was instantaneous or lasted for a considerable time. The case was different with the imperfect insulators, glass, stearine, and gutta percha, for which he has given no results. To test still farther the influence of the time, Boltzmann measured the attraction between a sulphur and a metal sphere—first, when the latter was charged continuously positive or negative, and, secondly, when it was charged positive for $\frac{1}{200}$th of a second, negative for the next $\frac{1}{200}$th, and so on; he found the attraction to be the same in both cases, provided the charges without respect to sign were equal. This experiment establishes beyond a doubt the existence, in the case of sulphur, of a specific dielectric action, which is fully developed in less than $\frac{1}{200}$th of a second. From experiments of this kind values of K were deduced, which agreed fairly well with those obtained by other methods. A very important result which he obtained was, that for a certain crystalline sphere of sulphur the values of K were different in the directions of the axes, being 4·773, 3·970, and 3·811 respectively. The result realizes an expectation of Faraday.[1]

Schiller' Method of electrical oscillations.

Schiller employed two methods—the method of Siemens, which we have already described, in which the duration of charge was from $\frac{1}{40}$th to $\frac{1}{50}$th of a second, and the method of electrical oscillations devised by Helmholtz. In the latter method K is given by the equation $K = (T^2 - T_0^2) \div (T'^2 - T_0^2)$, where T_0, T, T′, are the periods of oscillation of a certain coil, firstly, by itself, secondly, when connected with an air-condenser, and thirdly, with the same condenser when the air is replaced by the insulator to be tested (see below, p. 82). In this method the duration of charge varied from $\frac{1}{20000}$th to $\frac{1}{5000}$th of a second.

The following table gives some of the results of Boltzmann and Schiller:—

	Boltzmann.	Schiller.	
Ebonite	3·15	2·76	2·21
Paraffin (clear)	2·32	1·92	1·68
Do. (milky)		2·47	1·81
Sulphur	3·84	...	..
Rosin	2·55	...	...
Indiarubber (pure)	..	2·34	2·12
Do. (vulcanized)	..	2·94	2·69
White mirror glass	...	6·34	5·88

The first column of Schiller's results was obtained by Siemens's method, the second by the method of oscillations. It will be seen that the shortness of the time of charge has affected the value of K in the last column, reducing it considerably in all cases. Boltzmann's results are on the whole the largest obtained by any physicist; he attributes this to the care with which he constructed his plates. Gibson and Barclay found 1·97 for paraffin, and Siemens 2·9 for sulphur.

Among the more recent researches on the theory of dielectrics may be mentioned those of Rood,[2] whose results for crystals are interesting, and Wüllner,[3] who has studied the course of induction when the charge is maintained for a considerable time.

There are very few fluids which are sufficiently good insulators to allow an easy determination of their specific inductive capacity. Measurements have, however, been made by Silow.[4] He used (1) Siemens's method, and (2) a method in which he observed the deflection of a quadrant electrometer corresponding to the same potential, first, when the quadrants were filled with air, and secondly, when they were filled with the fluid to be examined; the ratio of the latter deflection to the former is the specific inductive capacity of the liquid.

The instrument actually used was a glass vessel, inside which were pasted pieces of tinfoil corresponding to the quadrants of Thomson's electrometer. The shape of the needle was also slightly different. A fine silver wire replaced the bifilar suspension, and the deflections were read off by means of a scale and telescope. The needle and one pair of quadrants were connected with the earth, and the other pair of quadrants charged to a constant potential by connection with a battery. The results were for oil of turpentine by method (1), 1·468; by (2), 1·473; for a certain specimen of petroleum, by (1), 1·439; for another specimen, by (2), 1·428; for benzol, by (1), 1·483.

In the researches in which Siemens's method was used, the speed of the commutator was varied considerably, but no effect was thereby produced on the value of K, which is therefore, within certain limits at least, independent of the duration of the charge.

Perhaps the most important of all the recent additions to our knowledge in this department is due to Boltzmann,[5] who has succeeded in detecting and measuring the decrease of the specific inductive capacity of gases when rarefied.

The principle of his method is as follows. Suppose we have an ordinary air-condenser inside a receiver, which we can exhaust at will. Let one of the armatures A of the condenser be connected with a battery of a large number n of cells (Boltzmann used about 300 Daniell's), while the other armature B is connected with the earth. If we now insulate B, and if the condenser does not leak, then on connecting B with the electrometer no deflection will be indicated. If, however, we increase the number of cells by one, the potential of A will increase from np to $(n+1)p$, while that of B will rise from 0 to an amount which is proportional to p. Let the corresponding electrometer reading be β. Suppose now that we altered the specific inductive capacity of the gas from K_1 to K_2, both armatures being insulated, A originally at potential np, and B at potential zero; the potential of A will, by the mathematical theory, become $\frac{K_1}{K_2}np$, while that of B remains zero. If now we reconnect A with the battery of n cells, the potential of A becomes again np. If we then connect B with the electrometer we shall get a deflection α proportional to

$$np\left(1 \quad \frac{K_1}{K_2}\right); \text{ hence we have } \frac{-\alpha}{\beta} = n\left(1 - \frac{K_1}{K_2}\right)$$

Let us now assume, what experiment shows to be the case, that the increase of K is very nearly proportional to the pressure, then, b_1 and b_2 denoting the manometric reading in millimetres corresponding to K_1 and K_2, we may write

$$K_1 = c\left(1 + \frac{\lambda b_1}{760}\right) \qquad K_2 = c\left(1 + \frac{\lambda b_2}{760}\right).$$

Here λ is a constant, the meaning of which is very simple, if we assume our law of proportionality to hold up to absolute vacuum; in fact, $1+\lambda$ is in that case the specific inductive capacity[6] of the gas at 760 mm. pressure, at the temperature t of observation, and $1+\lambda(1+at)$ is the corresponding coefficient at 0° C. The formula written above becomes therefore

$$\lambda = \frac{a.760}{\beta n(b_1 - b_2)}.$$

In this way Boltzmann arrived at the following values for $\sqrt{K}$ at 760 mm. pressure, and temperature 0° C.:—for air, 1·000295; carbonic acid, 1·000473; hydrogen, 1·000132; carbonic oxide, 1·000345; nitrous oxide, 1·000497; olefiant gas, 1·000656; marsh gas, 1·000472. These results are of great importance in connection with the electromagnetic theory of light.

Residual Discharge.

When an accumulator, whose dielectric is glass or shellac, is charged up to a moderately high potential, and one armature insulated, a gradual fall of the potential occurs. This fall is tolerably rapid at first, but it gets slower and slower till at last it reaches a certain limit, after which it remains sensibly constant for a considerable time. This fall is not entirely due to loss by conduction or convection of the ordinary kind, for we find that if an accumulator that has been charged to potential V, and has been allowed to stand till the potential has fallen considerably, be again charged up to potential V, then

[1] *Exp. Res.*, 1689.
[2] *Pogg. Ann.*, clviii., 1876.
[3] *Pogg. Ann.*, N.F. i., 1877.
[4] *Pogg. Ann.*, clvi., 1875; clvii., 1876.
[5] *Pogg. Ann.*, lv., 1875.
[6] K is now taken to be =1 for absolute vacuum.

the rate of loss is much less than before, being now very nearly constant, and not far from the limit above mentioned. It would appear, therefore, that this constant limit, which on favourable days is very small, represents the loss due to convection and conduction in the usual way, and that the larger varying loss is due to some other cause. When an accumulator, let us say a Leyden jar, has been repeatedly charged up to potential V, until the rate of dissipation has become constant, we shall say that it is saturated. If we discharge a saturated jar, by connecting the knob for a fraction of a second with a good earth communication, and then insulate the knob, the outer coating being supposed throughout in connection with the earth, we find that the instant after the discharge the potential of the knob is zero; after a little, however, it begins to rise, and by and by it reaches a value which is a considerable fraction of V, and has the same sign. This phenomenon justifies the assumption we made as to the peculiar nature of the variable loss of potential experienced by a freshly charged jar. The charge which reappears in this way subsequent to the instantaneous discharge is called the residual discharge.[1] If at any time during the appearance of the residual charge the jar be discharged, the potential of the knob becomes for a short time zero, but begins to rise again; and this may be repeated many times before all trace of charge disappears. Faraday made a variety of experiments on the subject, and established that whenever a charge of positive electricity disappeared or became latent in this way, an equal negative charge disappeared in a similar way. He concluded that the cause of the phenomenon was an actual penetration of the two electricities (*Exp. Res.*, 1245) by *conduction* into the dielectric. This is not the view which is favoured by the best authorities of the present day; it is indeed (see Maxwell, *Elect. and Mag.*, vol. i. § 325) at variance with the received theories of conduction, and alike untenable, as far as we know, whether we adopt the theories of Weber, of Maxwell, or of Helmholtz. Faraday established that time was a necessary condition for the development of the phenomenon; and he was thus enabled to eliminate its influence in the experiments on the specific inductive capacity of sulphur, glass, and shellac. The phenomenon is most marked in the last of these; and in spermaceti, which relatively to these is a tolerably good conductor, the phenomenon is very marked, and develops very rapidly.

Investigations of Kohlrausch.

Kohlrausch[2] studied the residual disharge in an ordinary Leyden jar, in a jar whose outside and inside coatings were at one time quicksilver and at another acidulated water, and in a Franklin's pane, one side of which was coated with tinfoil in the usual way, while the other was silvered like a piece of looking-glass. He showed, by taking measurements with an electrometer and a galvanometer, *that the ratio of the free or disposable charge to the potential is constant.* By the disposable charge is meant the charge which is instantaneously discharged when the knob of the jar is connected with the earth. This ratio is the capacity of the jar, and it appears that it is independent of the "residual" or "latent" charge. He showed that the "latent" charge is not formed by a temporary recession of the electricity to the uncovered glass about the neck and upper part of the jar; and that it does not to any great extent depend on the material used to fasten the armature to the glass, or on the air or other foreign matter between them. On the other hand, his results led him to suspect that the "latent" charge depended on the thickness of the glass, being greater for thick plates than for thin. This conclusion has been questioned, however.[3] He separated by a graphical method the loss by latent charge from the loss by conduction, &c., and found *that the amount of charge which becomes latent, or, which amounts to the same thing, the loss of potential owing to the forming of latent charge in a given time, is proportional to the initial potential so long as we operate with the same jar.*

Kohlrausch recognized the insufficiency of Faraday's explanation of the residual charge, and sought to account for it by extending Faraday's own theory of the polarization of the dielectric. The residual charge is due according to him to a residual polarization of the molecules of the dielectric, which sets in after the instantaneous polarization is complete, and which requires *time* for its development. This polarization may consist in a separation of electricity in the molecules of the dielectric, or in a setting towards a common direction of the axes of a number of previously polarized molecules, analogous to that which Weber assumes in his theory of induced magnetism. It is easy to see that such a theory will to a great extent account for the gradual reduction of the potential of a freshly charged jar, and the gradual reappearance of the residual charge.

If the charge, and consequently the potential, of the jar were kept constant at Q_0, the residual charge tends to a limit pQ_0 (p const.) Kohlrausch assumes that the difference $r_t - pQ_0$ between the residual charge actually formed and the limit decreases at a rate which is at each instant proportional to this difference, and furthermore, to a function of the time, which he assumes to be a simple power. In any actual case, where the jar is charged and then insulated, the charge varies, owing to conduction, &c., and to the formation of residual charge, so that the limit of r^t is continually varying, and we must write Q_t for Q_0, Q_t denoting the charge at time t. The equation for residual charge is then

$$\frac{d}{dt}(r_t - pQ_t) = -bt^m(pQ_t - r_t).$$

From this he deduces the formula

$$r_t = p\left(Q_t - Q_0 e^{-\frac{b}{m+1}t^{m+1}}\right),$$

which he finds to represent his results very closely. m has very nearly the same value ($-0·5744$, or $-\frac{1}{2}$ nearly) in all his experiments, p had the values 0·4289, 0·5794, 0·2562; and b 0·0397, 0·0223, 0·0446 in his three cases.

Analogous phenomena.

Kohlrausch called attention to the close analogy between the residual discharge and the "elastic recovery" (*elastische Nachwirkung*) of strained bodies, which had been investigated by Weber[4] in the case of a silk fibre, and which has of late excited much attention. The instantaneous strain which follows the application of a stress is analogous to the initial charge of the jar, and the gradually increasing strain which follows to the gradual formation of the latent or residual charge. The sudden return to a position near that of unconstrained equilibrium corresponds to the instantaneous discharge, and the slow creeping back to the original state of equilibrium to the slow appearance of the residual discharge. Another analogy may be found in the temporary and residual or subpermanent magnetism of soft iron or steel. If we wish to make the analogy still more complete, we have only to introduce the permanent polarity of tourmaline, the permanent set of certain solids when strained, and the permanent magnetism of hard steel. The phenomena of polarization furnish yet another analogy.

In justifying the introduction of a power of the time into his equation for the residual discharge, Kohlrausch makes the important remark that the time which a residual charge of given amount takes to reappear fully may be different according to the way that charge is produced. The charge reappears more quickly when it is produced in a short time by an initial charge of high potential, than when produced by a charge of lower potential acting

[1] When we think of the part of the charge that has disappeared, *i.e.*, ceased to effect the potential of the knob, we may talk of the "latent charge." This part of the charge is sometimes said to be absorbed.
[2] *Pogg. Ann.*, xci., 1854.
[3] Wüllner, *Pogg. Ann.*, N.F. i. pp. 272, 369.
[4] *De fili bombycini vi elastica*, Gottingæ, 1841

longer. He suggests that the same thing may be true of elastic recovery. He does not allude to the fact (possibly he was unware of it) that two residual charges of different sign may be superposed and reappear separately, although the possibility of this is to a certain extent involved in his remark. The analogous elastic phenomenon has recently been observed by F. Kohlrausch.

Maxwell[1] has shown that phenomena exactly like the residual discharge would be caused by *conduction* in a *heterogeneous* dielectric; each constituent of which by itself has not the power of producing any such phenomenon, so that the phenomenon in general might be due to "heterogeneity" simply.

Hopkinson has lately made experiments on the residual discharge of glass jars. He observed the superposition of residual charges of opposite signs, and he suggests theories analogous to those of Kohlrausch and Maxwell. He finds that his results cannot be represented by the sum of two simple exponential functions of the time, and concludes, therefore, that heterogeneity must be an important factor in the cause of the phenomenon.

The polarities of the different silicates of which the glass is composed rise or decay with the time at different rates, so that during insulation the difference of potential between the armatures E would be represented by a series $\Sigma_0^n A_r e^{-\lambda_r t}$. If, therefore, we charge a jar positively for a long time, and then negatively for a shorter time, the second charge will reverse the more rapidly changing polarities, while the sign of the more sluggish will not be changed; when, therefore, the jar is discharged and insulated, the first-mentioned polarities will decay more quickly at first and liberate a negative charge, and, finally, as the more sluggish also die away, a positive charge will be set free. Hopkinson also made the important observation that agitation of the glass by tapping accelerates the return of the residual discharge.

ON THE PASSAGE OF ELECTRICITY THROUGH BODIES.

We have hitherto supposed electricity to be either immovably associated with perfectly non-conducting matter, or collected on the bounding surfaces of conducting and non-conducting media in such a way that the force tending to cause it to move is balanced by an invincible resistance. **Electric currents.** We have now to consider what happens when there is a finite unbalanced resultant force at any point in a conducting medium. If a conducting sphere of radius a be charged with Q units of positive electricity, its potential will be $\frac{Q}{a}$. Connect this sphere by a long thin wire, whose capacity may be neglected, with another uncharged sphere of radius b, then we know that the potentials of the two spheres become equal; and since what we call electricity is subject to the law of continuity, the whole charge on the two spheres must be the same as before. Hence if U be the common potential, we must have $U=\frac{Q}{a+b}$. It appears, therefore, that the potential of a has fallen by $\frac{b}{a+b}\frac{Q}{a}$, and an amount $\frac{b}{a+b}Q$ of positive electricity has passed from a to b, and also a $\frac{b}{a+b}$th part of the lectric potential energy has disappeared. In accordance with our hypothesis that electricity obeys the law of continuity like an incompressible fluid, we explain this transference of electricity by saying that an electric current has flowed through the wire from the place of higher to the place of lower potential. We define the intensity or strength C of the current as the quantity of electricity which crosses any section of the wire in unit of time.

Owing to the law of continuity the current intensity is of course the same at every point of a linear conductor. In the case which we have just given, the whole transference takes place in so short a time that we cannot study the phenomenon in detail. It is obvious that C will vary rapidly from a large initial value, when the difference between the potentials of the spheres is $\frac{Q}{a}$, to zero when they are at equal potentials. It is possible, by replacing the wire by wetted string or other bad conductor, to prolong the duration of the phenomenon to any extent, so that C should vary very slowly; and we can imagine cases where C would remain constant for a long time. Machines for producing a continuous or "steady" current have been invented in considerable variety, the first of the kind having been the Pile of Volta. Of such machines we shall have more to say when we come to discuss Electromotive Force. We have seen, in the case of our spheres, that the passage of the electric current was accompanied by a loss of potential energy. **Application of principle of conservation of energy.** The question thus arises, what becomes of the energy after the current dies away, and the equalization of potential is complete? This leads us to look for transformations of energy depending on the electric current, or, in other words, to look for dynamical effects of various kinds due to it. Accordingly we find the passage of the electric current accompanied by magnetic phenomena, sparks, heating of the circuit, chemical decompositions, mechanical effects, &c. All these are observed in the discharge of the Leyden jar and other electrostatic reservoirs of potential energy. Exactly similar effects, some more, others less powerful, are observed accompanying the current of the voltaic battery and other machines which furnish a steady flow of electricity. In all such cases we have (1) a source of energy, (2) a flux of electricity, (3) an evolution of energy in different parts of the circuit. We reserve the consideration of (1) for the present, as being the most difficult, and devote our attention to (2) and (3).

Ohm's Law applied to Metallic Conductors.

Measurement of currents. We have already seen how to measure the strength of an electric current in a linear conductor. According to the definition we gave above, the unit current strength would be that for which a unit of electricity passes each section of the conductor in unit of time. If the unit of electricity is the electrostatic unit, this is called the electrostatical unit of current. We have supposed above that the current consists in the transfer of a certain amount of + electricity in a certain direction, which we shall call the *positive direction* of the current, and this for most purposes is convenient. We must remember, however, that no distinction can be drawn between the transference of +Q units of electricity in one direction and the transference of −Q units in the opposite direction; for we have no experimental evidence on which such a distinction can be founded.

Electromagnetic measure. We may measure the current by any one of its various effects. The method most commonly used, both for indicating and measuring currents, is to employ the magnetic effect. According to Oersted's discovery, a magnetic north pole placed in the neighbourhood of a straight current is acted on by a force such that, if the pole were to continually follow the direction of the force, it would describe a circle round the current as an axis, the direction of rotation being that of the rotation of a right-handed cork-screw which is traversing a cork in the positive direction of the current. If, therefore, we have currents of different strength in the same wire, the force exerted on a magnet which always occupies the same position relatively to the wire will be a measure of the current. The force exerted on the magnet may be found by balancing it against known forces, or by allowing the magnet to oscillate under it and finding the time of oscillation. It

[1] *Electricity and Magnetism*, §§ 327 *sqq.*

is easy, by applying the law of continuity to multiple circuits, to verify that the measure of current intensity thus got is proportional to the electrostatic measure.

Thus let AB (fig. 18) be a circuit splitting up into two exactly similar branches BCDG, BEFG, and uniting again at G. Then, since electricity behaves like an incompressible fluid, it is obvious that any current of intensity C in AB will split up into two currents each of intensity $\frac{1}{2}$C in CD and EF. By placing a magnet in similar positions at the same distance with respect to AB, CD, and EF, it will be found that the magnetic action in the last two positions is just half that in the first.

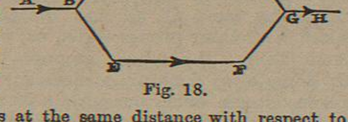

Fig. 18.

Electromagnetic unit. The appropriate unit in magnetic measurements of current intensity is that current which, when flowing in a circular arc of unit radius and unit length, exerts unit of force on a unit north pole placed at the centre of the arc, the unit north pole being such that it repels another equal north pole at unit distance with unit force. This is called the *electromagnetic unit* of current intensity. Unless the contrary is stated, all our formulæ are stated in terms of this unit.

Galvanometer. To facilitate the detection and measurement of currents by magnetic means, an instrument called a galvanometer is used. It consists of a coil of wire, of rectangular, elliptical, or circular section, inside which is suspended a magnetic needle, so as to be in equilibrium parallel to the coil windings under the magnetic action of the earth, or of the earth and other fixed magnets. When a current passes through the coil a great extent of the circuit is in the immediate neighbourhood of the magnet, and the magnetic action is thus greatly accumulated. See article GALVANOMETER.

Electromotive force, resistance, and current strength. If we connect two points A and B of a homogeneous linear conductor, every point of which is at the same temperature, by two wires of the same metal to the electrodes of a quadrant electrometer, then, if a steady current C (measured in *electrostatic* units) be flowing from A to B, we shall find that the potential at A is higher than that at B by a certain quantity E, which we may call the *electromotive force* between A and B, and we may suppose E for the present to be measured in electrostatic units. If we examine the value of the ratio $\frac{E}{C}$ for different positions of the points AB, we shall find that it varies directly as the length of linear conductor between A and B, provided the section of the conductor is everywhere the same. If we try wires of different section, but of the same length and the same material, we find that $\frac{E}{C}$ is inversely proportional to the sectional area; in fact we may write

Ohm's law, $\frac{E}{C}=R$.

$$\frac{E}{C}=R=\frac{kl}{\omega} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

where l denotes the length of the wire, ω its section, and k a constant depending on its material, temperature, and physical condition generally. This is Ohm's law.

In whatever unit measured, R is called the resistance of the conductor. The unit of resistance can always be conceived as established by means of a certain standard wire. The unit of electromotive force is then such that if applied at the end of the standard wire it would generate a unit current in the wire. The constant k is called the *specific resistance* of the material of which the wire is made; it is obviously the resistance of a wire of the material of unit length and unit section

In the electrostatic system of unitation the unit of E = the work done by a unit particle of + electricity in passing to infinity from the surface of an isolated sphere of radius unity charged with an electrostatic unit of + electricity. The dimension of E is $[QL^{-1}]$, where [Q] is the dimension of the electrostatic unit of quantity (see p. 22), $[Q]=[L^{\frac{3}{2}}M^{\frac{1}{2}}T^{-1}]$. Hence the dimension of E is $[L^{\frac{1}{2}}M^{\frac{1}{2}}T^{-1}]$. The unit of C we have already discussed; its dimension is $[QT^{-1}]=[L^{\frac{3}{2}}M^{\frac{1}{2}}T^{-2}]$. From these results, and equation (1), it follows that the dimension of R is $[L^{-1}T]$, *i.e.*, that of the reciprocal of a velocity. We shall show hereafter that, if C be measured in electromagnetic units, its dimension is $[L^{\frac{1}{2}}M^{\frac{1}{2}}T^{-1}]$; hence that of Q is $[L^{\frac{1}{2}}M^{\frac{1}{2}}]$, the unit of Q being the quantity of electricity conveyed across any section by the unit current. Also ECT = work done in time T in conveying C units of + electricity from potential V + E to potential V, whence [ECT] = dimension of energy $=[L^2MT^{-2}]$. Hence $[E]=[L^{\frac{3}{2}}M^{\frac{1}{2}}T^{-2}]$. In this case then $[R]=[LT^{-1}]$; so that in electromagnetic measure R has the dimension of a velocity.

We can put the equation (1) into another form, which suggests at once the generalization of Ohm's law for any conductor. Consider two points P and Q on a linear conductor, at a distance dx from each other, x being measured in the direction of the current. Let the potentials at P and Q be V and V + dV, then E = −dV. If u denote the current per unit of area of the section, then $C=u\omega$, and since $l=dx$ we have $R=\frac{kdx}{\omega}$. Substituting these values in (1) we get

$$u=-\frac{1}{k}\frac{dV}{dx}=\frac{X}{k} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2),$$

where X is the component electric force at P in the direction of the current. Since the electric current is of the nature of a flux, it is determined at any point of a conductor by the flux components uvw, representing the quantities of electricity which in unit of time cross three unit areas perpendicular to three rectangular axes drawn through P. If X, Y, Z be the components of the electric force at P, then the general statement of Ohm's law for a homogeneous isotropic conductor is

$$u=\frac{X}{k} \quad v=\frac{Y}{k} \quad w=\frac{Z}{k} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3).$$

In such a conductor the resistance of a small linear portion of given dimensions, cut out of the substance any where or any how, will be the same. It is conceivable, however, that the resistance of such a small portion would be different if cut in different directions at any point, in which case the conductor would be æolotropic. The most general statement of Ohm's law would then be

Equations of conduction.

$$\left.\begin{aligned} u&=r_1X+p_3Y+q_2Z \\ v&=q_3X+r_2Y+p_1Z \\ w&=p_2X+q_1Y+r_3Z \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4),$$

where r_1, &c., p_1, &c., q_1, &c., are constants for any one point. If they are the same for all points, the body is said to be homogeneous; if they vary from point to point, the body is said to be heterogeneous. If we may liken our conductor to an arrangement of linear conductors (see Maxwell, §§ 297, 324, vol. i.), then it may be shown that the skew system of (4) becomes symmetrical, inasmuch as $p_1=q_1$, $p_2=q_2$, $p_3=q_3$. The great majority of the substances with which the electrician has to deal are, however, isotropic; and unless the experiments of Wiedemann on certain crystals point to æolotropic conduction, we do not know of any case which has been experimentally examined. The reader will find interesting developments of the subject in Maxwell, vol. i. § 297 *sqq.*

A very important remark to be made with regard to the equations (4) is that, being linear, the principle of superposition applies. Thus, if u,v,w be the current components due to electric forces X, Y, Z and u', v', w' similar components for X′, Y′, Z′, then the current for X + X′, Y + Y′, Z + Z′ is given by $u+u'$, $v+v'$, $w+w'$. It is obvious, moreover, that (4) are the most general equations that can be written down to connect current with electromotive force, subject to the condition that the currents due to superposed electric forces are to be found by the superposition of the currents due to the separate forces.

Besides the equations (4), u,v,w are subject like any other flux components to an equation of continuity. This equation, investigated in the usual manner, is

$$\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}+\frac{d\rho}{dt}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (5),$$

where ρ is the electric volume density at the time t. At a surface of discontinuity (5) must be replaced by

$$(u-u')l+(v-v')m+(w-w')n-\frac{d\sigma}{dt}=0 \quad . \quad . \quad (6),$$

where u,v,w, and u',v',w' are components of flux on the first and second sides of the surface, l,m,n the direction cosines of the normal drawn from the first to the second side, and σ the electric surface density at time t.

If we consider the particular case of homogeneous isotropic media, and suppose further that $X=-\frac{dV}{dx}$, $Y=-\frac{dV}{dy}$, $Z=-\frac{dV}{dz}$, these equations reduce to

$$\frac{d^2V}{dx^2}+\frac{d^2V}{dy^2}+\frac{d^2V}{dz^2}=k\frac{d\rho}{dt} \quad . \quad . \quad . \quad . \quad . \quad (7),$$

$$-\frac{1}{k_1}\frac{dV_1}{d\nu}+\frac{1}{k_2}\frac{dV_2}{d\nu}=\frac{d\sigma}{dt} \quad . \quad . \quad . \quad . \quad . \quad (8).$$

In the last equation V_1 and V_2 are the potentials on the two sides of a boundary between media of specific resistance k_1 and k_2.

In the particular case of steady motion, the right-hand sides of (7) and (8) are zero. The analytical treatment of problems about steady currents is therefore precisely analogous to that of problems about electrostatical equilibrium, steady flow of heat, hydrodynamics, &c.: to every solution in one such physical subject corresponds a solution in each of the others. Many valuable details on this subject are to be found in Thomson's papers on electrostatics and magnetism.

Results of Ohm's law.

The consequences of Ohm's law have been followed out mathematically, and verified in a variety of cases. We shall notice a few which are interesting, either from the accuracy of the experimental results, or from the interest or practical importance of some method or principle involved.

Application to uniform linear conductors.

In the case of a steady current in a uniform linear conductor, say a wire, it is obvious that the potential must fall uniformly in the direction in which the current is flowing. Hence, if we suppose the wire stretched out straight, and erect at different points lines perpendicular to it, representing the potential at each point, the locus of the extremities of these lines will be a straight line.

This may be arrived at by integrating equation (5), which becomes in this case $\frac{d^2V}{dx^2}=0$, x being measured along the wire supposed to be straight; we thus get for the potential V, at any point distant x from the origin, at which potential is V_0,

$$V=V_0-\frac{Ck}{\omega}x \quad . \quad . \quad . \quad . \quad . \quad . \quad (9).$$

If V be taken as ordinate, this represents a straight line, the tangent of whose inclination to the x-axis is $-\frac{Ck}{\omega}$, or $-uk$.

Voltaic circuit.

We cannot apply Ohm's law at the junction of two different substances. The condition of continuity of course applies; in other words, if the flow has become steady, the current is the same at all points of the circuit, whether homogeneous or not. We shall see, when we come to discuss electromotive force, that there is a constant difference between the potentials at two points infinitely near each other, but on opposite sides of the boundary between two conductors of different material. If we knew this potential difference for each point of heterogeneous contact in the circuit, we could draw the complete potential curve for the circuit by applying Ohm's law to each conductor separately. The diagram (fig. 19) represents (on the contact theory, as held by Ohm, see Origin of Electromotive Force) the fall of potentials and the discontinuities in a voltaic circuit, consisting of zinc, water, and copper, in which the current flows from Cu to Zn across the junction of the metals. We assume for the present that Ohm's law applies to the liquid conductor.

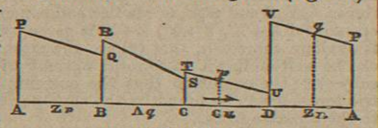

Fig. 19.

Let us denote by V_Q, V_R, &c. the potentials at Q and R, &c., or what is the same thing, the ordinates BQ, BR, &c., in our diagram. Then applying Ohm's law to the homogeneous parts of the circuit, we have $V_V-V_Q=CR'$, $V_R-V_S=CS$, $V_T-V_U=CR''$, where R′, S, R″, denote the resistances of the zinc, the water, and the copper respectively. Now, denoting V_V-V_U, the potential difference, or as it is sometimes called, the "contact force" between Zn and Cu by E_{ZC}, and so on, let us add the above three equations; we thus get

$$E=E_{ZC}+E_{AZ}+E_{CA}=C(R'+R''+S).$$

Here E is called the *whole electromotive force* of the circuit, being the sum of all the discontinuities of potential, taken with their proper signs, or, what is equivalent to the same thing, the whole amount of work which would be done by a unit of + electricity, in passing round the whole circuit once, supposing it to get over the discontinuities without gain or loss of work. Defining E in this way, we may extend Ohm's law to a heterogeneous circuit, the resistance R being now the sum of all the resistances of the different parts, or the whole resistance. In accordance with this definition, if we take two points, p and q (fig. 19) in the Cu and Zn respectively, the whole electromotive force will be $V_p-V_q+E_{ZC}$ and the current will be given by

$$V_p-V_q+E_{ZC}=RC \quad . \quad . \quad . \quad . \quad . \quad . \quad (10),$$

where R is the whole resistance of pq. V_p-V_q is sometimes called the "external," and E_{ZC} the "internal" electromotive force. If p, q include more than one contact of heterogeneous metals, we have only to add on the left-hand side of (10) the corresponding internal electromotive force for each discontinuity.

If p and q be connected by wires of the same metal, say copper, to the electrodes of a Thomson's electrometer, then the electrometer will indicate a potential difference, $V_p-V_q+E_{ZC}$, and not V_p-V_q as might at first sight be suspected.[1] No electricity can flow through the electrometer, hence the copper wire attached at p, and the pair of quadrants to which it leads (we may suppose the quadrants made of copper, but in reality it does not matter, see below, Origin of Electromotive Force), will be at potential V_p. But owing to the contact force between the Zn and Cu at q, the wire from q and the quadrant to which it leads will be at potential V_q-E_{ZC}. It appears, therefore, that the electrometer indication corresponds to the *whole* electromotive force between p and q, and is proportional to the whole resistance between p and q, no matter what metals the circuit may include.[2] This conclusion was verified by Kohlrausch. His method rested on the principle of Volta's condensing electroscope.

Verification by Kohlrausch.

He used an accumulator consisting of a fixed plate B, and an equal movable plate A, which could be lowered to a very small fixed distance from B, and raised to a considerable distance, so as to touch a fixed wire leading to a Dellmann's electrometer. The plate A was lowered and connected with p, while q and the fixed plate were connected with the ground; the connection with p was then removed, and A raised, its potential thereby greatly increasing owing to its greatly diminished capacity. This increased potential was measured by the electrometer, with which A was in connection through the fixed wire. In one of Kohlrausch's experiments, he found for the electromotive force between a fixed point of the metallic circuit and four points, such that the resistance between each adjacent pair was very nearly equal, the values 0·85, 1·81, 2·69, 3·70; the values calculated by Ohm's law were 0·93, 1·86, 2·80, 3·73. He also examined the fluid part of the circuit, and still found a good agreement between theory and experiment. (See Wiedemann, § 102.)

The laws of current distribution in a network of linear circuits were first studied by Kirchhoff. He laid down two general principles which are very convenient in practical calculations.

I. The algebraical sum of all the currents flowing from any node of the network is zero.

II. If we go round any circuit of the network, then no

[1] It is supposed that all the wires are at the same temperature.

[2] This more general statement follows at once from the above reasoning in conjunction with Volta's law (*cf.* below, Origin of Electromotive Force).

matter how many meshes it may include, or what conductors may branch off at different parts, we have

$$E=R_1C_1+R_2C_2+\ \ldots\ldots\ R_nC_n,$$

where E is the *whole internal electromotive* force, and R_1, R_2 .... C_1, C_2 ... are the resistances and current strengths in the different parts of the circuit.

The first of these principles is simply the law of continuity, and the second is got at once by applying equation (10).

We give here an investigation of the currents and potentials in a network of conductors. The method and notation are taken from Maxwell, vol. i. § 280. Let A_1, A_2, ... A_n be n points, connected by a network of $\frac{1}{2}n(n-1)$ conductors (that being the number of different pairs of conductors that can be selected from the n). Let C_{pq}, E_{pq}, K_{pq} denote the current strength, internal electromotive force, and *conductivity*, *i.e.*, the reciprocal of the resistance, for the conductor $A_p A_q$. Let, moreover, the potential at A_p be P_p, and the current of electricity which enters the system there be Q_p. It is obvious from our definitions of the symbols that

$$K_{pq}=K_{pq}, \quad C_{pq}=-C_{qp}, \quad E_{pq}=-E_{qp},$$

and, by the condition of continuity, that

$$Q_1+Q_2+\ \ldots\ +Q_n=0.$$

At the point A_p we have

$$C_{p1}+C_{p2}+\ \ldots\ +C_{pn}=Q_p \quad . \quad . \quad . \quad (\alpha).$$

Now

$$C_{pq}=K_{pq}(P_p-P_q+E_{pq}) \quad . \quad . \quad . \quad . \quad (\beta).$$

Hence (α) becomes

$$K_{p1}(P_1-P_p)+K_{p2}(P_2-P_p)+\ \ldots\ +K_{pn}(P_n-P_p)$$
$$=K_{p1}E_{p1}+\ \ldots\ +K_{pn}E_{pn}-Q_p \quad . \quad . \quad . \quad (\gamma).$$

The symbol K_{pp} does not occur in this equation, and has no meaning as yet. Let us define it to mean $-(K_{p1}+K_{p2}\ \ldots\ K_{pn})$, where K_{pp} does not occur. Then we have

$$K_{p1}+K_{p2}+\ \ldots\ +K\ \ +\ \ldots\ +K_{pn}=0, \quad . \quad . \quad . \quad . \quad (\delta)$$

and, multiplying by P_p-P_r,

$$K_{p1}(P_p-P_r)+\ \ldots\ +K_{pp}(P_p-P_r)\ \ldots\ K_{pn}(P_p-P_r)=0.$$

Adding this last equation to (γ) we get

$$K_{p1}(P_1-P_r)+K_{p2}(P_2-P_r)+\ \ldots\ +K_{pn}(P_n-P_r)$$
$$=K_{p1}E_{p1}+\ \ldots\ +K_{pn}E_{pn}-Q_p \quad . \quad . \quad (\epsilon).$$

In this equation the term whose coefficient is K_{pr} of course vanishes. By giving p all possible values except r, we get a set of $n-1$ equations to determine the $n-1$ quantities P_1-P_r, P_2-P_r, &c. Hence if M_{rr} denote the minor of K_{rr} in the determinant $\Delta=(K_{11}K_{22}\ \ldots\ K_{nn})$;[1] and if M_{rrp} denote the minor of K_{pq} in M_{rr}, we have

$$(P_p-P_r)M_{rr}=\{K_{11}E_{11}+K_{12}E_{12}+\ \ldots\ +K_{1n}E_{1n}-Q_1\}M_{rr1p}$$
$$+\{K_{21}E_{21}+K_{22}E_{22}+\ \ldots\ldots\ldots\ _2-Q\}M_{rr2p}$$
$$+\ \&c. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\zeta),$$

where of course E_{11} and E_{22} are zero, and M_{rrrp} does not occur. This expression is linear in the letters E and Q, and the principle of superposition holds, as we saw it ought to do in all applications of Ohm's law.

Consider the particular case in which all the Qs and Es vanish, except E_{lm} and E_{ml} $(=-E_{lm})$, we then have the case of a linear circuit in which an electromotive force E_{lm} is introduced into A_lA_m. We get from (ζ)

$$P_p-P_r=\frac{K_{lm}E_{lm}}{M_{rr}}(M_{rrlp}-M_{rrmp}),$$

and

$$P_q-P_r=\frac{K_{lm}E_{lm}}{M_{rr}}(M_{rrlq}-M_{rrmq}).$$

Hence

$$P_p-P_q=\frac{K_{lm}E_{lm}}{M_{rr}}(M_{rrlp}-M_{rrlq}-M_{rrmp}+M_{rrmq}),$$

and

$$C_{pq}=\frac{K_{pq}K_{lm}E_{lm}}{M_{rr}}(M_{rrlp}-M_{rrlq}-M_{rrmp}+M_{rrmq}) \quad . \quad . \quad . \quad (\eta).$$

Similarly, if C_{lm} be the current in A_lA_m due to an electromotive force E_{pq} in A_{pq}, we get

$$C_{lm}=\frac{K_{lm}K_{pq}E_{pq}}{M_{rr}}(M_{rrpl}-M_{rrpm}-M_{rrql}+M_{rrqm}) \quad . \quad . \quad . \quad . \quad (\theta).$$

[1] This determinant has many properties of interest to the mathematical student; *e.g.*, in our notation $M_{11}=M_{22}\ \ldots\ =M_{nn}$, $M_{rrpq}-M_{rrlp}=M_{pprr}-M_{pptr}$, &c. &c.

Now, since Δ is a symmetrical determinant, $M_{rrlp}=M_{rrpl}$, &c., and the expressions within brackets in (η) and (θ) are identical. Hence follows the important proposition:—

If an electromotive force equal to unity, acting in any conductor A_lA_m of a linear system, cause a current C to flow in the conductor A_pA_q, then an electromotive force equal to unity, acting in A_pA_q, will cause an equal current C to flow in A_lA.

If we suppose all the conductors of the system except A_lA_m and A_pA_q removed, and A_lA_p and A_mA_q joined by two wires, in such a way that for electromotive force unity in A_lA_m the current in A_pA_q is C then the conductivity of the circuit which we have thus constructed would be

$$\frac{K_{lm}K_{pq}}{M_{rr}}(M_{rrpl}-M_{rrpm}-M_{rrql}+M_{rrqm});$$

this might be called the reduced conductivity of the system with respect to A_pA_q and A_lA_m. When the expression within brackets vanishes, the conductors A_pA_q and A_lA_m are said to be conjugate. The reduced resistance in this case is infinite, and no electromotive force in A_lA_m, however great, will produce any current in A_pA_q, and reciprocally.

Conjugate conductors.

Similarly, we may prove that if unit current enter a linear system at A_l and leave it at A_m, the difference of potential thereby caused between A_p and A_q is the same as that caused between A_l and A_m, when unit current enters at A_p and leaves at A_q. (See Maxwell.)

Multiple arc.

The case of several wires forming a multiple arc very often occurs in practice. Let AB, CD (fig. 20) be two parts of a circuit whose resistances are R and S, and let the circuit branch out between B and C into three branches of resistances R_1, R_2, R_3.

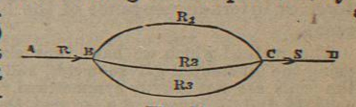

Fig. 20.

We have $V_B-V_C=R_1C_1=R_2C_2=R_3C_3$, and

$$C_1=\frac{\frac{1}{R_1}C}{\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}}, \quad C_2=\&c.$$

Also

$$V_A-V_D=V_A-V_B+V_B-V_C+V_C- \qquad (R+\rho+S)C,$$

where

$$\frac{1}{\rho}=\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}.$$

Hence current in each branch is inversely proportional to the resistance, that is directly proportional to the *conductivity;* and the reduced conductivity of the multiple arc is equal to the sum of the conductivities of its branches. These statements are obviously true for any number of branches.

Resistance measurement.

Some of the most important applications of the theory of linear circuits occur in the methods for comparing resistances. The earliest method for doing this consisted simply in putting the two conductors, whose resistance it was required to compare, into a circuit which remained otherwise invariable; if the current, as measured by a galvanometer, was the same, whichever conductor was in the gap, it was concluded that their resistances were equal. The difficulty in this method is that the electromotive force and internal resistance of the battery are supposed to remain constant, a condition which it is excessively hard to fulfil.

This difficulty can be avoided by using a differential galvanometer, or the arrangement of conductors called Wheatstone's bridge. The differential galvanometer differs from an ordinary one simply in having two wires wound side by side instead of a single wire. If we pass equal currents in opposite directions through the two wires, the action on the needle is zero, provided the instrument be perfectly constructed. If the currents are unequal, the indication will be proportional to the difference of the current strength.

If the coils are not perfectly symmetrical, but such that