and the box and discharge the box, we are left with a known quantity of electricity on C, viz. $\frac{AV}{4\pi d}$.

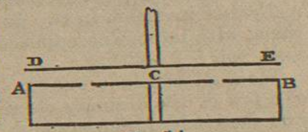

Fig. 14.

The most usual and for many purposes the most convenient form of accumulator is the Leyden jar. This is merely a glass jar (fig. 15) coated to a certain height outside and inside with tinfoil. The mouth of the jar is stopped with a cork or wooden disc, which serves the double purpose of keeping dirt and moisture from the uncovered glass inside, and of carrying a wire in metallic connection with the inside coating, which passes up through the stopper and ends in a metal knob. If the glass of the jar be very thin, we may find the distribution on the two coatings by neglecting the curvature; the electric density on the inner surface of the two coatings will then be the same as in the case of parallel plates. If, therefore, the inner coating be at potential V, and the outer at potential zero, the density on the inner coating will be $\frac{V}{4\pi d}$, and that on the outer $-\frac{V}{4\pi d}$. In the particular case we are considering the inner coating forms very nearly a closed conductor, so that there will be very little electricity on its inner surface, and there will also be very little on the wire and knob compared with the amount on the surface of the inner coating which is next the glass. We may therefore put for the whole electricity on the inner coating $\frac{SV}{4\pi d}$, where S is the extent of its surface. The capacity C of the jar is then given by

Fig. 15.

$$C = \frac{S}{4\pi d} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (45).$$

Green calculated to a first approximation the effect of the curvature on the capacity, and found that, if R and R′ be the greatest and least radii of curvature of the inner coating at any point, then the densities on the inner and outer coatings are given by

$$\frac{V}{4\pi d}\left\{1 \pm \frac{d}{2}\left(\frac{1}{R} + \frac{1}{R'}\right)\right\} \quad . \quad . \quad . \quad . \quad (46),$$

and consequently the capacity of the inner coating by

$$\frac{1}{4\pi}\left\{\iint \frac{dS}{d} + \tfrac{1}{2}\iint\left(\frac{1}{R} + \frac{1}{R'}\right)dS\right\} \quad . \quad . \quad (47).$$

Battery of jars abreast. In any case, C being a constant, we have charge $E = CV$ and energy $Q = \tfrac{1}{2}CV^2$. Hence if we connect the inner coatings of n similar jars, and charge them to potential V, all the outer coatings being at the same time connected with the earth, we have, E and Q representing the whole charge and energy,

$$\left.\begin{aligned} E &= nCV \\ Q &= \frac{n}{2}CV^2 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (48).$$

If we discharge such a battery of n jars into another of n' similar jars, by connecting the knobs together, and the outer coatings to earth in each case, we have, U being the common potential after discharge,

$$nCV = nCU + n'CU$$

$$\text{and} \quad U = \frac{n}{n+n'}V \quad . \quad . \quad . \quad . \quad . \quad . \quad (49).$$

There is therefore a loss of energy represented by

$$\tfrac{1}{2}nCV^2 - \tfrac{1}{2}(n+n')CU^2,$$

$$\text{that is} \quad \frac{nn'}{2(n+n')}CV^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (50).$$

In other words, an $\frac{n'}{n+n'}$th part of the potential energy is lost. When a battery of jars is discharged through a circuit in which there is a fine wire of large resistance, the greater part of the potential energy lost in the discharge appears as heat in the fine wire. Riess made elaborate experiments on the heating of wires by the discharge in this way, and the results of his experiments are in agreement with the formulæ which we have just given. (See Heating Effects.)

We may also arrange a battery of jars by first charging each separately to potential V in the usual way, and then connecting them in series, so that the outer coating of each jar is in metallic connection with the inner coating of the next. In such an arrangement of jars, it is obvious that in passing from the outer coating of the last at potential zero to the inner coating of the first, the potential will rise to nV. When we come to discharge such a series, the electromotive force to begin with is nV, so that for any purpose in which great initial electromotive force is required this combination has great advantages over n jars abreast. The "striking distance," for instance, *i.e.*, the greatest distance at which the discharge by spark will just take place through air, is much greater. On the other hand, the quantity of electricity which passes is less, being only CV instead of nCV; the whole loss of potential energy in a complete discharge is, however, the same.

The case which we have been discussing must be carefully distinguished from that of a series of jars charged by

Cascade. "cascade," where n uncharged jars are connected up in succession as in last case, and the first charged by connection with the electric machine to potential V, while the outer coating of the last of the series is connected to earth, and the rest of the jars insulated. The whole electromotive force in this case is clearly only V, and, if all the jars be similar, the potential difference between the coatings in each is $\frac{V}{n}$; the charge on the inner coating of the first is therefore $\frac{CV}{n}$, and the whole potential energy only $\frac{1}{2}\frac{CV^2}{n}$. The arrangement is, therefore, not so good as a single jar fully charged by the same machine. It was fancied by Franklin, who invented this method of charging, that some advantage was gained by it in the time of charging, the notion being that the overflow was caught by the successive jars and that electricity was thereby saved. Charging by cascade was treated by Green. Some of the experiments of Riess bear on the matter (*vide* Mascart, §§ 190, 191), which, after all, is simple enough.

Free and bound electricity. In the theory of accumulators, or condensers as they are often called, much stress has been laid on the difference between "*free*" and "*bound*" electricity. To illustrate the meaning of these terms, let us take a case where the calculations can be carried out in detail.

Suppose we have two concentric spherical shells, an inner, A, and an outer, B. Let the outer radius of A be a, and the inner and outer radii of B be b and c, so that the thickness of the latter is $c-b$. We shall suppose that we can, when we please, connect the inside sphere with the earth. It is clear that there can never be any electricity on the inner surface of A. Let the charges on the other surfaces in order be E, F, G. Let us suppose in the first instance that A is at potential V, and B at zero. Then we have to find E, F, G. Draw a surface in the substance of B; no lines of force cross it, therefore the whole amount of electricity within is zero. Hence $F = -E$. Also, considering the external space, which is inclosed between two surfaces of zero potential, we see that $G = 0$. Thus, since A is at potential V, we have $\frac{E}{a} - \frac{E}{b} = V$

$$E = \frac{ab}{b-a}V = pV\left(\text{where } p = \frac{ab}{b-a}\right) \quad . \quad . \quad (51).$$

In this case, then, there is no electrification on the outside of B, and an electric pendulum suspended there would give no indication.

Let us now connect A with the earth, so that its potential becomes zero; we have now to find the charges and potentials, our datum being that the whole charge on B is $-E$.

As before, we have $F' = -E'$, but G is no longer zero. We have however, $F' + G' = -E$. Hence $G' - E' = -E$.

Also, since A is at zero potential, $\frac{E'}{a} - \frac{E'}{b} + \frac{G'}{c} = 0$,

therefore $G' = \frac{-cE'}{p}$; $-F' = E' = \frac{pE}{p+c}$; $G' = \frac{-cE}{p+c}$.

The potential of B is $\frac{G'}{c}$, or $\frac{-cpV}{p+c}$.

In this process, therefore, a quantity $E - E'$, or $\frac{cp}{p+c}V$, of electricity has flowed away to earth from A, and a quantity $\frac{-cp}{p+c}V$ has passed from the inner to the outer surface of B, while the potential has altered, on A from V to 0, and on B from 0 to $\frac{-cp}{p+c}V$.

Suppose now we connect B with the earth, thus reducing it to zero potential. Since the charge on A remains the same, and that on the inner coating of B is equal and opposite to it, it follows that now the charges on A, &c., are $\frac{pq}{c}V$, $-\frac{pq}{c}V$, 0, where q denotes $\frac{cp}{p+c}$; and the potentials of A and B are $\frac{q}{c}V$ and 0. After another pair of such operations the charges will be $\frac{pq}{c}\frac{q}{c}V$, &c., and the potential, $\left.\frac{q}{c}\right|^2 V$; after a third, charges, $\frac{pq}{c}\left.\frac{q}{c}\right|^2 V$, &c., and potential, $\left.\frac{q}{c}\right|^3 V$. Hence the charges and potentials go on decreasing in geometrical progression. Amounts of electricity flow away from A equal to qV, $q\frac{q}{c}V$, $q\left.\frac{q}{c}\right|^2 V$, $q\left.\frac{q}{c}\right|^3 V$ &c., in the successive operations, and equal amounts of opposite signs are discharged from B. The sum of all these discharges is the whole original charge on A, for

$$qV\left(1 + \frac{q}{c} + \frac{q}{c} + \text{\&c., } ad.\ inf.\right) = \frac{q}{1-\frac{q}{c}}V = pV.$$

Hence by an infinite number of alternate connections we shall finally discharge the jar completely. The electricity which flows out at each contact is called the "free electricity," and that which remains behind the "bound electricity." The quantity which we have denoted by p is clearly the capacity of a spherical Leyden jar; it increases indefinitely as the distance between the conducting surfaces decreases, and is very nearly proportional to the surface of the inside coating, when the distance is small compared with the radius of either surface. Capacity of spherical jar.

It is very easy to extend our reasoning to any condenser.

If, in fact, q_{11}, q_{12}, q_{22} be the coefficients of self and mutual induction for the armatures, then this potential after operating n times as above is $\left(\frac{q_{12}^2}{q_{11}q_{22}}\right)^n V$, the charges, $q_{11}\left(\frac{q_{12}^2}{q_{11}q_{22}}\right)^n V$ and $q_{12}\left(\frac{q_{12}^2}{q_{11}q_{22}}\right)^n V$ and the amounts of electricity which leave 1 and 2 in the nth operation are $\pm qq_{11}\left(\frac{q_{11}q_{22} - q_{12}^2}{q_{11}q_{22}}\right)\left(\frac{q_{12}^2}{q_{11}q_{22}}\right)^n V$ respectively.

We must not omit one more interesting case. If we have two infinite coaxial cylinders of radii a and b $(b > a)$, then obviously the potential is symmetrical about the common axis, and Laplace's equation becomes

$$\frac{d^2V}{dr^2} + \frac{1}{r}\frac{dV}{dr} = 0.$$

The integral of this is $V = C\log r + D$. Let the inner cylinder be at potential V_1, the outer at potential V_2, then

$$V = (V_1 - V_2)\frac{\log r}{\log a - \log b} + \frac{V_2\log a - V_1\log b}{\log a - \log b} \quad . \quad (52)$$

Hence the surface density on the inner cylinder is given by

$$= -\frac{1}{4\pi}\frac{dV}{dr} = \frac{V_1 - V_2}{4\pi a\log\frac{b}{a}};$$

and the capacity per unit of length of same is

$$\frac{1}{2a \log \frac{b}{a}} \quad . \quad . \qquad . \quad (53).$$

This result has important applications in the theory of telegraph cables, and to a form of graduated accumulator, invented by Sir William Thomson, and used by Messrs Gibson and Barclay in their experiments on the specific inductive capacity of paraffin (see Maxwell, vol. i. § 127).

ON THE INSULATING MEDIUM.

It has been assumed hitherto that the medium interposed between the conductors in the electric field is in all cases air—the most prevalent of all dielectric media; or, where any other medium actually occurred, as in the case of the Leyden jar, it has been assumed that the result is the same as if the glass were replaced by air. Experimenters soon recognized, however, that the capacity of a Leyden jar depends very much on the quality of the glass of which it is made. But the nature of this action was very little understood, until Faraday showed by a number of striking experiments that the dielectric has a specific function in all phenomena of induction.

Faraday's experiments.

Faraday used in his experiments two identical pieces of apparatus, which were virtually two spherical Leyden jars. The outer coating EF (fig. 16) was divided into two hemispheres, which could be fitted together air-tight. The lower hemisphere F was fitted to a perforated stem, provided with a stop-cock G, so that it could be screwed to an air-pump while the apparatus was being exhausted, and afterwards screwed into a foot H. The upper hemisphere was pierced by a tube, into which was cemented a shellac plug B. C is a metal wire passing down through B, which supports the hollow metal sphere D, forming the inside armature, and carries the metal ball A, by means of which D can be charged and discharged. To give an idea of the size of the apparatus, it may be mentioned that the diameters of the inner and outer spheres were 2·33 in. and 3·57 in. respectively. Two jars were made on the above pattern, as nearly alike as possible. The equality of their capacities was tested as follows. Both were filled with air at the same temperature and pressure. Apparatus I. was then charged, by bringing A in communication with the knob of a Leyden jar, while the coating EF was connected to earth. I. and II. were then placed at a moderate distance from each other, as symmetrically as possible with respect to the observer and other external objects, the outer armatures in both cases being in conducting communication with the earth. The ball of I. was touched by a small proof sphere, the repulsion of which on the movable ball of a Coulomb balance was measured; after a short interval this measurement was repeated. The balls of I. and II. were then brought into communication, and the charge divided between the internal armatures. The ball of II. was immediately tested as before, and then the ball of I. again. Finally I. and II. were discharged and tested for permanent "stem effect." The result of one such series of measurements was

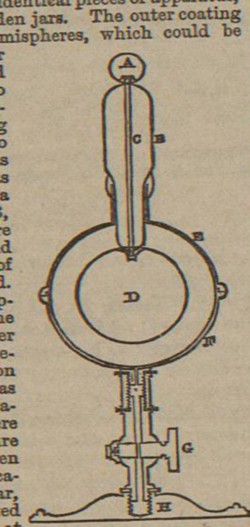

Fig. 16.

I.	 254,250	 124	1 . . .
II.	0, ...	122	. . . 2.

Neglecting the slight dissipation of the charge, and taking account only of the "stem effect" in I., we see that the charges on I. and II. after division are represented by 122 and 124, each of which is not far from the half of the whole disposable charge in I., viz., 124·5; so that the capacities of the two jars must be equal. This will perhaps be clearer if we consider what would happen were the capacities unequal. Let the capacities be C and C′, the potential of I. before division V, and the common potential after U, the charge on I. Q, and on I. and II. q and q' after division. Then Q = CV, q = CU, q' = C′U, and $q + q'$ = Q. The indication of the torsion balance is proportional to the charge of the proof sphere, that is (owing to the symmetry of the arrangements), to the potential of the knob with which it was in contact; or at all events this is true if we consider only readings taken from the knob of the *same* jar, and that is all we shall ultimately want. But (C + C′) U = CV; hence

$$\frac{C'}{C} = \frac{V-U}{U}$$

Hence the ratio of the capacities is equal to the ratio of the excess of the first over the last reading to the last reading, both being taken from the knob of I. Thus, taking the uncorrected values in the above experiment, the ratio of the capacities would be (250 − 124) ÷ 122, *i.e.* 1·02. By various experiments of this kind, Faraday convinced himself of the equality of his two jars. To test the sensibility of his method, he reduced the distance between the lower hemispheres and the ball in II. from ·62 in. to ·435 in., by introducing a metal lining. The capacity of II. was then found to be 1·09 (the mean of two observations). He next compared the capacities of the jars when the lower half of the space between the armatures of one of them was filled with shellac. The ratio of the capacities was found to be 1·5 (mean of several experiments), the shellac jar having the greater capacity.

Specific inductive capacity.

It appears, therefore, that, other things being equal, the capacity of an accumulator is greater when the insulating medium, or, as it is called, the "dielectric," is shellac, than when it is air. The ratio of the capacity in the former case to that in the latter[1] is called the *Specific Inductive Capacity* of shellac. This we shall in general denote by K. According to this definition, air is taken as the standard, and its specific inductive capacity is unity. Properly speaking, we ought to state the temperature and pressure of the air; we may assume 0° C. as our temperature, and the average atmospheric pressure (760 mm.) as our standard barometric pressure.

It is easy to obtain an approximate value of K from the above result for the shellac apparatus. Remembering that the shellac occupies only one hemisphere, and assuming that the lines of force are not disturbed at the junction of the air and shellac, we have, if ρ denote the ratio of the capacities,

$$\frac{1+K}{1+1} = \rho, \text{ and } K = 2\rho - 1 .$$

This gives for shellac K = 2·0, the real value being probably greater. Similar experiments gave for glass and sulphur K = 1·76 and 2·24 respectively.

Thus the specific inductive capacities of shellac, glass, and sulphur are considerably larger than that of air. Faraday was unable to find any difference in this respect between the different gases, or in the same gas at different temperatures and pressures, although he made careful experiments in search of such differences.

It would lead us too far to discuss in detail the precautions taken by Faraday to remove uncertainty from his experimental demonstration of the existence of a specific dielectric action. The reader will find a minute description in Faraday's own surpassingly lucid manner in the eleventh series of the *Experimental Researches.*

His discovery of the action of the medium led Faraday to invent his well-known theory of the dielectric. According to him, the fundamental process in all electrical action is a polarization of the ultimate particles of matter; this polarization consists in the separation of the positive and negative electricities *within* the molecules, exactly as the two magnetic fluids are supposed to separate in the theory of magnetic induction. In this view a dielectric is supposed to consist of a number of perfectly conducting particles, immersed in a medium or menstruum, which is either a non-conductor or a very imperfect conductor. When electrical action starts, the two electricities separate in the molecules; but, in the first instance at least, there is no interchange of electricity between different molecules.

[1] It must be noticed that the assumption is tacitly made that the air is to be replaced by shellac *everywhere*, or at least wherever there are lines of force.

Faraday assumed that the electrical action is propagated from molecule to molecule by actions whose sphere of immediate activity is very small. He denied the existence of "action at a distance," and regarded his results about induction in curved lines as at variance with it. Thomson[1] showed, however, that Faraday's results were perfectly consistent with the theory of action at a distance, provided the polarization of the dielectric be taken into account, and that the mathematical treatment of the subject is identical with Poisson's theory of induced magnetism. The theory of action at a distance as applied to this subject will be found under MAGNETISM. Helmholtz, whose memoirs we ha already mentioned, takes this view of the matter. We do not propose to follow Faraday's theory any further at present; its main features are involved in Maxwell's theory, to which we shall afterwards allude.

W. Siemens[2] examined and confirmed the conclusions of Faraday. He used voltaic electricity in comparing the capacities of condensers. By means of a kind of self-acting commutator[3] (*Selbstthätige Wippe*), the armatures of the condenser were connected alternately with a battery of Daniell's cells and with each other; so that the condenser was charged and discharged about 60 times per second.

Figure 17 gives a scheme of the arrangement. F and G are two insulated metal screws, with which the vibrating tongue E of the Wippe comes alternately into contact; CD and AB are the armatures of the condenser, H the battery, and K the galvanometer. Theory indicates, and experiment confirms, that the deflection will be the same whether the galvanometer is put in the charge or in the discharge circuit. The former arrangement is that indicated in the figure.

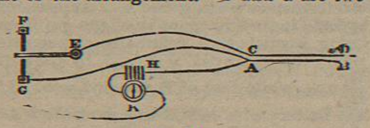

Fig. 17.

The amount of electricity which flows through the galvanometer each time the condenser is charged, is proportional to the product of the capacity C of the condenser and the electromotive force E of the battery. E is proportional to the number of cells in the battery. If, therefore, the speed of the Wippe be constant, the galvanometer deflection, or its sine or tangent as the case may be, will be proportional to EC. By varying E and C independently, we can verify the laws that regulate the charge of condensers. If we keep E the same, and the speed the same, we can compare the capacities of two condensers, or of the same condenser with two different dielectrics, and thus find the specific inductive capacities of various substances with respect to air. Siemens found that C is independent of E, and concluded that the effect of solid dielectrics on the capacity of a condenser is not to be explained by a penetration of the electricity into the dielectrics. We shall give some of his values of the specific inductive capacity farther on.

Gaugain[4] studied the effect of the insulator on the capacity of condensers. He used in his researches the discharging electroscope (see art. ELECTROMETER), an instrument which does not at first sight look likely to lead to very accurate results, but which seems to have worked satisfactorily in his hands. Many of Gaugain's results concerning the gradual increase of the charge are very interesting; their bearing on theory is difficult to estimate, however, owing to the mixture of effects due to surface and body conduction. His results concerning the "limiting" value of the specific inductive capacity are at variance with those of subsequent experimenters who have worked with more delicate instruments.

In their experiments on the specific inductive capacity of paraffin, Gibson and Barclay[5] employed a method due to Sir William Thomson, in which an instrument called the Platymeter is used in conjunction with the quadrant electrometer. They found for the specific inductive capacity of paraffin 1·97, and showed that this value alters very little, if at all, with the temperature.

The most extensive measurements of this kind that have been made of late are those of Boltzmann[6] and Schiller.[7] Boltzmann used a sliding condenser, whose plates could be placed at measured distances apart. Plates of different insulating materials were introduced between the parallel plates of the condenser, so as to be parallel with them and at different distances from one of them.

According to the mathematical theory, the capacity of the condenser is independent of the position of the plate, and varies inversely as $m - n + \frac{n}{K}$, where m is the distance between the plates of the condenser, and n the thickness of the plate of insulating material whose specific inductive capacity is K. In other words, the plate may be supposed replaced by a plate of air of thickness $\frac{n}{K}$. If therefore λ denote in absolute measure the reciprocal of the capacity of the condenser, then

$$\lambda = G\left(m - n + \frac{n}{K}\right),$$ [8]

where G is a constant. The capacity of the condenser was measured by charging it with a battery of 6 to 18 Daniell's cells, and then dividing its charge with the electrometer. One pole of the battery and one armature of the condenser are connected to earth. The other pole of the battery is first connected with the electrode A of the electrometer, whose other electrode B is connected to earth. Let the reading thus obtained be E, then E is proportional to the potential of the battery pole. The condenser is next charged by connecting its insulated armature with the battery; the battery connection is then removed, and the electrode A of the electrometer, which has meanwhile been connected with the earth, is now connected with the condenser. If C be the capacity of the condenser, C′ that of the electrometer (in certain cases artificially increased), we have, if F be the common potential of the condenser and connected parts of the electrometer, (C + C′)F = CE, and

$$C = \frac{FC'}{E-F}, \quad \text{or} \quad \lambda = \frac{E-F}{F} \cdot \frac{1}{C'} .$$

But F is proportional to the second reading of the electrometer, hence λ is known in terms of C′. As only relative measures are wanted, C′ is not required. Boltzmann made a variety of experiments, all of which confirmed the theory, and showed the applicability of the above formula.

If we make three measurements, first with the plates at distance m_1; secondly, at distance m_2, with only air between in each case; and thirdly, at distance m_3, with an insulating plate of thickness n between, we have, if $\lambda_1, \lambda_2, \lambda_3$ be the corresponding values of λ,

$$G = \frac{\lambda_2 - \lambda_1}{m_2 - m_1}, \quad \text{and} \quad \frac{1}{K} = \left(\frac{\lambda_3 - \lambda_1}{G} - m_3 + m_1 + n\right) \div n .$$

The advantage of this procedure is that only differences of m_1, m_2, m_3 come in, and no absolute length has to be measured. Measurements were also made with condensers, in which there was no air between the armatures and the insulating plates; in them the armatures were formed by means of mercury. To give an idea of the agreement of the results by different methods, we give K for paraffin as determined on plates of different thickness; with the ordinary condenser, K = 2·28, 2·34, 2·31 for plates I., II., and III.; and K = 2·31, 2·33 for plates I. and II. used with mercury armatures.

Boltzmann convinced himself that, in the case of ebonite paraffin, sulphur, and rosin, the time during which th condenser was charged was without sensible influence. He found that the result was the same whether the charge

[1] *Camb. and Dub. Math. Journ.*, 1845, or *Reprint of Papers*, p. 15.
[2] *Pogg. Ann.*, cii., 1857.
[3] For a description of this instrument, see Wiedemann's *Galvanismus*, Bd. i. § 451.
Ann. de Chim. et de Phys., 4 ser. t. ii. (1862).

[5] *Phil. Trans.*, 1871.
[6] *Pogg. Ann.*, cli., 1874, or *Sitzb. der Wiener Akad.*, lxvii.
[7] *Pogg. Ann.*, clii.
[8] It is supposed that the plates are near enough to allow us to neglect the effect of the rims.