shell. In passing from P to Q, without cutting the shell, the solid angle ω decreases by 4π infinitely nearly. Now, during the passage from Q to P we may not represent the action of the current by S, but nothing hinders us from representing its action by another shell S′, which does not pass between Q and P, but is at a finite distance from either of them; for it will be remembered that the shell which represents the action of a current i is definite to this extent merely—that its strength is i, its boundary is the circuit, and it does not pass through the point at which the action is being considered. But infinitely little work, owing to the action of S′, is done in passing from Q to P. Hence the work done by a unit pole in going once completely round any path which embraces the current once is $4\pi i$.

To reconcile this result with the continuity of the magnetic potential of a linear circuit, for the existence of which we have now furnished sufficient evidence, we must admit that the potential of a linear circuit at any point P is $V = i(\omega + 4n\pi)$, where n is any integer. In other words, V is a many-valued function differing from i times the solid angle subtended at P by a multiple of $4\pi i$. If we pass along any path from P and return thereto, the difference of the values of V, or the whole work done on the journey, is zero if the path does not embrace the circuit, $4n\pi i$ if it embraces[1] it n times

Linear circuit in magnetic field.

The considerations enable us to determine the action of any closed current on a magnetic pole, and consequently on any magnetic system. We have next to find the action on a linear circuit when placed in any given magnetic field, whether due to magnets or electric currents. This we do by replacing the circuit acted on by its equivalent magnetic shell.

If the potential at any point of the magnetic field be V, then the potential energy of a magnetic shell S, of strength i, placed in the field is given by

$$M = i\iint\left(l\frac{dV}{dx} + m\frac{dV}{dy} + n\frac{dV}{dz}\right)dS, \quad . \quad . \quad (3),$$

where (l, m, n) are the direction cosines of the positive direction (south to north) of the normal to the element dS. Since, so long as the magnetic force considered is not due to S itself, there is none of the magnetism to which V is due on S, we may write $-a, -b, -c$ for $\frac{dV}{dx}, \frac{dV}{dy}, \frac{dV}{dz}$, where a, b, c are the components of the magnetic induction.[2] Then, if $N = \iint(la + mb + nc)dS$ (*i.e.*, = the surface integral of magnetic induction, or the number of lines of magnetic force which pass through the circuit), we may write

$$M = -iN \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

From this expression for the potential energy of the equivalent magnetic shell we can derive at once the force tending to produce any displacement of the circuit regarded as rigid.

Thus let ϕ be one of the variables which determine the position of the system, then the force Φ tending to produce a displacement $d\phi$ is given by $\Phi d\phi + dM = 0$, or

$$\Phi = i\frac{dN}{d\phi} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5).$$

Hence the work done during any displacement of a closed circuit, in which the current strength is i, is equal to i times the increase produced by the displacement in the number of lines of force passing through the circuit. The force tends, therefore, to produce the displacement or to resist it, according as the displacement tends to increase or to diminish the number of lines of force passing through the circuit. It is evident, therefore, that a position of stable equilibrium will be that in which the number of lines of magnetic force passing through the circuit is a maximum. If that number is a minimum, we have a case of unstable equilibrium.

Action on element of circuit.

Maxwell[3] has shown how we may deduce from the above theory the force exerted on any portion of the circuit which is flexible or otherwise capable of motion. "If a portion of the circuit be flexible so that it may be displaced independently of the rest, we may make the edge of the shell capable of the same kind of displacement by cutting up the surface of the shell into a sufficient number of portions connected by flexible joints. Hence we conclude that, if by displacement of any portion of the circuit in a given direction the number of lines of induction which pass through the circuit can be increased, this displacement will be aided by the electromagnetic force acting on the circuit."

From these considerations we may find the electromagnetic force acting on any element ds of the circuit. Let PQ (fig. 30) be the element ds belonging to the arc AB of any circuit. Let P$\mathfrak{B}$ be the direction of the magnetic induction[4] at P, and $\mathfrak{B}$ its magnitude. It is obvious that no motion of PQ in the plane of PQ and P$\mathfrak{B}$ will increase or diminish the number of lines of force passing through the circuit; consequently no work will be done in any such displacement. Hence the resultant electromagnetic force R must be perpendicular to the plane QP$\mathfrak{B}$. Let PR be a small displacement perpendicular to this plane, the work done in the displacement is R.PR, and the number of lines of force cut through is i times the rectangular area PQR multiplied by the component $\mathfrak{B}\sin\theta$ of the magnetic induction perpendicular to it. Hence we have

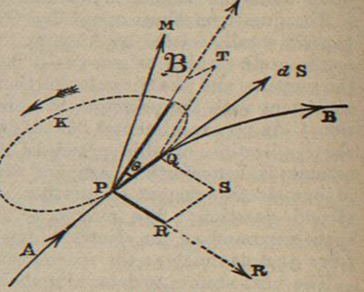


Fig. 30.

$$R \times PR = ids \times PR \times \mathfrak{B}\sin\theta,$$

i.e.

$$R = ids\mathfrak{B}\sin\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6).$$

Hence the resultant electromagnetic force on the element ds may be determined as follows:—Take P$\mathfrak{B}$ in the direction of the resultant magnetic induction (magnetic force) and proportional to $i\mathfrak{B}$, and take PQ in the direction of ds and proportional to it; the electromagnetic force[5] on the element of the circuit is proportional to the area of the parallelogram whose adjacent sides are P$\mathfrak{B}$ and PQ, and is perpendicular to it. The force in any direction making an angle ϕ with the direction of the resultant is of course $R\cos\phi$. The following consideration is convenient for determining which way the resultant force acts. It is obvious that the force on the element will be the same to whatever circuit we suppose it to belong, so long as the direction and strength of the current in it is the same. Take, then, a small circuit PQK perpendicular to the lines of magnetic induction (magnetic force) near PQ, in such a way that the direction of the current in PQK (as determined by the direction in PQ) is related to the direction of the magnetic induction in the same way as rotation and translation in right-handed screw motion; then the element PQ tends to move so that the number of lines of force passing through PQK increases.[6]

[1] On the space relations involved here see Maxwell, vol. i. § 17, &c.

[2] Magnetic induction is used here in Maxwell's sense. It coincides in meaning with "magnetic force" at points where there is no magnetism. "Line of force" in Faraday's extended sense is synonymous with "line of induction" in Maxwell's sense.

[3] *Electricity and Magnetism*, vol. ii. § 490.

[4] "Resultant magnetic force," if there is none of the magnetism producing it at P.

[5] We need scarcely remind the reader that this is a ponderomotive force acting on the matter of the element of the circuit. There is no question of force acting on the current or the electricity in it.

[6] From this may be derived the following, which is often very convenient. Stand with feet on PQ and body along the positive direction of the line of magnetic force and look in the direction of the current, then the force is towards the right hand.

Several other ways of remembering this direction might be given. Although the above may sound arbitrary and look clumsy at first, yet we have found it more convenient in practice than some others we have tried.

We may extend what has been said above to the case where part of the magnetic force, it may be the whole of it, is due to the current in the circuit itself; for we might suppose the magnetic field to be that due to a shell whose boundary coincides infinitely nearly with the circuit. If the circuit is rigid, there will of course be no motion caused by its own action; but if it be flexible, there may be relative motions; in fact each portion will move until the number of lines of force that pass through the circuit is the greatest possible consistent with the geometrical conditions.

Vector potential, &c.

It is an obvious remark, after what has been said, that the potential energy of the magnetic shell which represents a current depends merely on its boundary, or, in other words, that the magnetic induction or the number of lines of magnetic force which pass through a circuit depends merely on its form. Hence we should expect to find some analytical expression for the surface integral of magnetic induction depending merely on the space relations of the circuit; in other words, we should expect to find a line integral to represent it. And when the field is that of another circuit, we should expect to find a double line integral for the mutual potential energy of the two representative shells.[1] We shall describe briefly how these expectations are realized.

In the first place, a vector may be found which has the property that its line integral taken round any circuit is equal to the surface integral of magnetic induction taken over any surface bounded by the circuit.[2] This vector has been called by Maxwell the "vector potential" ($\mathfrak{A}$). Let its components be F, G, H. Then applying the definition to small areas $dydz$, $dzdx$, $dxdy$, at the point xyz perpendicular to the three axes,[3] a, b, c being components of magnetic induction as before, we get

$$a = \frac{dH}{dy} - \frac{dG}{dz}, \quad b = \frac{dF}{dz} - \frac{dH}{dx}, \quad c = \frac{dG}{dx} - \frac{dF}{dy} \quad . \quad . \quad (7).$$

These equations might be used to determine F, G, H, and would lead to a much more general solution than is here required. The following synthetical solution is simpler.

Consider a magnetized particle sn at O (fig. 31). Let the positive direction of its axis be OK, and let its moment be m. The resultant force due to sn at any point P is in a plane passing through OK; hence the vector potential $\mathfrak{A}$ at P must be perpendicular to this plane. Let its direction be taken so as to indicate a rotation round OK, which with translation along OK would give right-handed screw motion. Describe a sphere with O as centre and OP (= D) as radius. Let PQ be a small circle of this sphere whose pole is K. Consider the line integral round PQ, and the surface integral over the spherical segment PKQ. Since $\mathfrak{A}$ is the same at all points of PQ by symmetry, the former is $2\pi D\sin\theta\mathfrak{A}$, and the latter is $\frac{2\pi m\sin^2\theta}{D}$. Equating these we get for vector potential of sn at P

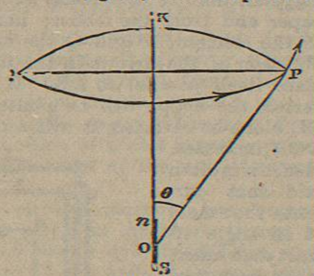


Fig. 31.

$$\mathfrak{A} = \frac{m}{D^2}\sin\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8),$$

its direction being that already indicated.

Suppose now the particle sn placed at $Q(xyz)$ so that the direction cosines of sn are λ, μ, ν. Let the coordinates of P be ξ, η, ζ; also let $QP = D = +\sqrt{(\xi - x)^2 + (\eta - y)^2 + (\zeta - z)^2}$. Then the direction cosines of QP are $D^2\frac{dp}{dx}, D^2\frac{dp}{dy}, D^2\frac{dp}{dz}$, where $p = \frac{1}{D}$; and we get for the component of the vector potential at P

$$F = m\left(\mu\frac{dp}{dz} - \nu\frac{dp}{dy}\right) \quad . \quad . \quad . \quad . \quad . \quad (9),$$

and two similar expressions for G and H.

The vector potential of a magnetized body may be got by compounding the vector potentials of the different elements; hence, $I\lambda$, $I\mu$, $I\nu$ being the components of magnetization at any point of the body, we get

$$F = \iiint I\left(\mu\frac{dp}{dz} - \nu\frac{dp}{dy}\right)dxdydz \quad . \quad . \quad . \quad (10),$$

and two similar expressions for G and H. The first part of our problem is thus solved.

Let us, in the second place, apply the above result (10) to the case of the two shells which are equivalent to two currents. In a *lamellar* distribution of magnetism $\frac{d(I\mu)}{dz} = \frac{d(I\nu)}{dy}$, &c.; hence the volume integral in (10) reduces to a surface integral, and

$$F = \iint\frac{1}{D}I(\mu n - \nu m)dS \quad . \quad . \quad . \quad . \quad (11);$$

where l, m, n are the direction cosines of the outward normal to dS.

Now the magnetic shell of thickness τ and strength i is a lamellarly magnetized body of constant intensity $= i \div \tau$. It may be looked upon as bounded by two parallel surfaces normal everywhere to the lines of magnetization, and by an edge generated by lines of magnetization. At every point on either of the parallel surfaces we have therefore $l = \lambda$, $m = \mu$, $n = \nu$; and at the edge $l = \nu\frac{dy}{ds} - \mu\frac{dz}{ds}$, and similarly for m and n. Hence every element of the double integral in (11) belonging to either of the parallel surfaces vanishes, and there remain only the parts on the edge which give

$$F = \Sigma\frac{I\tau}{D}\left\{\mu\left(\mu\frac{dx}{ds} - \lambda\frac{dy}{ds}\right) - \nu\left(\lambda\frac{dz}{ds} - \nu\frac{dx}{ds}\right)\right\}ds = i\int\frac{1}{D}\frac{dx}{ds}ds \quad (12),$$

since $\lambda\frac{dx}{ds} + \mu\frac{dy}{ds} + \nu\frac{dz}{ds} = 0$. (12) gives the vector potential at $(\xi\eta\zeta)$ due to a magnetic shell S. Let $(\xi\eta\zeta)$ be any point on the boundary of another shell S, of strength i, and let $d\sigma$ be the element of arc of the boundary, then

$$-i'\int\left(F\frac{d\xi}{d\sigma} + G\frac{d\eta}{d\sigma} + H\frac{d\zeta}{d\sigma}\right)d\sigma \quad . \quad . \quad . \quad . \quad (13)$$

is the magnetic induction through S′ due to S with the sign changed, in other words, the mutual potential energy M. Putting for F, G, H their values by (12), we have

$$M = -ii'\iint\frac{1}{D}\left(\frac{dx}{ds}\frac{d\xi}{d\sigma} + \frac{dy}{ds}\frac{d\eta}{d\sigma} + \frac{dz}{ds}\frac{d\zeta}{d\sigma}\right)dsd\sigma$$

$$= -ii'\iint\frac{\cos\epsilon}{D}dsd\sigma \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14),$$

where ϵ is the angle between ds and $d\sigma$.

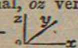
The result of (14) realizes the second of our expectations. The double integral arrived at is of great importance, not only in the theory of electrodynamics, but also as we shall see in the theory o the induction of electric currents.

Hitherto we have spoken only of closed circuits, and considered merely the action of a circuit regarded as a whole. When we did speak of the force on an element of a circuit, we deduced this force directly from the state of the magnetic field in its immediate neighbourhood. There is an order of ideas, however, in which the mutual action of two circuits is considered to be the sum of all the mutual actions of every element in one circuit on every element in the other. Now, we can easily show, by means of (14), that a system of elementary forces of this kind can be found which will lead to the same result for closed circuits as the theory given above.

Let the circuit S′ be supposed rigid and fixed, and let the circuit S be movable in any way with respect to S′; it may even be flexible.

[1] It is important to remark here that we say "of the two representative shells," not "of the two circuits," or "of the two currents" (see below, p. 76).

[2] The mathematical idea concerned here seems to have been originally started by Prof. Stokes; it is deeply involved in the improvements effected in the theories of hydrodynamics, elasticity, electricity, &c., by Stokes, Thomson, Helmholtz, and Maxwell.

[3] It is to be noted that the rectangular axes here used are drawn thus:—ox horizonal, oz vertical (in plane of paper say), and oy from the reader; thus— 

In this way rotation from y to z and translation along ox give right-handed screw motion, and so on in cyclical order.

Denote the angles between the positive directions of $d\sigma$ and ds and the direction of D from $d\sigma$ to ds by θ' and θ, then we have

$$\left.\begin{aligned}\cos\theta = \frac{dD}{ds},\ \cos\theta' = -\frac{dD}{d\sigma},\\ \cos\epsilon = -\frac{dD}{ds}\frac{dD}{d\sigma} - D\frac{d^2D}{ds\,d\sigma}\end{aligned}\right\} \quad . \quad . \quad (15).$$

By means of these we get

$$M = ii'\iint \frac{1}{D}\frac{dD}{d\sigma}\frac{dD}{ds}d\sigma ds, \quad . \quad . \quad . \quad (16).$$

The part which is a complete differential has been left out, because it disappears when the integration is carried round closed circuits, as we always suppose it to be. Consider now the work done in a small displacement which alters D and S, $\frac{dD}{d\sigma}$, $\frac{dD}{ds}$, and ds, but not $d\sigma$; we have

$$\begin{aligned}\delta M = -ii'\iint \frac{1}{D^2}\frac{dD}{d\sigma}\frac{dD}{ds}\delta D d\sigma ds + ii'\iint \frac{1}{D}\frac{d\delta D}{d\sigma}\frac{dD}{ds}d\sigma ds\\ + ii'\iint \frac{1}{D}\frac{dD}{d\sigma}\left(\frac{d\delta D}{ds} - \frac{dD}{ds}\frac{d\delta s}{ds}\right)d\sigma ds\\ + ii'\iint \frac{1}{D}\frac{dD}{d\sigma}\frac{dD}{ds}\frac{d\delta s}{ds}d\sigma ds\,.\end{aligned}$$

The parts containing δs disappear in this expression, and if the rest be arranged by integration by parts as usual, we get

$$\delta M - \iint R\delta D ds d\sigma = 0 \quad . \quad . \quad . \quad . \quad . \quad (17).$$

where $R = ii'\frac{2\cos\epsilon - 3\cos\theta\cos\theta'}{D^2}$.

Hence the electrodynamical action of the two circuits is completely accounted for by supposing every element $d\sigma$ to *attract* every element ds with a force

$$\frac{ii'dsd\sigma}{D^2}\left(2\cos\epsilon - 3\cos\theta\cos\theta'\right) \quad . \quad (18).$$

We may therefore use this elementary formula whenever it suits our convenience to do so.

It is very easy to obtain a similar elementary formula, which is very often useful, for the action of an element of a circuit on a unit north pole.

We have seen above how to find the action on an element PQ (ds) of a circuit in a given magnetic field. Let the field be that due to a unit north pole N (fig. 32). Then the magnetic induction at P is in the direction NPK, and is equal to $\frac{1}{D^2}$, if NP = D. Hence by (6) the force R on PQ[1] is perpendicular to NP and PQ, is in the direction PM shown in the figure, and is equal to $\frac{ids\sin\theta}{D}$. Now, by the princip of "action and reaction," the force on N is R in the

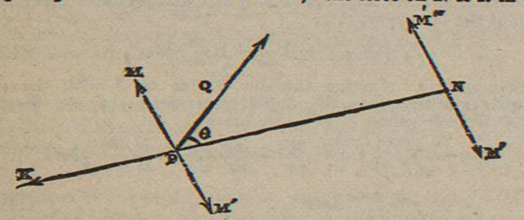


Fig. 32.

direction PM′ opposite to PM, *i.e.* is equal to a force R acting at N in a direction NM″ parallel to PM′, together with a couple whose moment is R × PN, and whose axis is perpendicular to NP and in the plane NPQ. Now a simple calculation, which we leave to the reader, will show that for any closed circuit the resultant of all the couples thus introduced is *nil;* hence, since we deal with closed circuits only, we may neglect the couple.

The force exerted by a closed circuit on a unit north pole may therefore be found by supposing each element ds to act on the pole with a force equal to

$$\frac{ids\sin\theta}{D^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19),$$

whose direction is perpendicular to the plane containing the pole and the element, and such that it tends to cause rotation round the element related to the direction of the current in it by the right-handed screw relation.

[1] PQ is supposed to be drawn *from* the reader.

Comparison of Theory with Experiment.—The best verification of the theory which has just been laid down consists in its uniform accordance with experience. We proceed to give a few instances of its application, adopting now one, now another, of the equivalent principles deduced from it.

We have already remarked that the lines of magnetic force in an electric field due to an infinite straight current are circles having the current for axis. It is easy to deduce from the fact that there is a magnetic potential that the force must vary inversely as the distance from the current.

This may also be proved by means of the formula (19); in fact, the resultant force at P is given by

$$R = i\int\frac{\sin\theta}{D^2}ds = i\int_0^\pi \frac{\sin\theta}{d^2\mathrm{cosec}^2\theta}\mathrm{cosec}^2\theta\, d\theta = \frac{2i}{d} \quad . \quad . \quad . \quad (20),$$

d being the distance of P from the current.

Parallel currents. Let AB (fig. 33) be a very long straight current, and poq an element ds of a parallel current, having the same direction as AB. If we draw the line of force (a circle with C as centre) though O, the tangent $\overrightarrow{OR}$ is the direction of the force at O; hence by (6) and (20), the force on poq is $\frac{2i}{d}ds$, and acts in the direction OC; poq is therefore attracted. If the current in poq be reversed, the force will have the same numerical value, but will act in the direction CO. Hence two parallel straight conductors attract or repel each other according as the currents in them have the same or opposite directions.

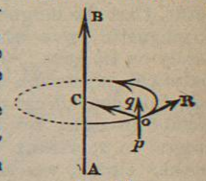


Fig. 33.

Inclined currents Let $\overrightarrow{AB}$ (fig. 34) be an infinitely long (or very long) current, $\overrightarrow{CD}$ a portion of a current inclined to it, and passing very near it at O. If the plane of the paper contain AB and CD, then at every point in OD the magnetic force is perpendicular to the plane of the paper and towards the reader, at every point in OC perpendicular to the plane of the paper and from the reader; hence at the elements P and Q the forces acting will be in the direction of the arrows in the figure, and CD will tend to place itself parallel to AB. If both the currents be reversed, the action will be unaltered; but if the current in CD alone be reversed, it will move so that the acute angle DOB increases.

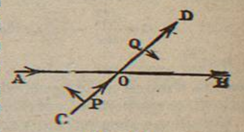


Fig. 34.

Hence it is often said that currents that meet at an angle attract each other, when both flow to or both flow from the angle, but repel when one flows to and the other flows from the angle.

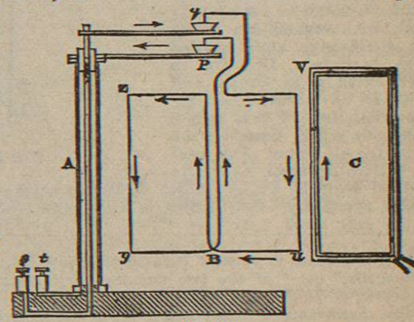


Fig. 35.

These actions may be demonstrated in a great variety of ways. Figure 35 shows an arrangement for demonstrating the attraction or repulsion of parallel currents, which is essentially that first used by Ampère. A is an upright consisting of a tube in good metallic connection with one of the binding screws t, and with a little cup p, containing a drop of mercury. A stout wire passes up the centre of the tube, and is insulated from it, but in metallic connec-

tion with the screw s and the cup q. B is a light conductor,[1] consisting of two parallelograms of wire, in which the current circulates in opposite directions, the object of which is to eliminate the magnetic action of the earth. The conductor is hung in the cups p and q, so as to be easily movable about a vertical axis. C is a frame on which several turns of wire are wound, so that when a current is passed through, we have a number of parallel conductors, all of which act in the same way on the vertical branch uv of the movable conductor. Owing to the opposite directions of the currents in the tube and the wire inside it, there is no action on yz due to that part of the apparatus. It is clear, therefore, that the action of C on uv will prevail and determine the motion.

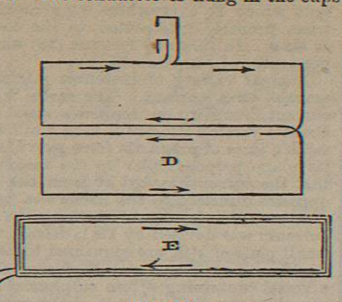


Fig. 36.

The action of straight conductors, making an angle with each other, may be shown by means of the conductor D, represented in fig. 36, which may be fitted to the stand shown in fig. 35.

Circular circuit. In a very large class of practical cases, circular circuits play an important part. The most convenient way of dealing with these, as a rule, is to replace them by the equivalent magnets or magnetic shells. The action of a circular circuit may be represented by two layers of north and south magnetism, whose surface densities are $\pm i \div \tau$, where i is the strength of the current and τ the distance between the layers. For details concerning the calculations in a variety of cases, we refer the reader to Maxwell's *Electricity and Magnetism*, vol. ii. cap. xiv.

We may calculate the force exerted (see fig. 37) by a circular current AB on a unit north pole at its centre C, as follows. Replace the current by two discs AB and A′B′, of north and south magnetism, the distance between which is τ; the surface densities are $+i\div\tau$ and $-i\div\tau$. The first of these exerts a repulsive force $2\pi i\div\tau$, the second an attractive force

$$2\pi i\div\tau(1 - \cos\tfrac{1}{2}A'C'B')\,;$$

hence the resultant repulsive force is

$$2\pi i\cos\tfrac{1}{2}A'C'B'\div\tau = 2\pi i\div r\,,$$

r being the radius of the disc. Hence a unit of length of the current exerts a force $i\div r^2$ at the distance r.

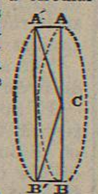


Fig. 37.

Unit of current strength. It follows therefore that the statement of our fundamental principle (p. 67) involves a unit of current strength such that unit length of the unit current, formed into an arc whose radius is the unit of length, exerts a unit of force on a unit pole placed at the centre of the arc. From this statement and the definition of a unit negative pole it follows at once that the dimension of the unit of current is $[L^{\frac{1}{2}}M^{\frac{1}{2}}T^{-1}]$.

Solenoid. One arrangement of circular currents has become famous from the part it plays in Ampère's theory of magnetism. A wire wound into a cylindrical helix, such as that represented in figure 38, the ends of the wire being returned parallel to the axis of the helix, and bent into pivots, so that it can be hung upon Ampère's stand (fig. 35), is called a solenoid. The conductor thus formed is obviously equivalent to a series of circular currents disposed in a uniform manner perpendicular to a common axis. In the case represented in figure 38, this axis is straight; but the name solenoid is not restricted to this particular case,

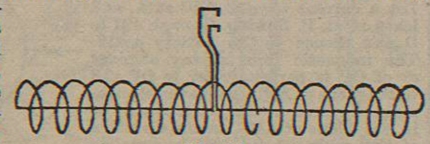


Fig. 38.

[1] Aluminium is often used.

and what we are about to advance will apply to a solenoid whose axis is a curve of any form.

Let there be nds of the circular currents (each of area λ) in the arc ds of the axis of the solenoid. As we suppose the distribution to be uniform, n is constant. We may suppose each current to be placed at the middle of a length $\frac{1}{n}$ of the axis, which it occupies for itself. Hence, if each circular current be replaced by a shell of thickness $\frac{1}{n}$, the surface densities of the magnetism on each of these shells will be $\pm ni$, and the north magnetism of each shell will coincide with the south magnetism of the next; so that the whole action *at points external to the solenoid* reduces to the action of a quantity $ni\lambda$ of magnetism spread over one end of the solenoid, and a quantity $-ni\lambda$ spread over the other. The positive or north end of the solenoid is obtained, as usual, from the direction of the current, by means of the right-handed screw relation. If λ be very small, or if the system acting on, or acted upon by, the solenoid is at a distance very great compared with the dimensions of λ, then we may suppose the representative magnetism concentrated at the ends of the axis of the solenoid.

Hence the particular arrangement of electric currents, which we have called a solenoid, acts and is acted on exactly like an ideal linear magnet (whose poles coincide with the ends of its axis).

Thus the north pole of a magnet or solenoid repels the north end and attracts the south end of a solenoid; a solenoid tends to set under the action of the earth, its north end behaving like a magnetic north pole, and so on.

Cylindrical bobbin. In a cylindrical bobbin wound to a uniform depth with silk-covered wire we have an arrangement which is equivalent to a number of solenoids all having a common axis. Each of these solenoids may be replaced by the equivalent terminal discs of positive and negative magnetism, and the external action of the whole thus calculated. The magnetic disc at each end will, of course, not be of uniform density,[2] but if the points acted on be at a distance which is infinitely great compared with the lateral dimensions of the bobbin, we may collect the magnetism at the ends of the axis; the quantities will be

$$\pm mni\frac{\pi}{3}(a^2 + ab + b^2)\,,$$

where a and b are the outer and inner radii of the shell of wire, m the number of layers in the depth, and n the number of turns per unit of length of each layer. The magnetic moment of the bobbin is therefore

$$mpi\frac{\pi}{3}(a^2 + ab + b^2)\,,$$

where p denotes the number of turns in each layer, and mp the whole number of turns on the bobbin.

The above is a simple case of the kind of calculation on which Weber founded his verification of Ampère's theory. He did not, however, replace the circular currents by the equivalent magnetic distributions, but calculated directly from Ampère's formula (18).

The instrument (electrodynamometer) which he used in his experiments was invented by himself. It consists essentially of a fixed coil and a movable coil, usually suspended in the bifilar manner, and furnished with a mirror, so that its motions about a vertical axis can be read off in the *subjective* manner (see art. GALVANOMETER) by means of a scale and telescope. Two varieties of the instrument were used by Weber. In one of these (A), the movable coil was suspended within the fixed coil; in the other (B), the movable coil was ring-shaped, and embraced the fixed coil, which, however, was so supported that it could be arranged either inside the movable coil or outside it at any distance and in any relative position with respect

[2] The reader will easily find the law for himself.

to it. We do not propose to go into detail respecting Weber's experiments, but merely to indicate their general character and give some of the results. Those desiring further information will find it in §§ 1–9 of the *Electrodynamische Maasbestimmungen.*

Weber first showed that the electrodynamic action between two parts of a piece of apparatus traversed by the same current varies as the square of the current. Apparatus A was arranged with the plane of its fixed coil in the magnetic meridian. The movable coil was concentric with the fixed one, but its plane was perpendicular to the magnetic meridian. The current of 1, 2, or 3 Grove's cells was sent through the fixed coil and through the suspended coil; but as the deflection with this arrangement was too great, the latter was shunted by connecting its terminals by a wire of small but known resistance. A measurement of the *first power* of the strength of the current was found by observing the deflection produced by the current in the fixed coil on a magnet suspended in its plane at a convenient distance north of it. After the necessary corrections were applied, the following results were obtained :—

n	D	M	M′	Diff.
3	440·038	108·426	108·144	−0·282
2	198·255	72·398	72·589	+0·191
1	50·915	36·332	36·786	+0·454

where n is the number of cells, D the electrodynamic force on the suspended coil, expressed in an arbitrary unit, M the force on the magnet, M′ the force on the magnet calculated from $\sqrt{D}$ by means of a constant multiplier. The agreement between M and M is within the limits of experimental error.

In another series of experiments Weber used the apparatus B described above. The suspended coil was arranged with its axis in the magnetic meridian, and the fixed coil set up with its axis perpendicular to the magnetic meridian. Experiments were made with the centres of the two coils coincident, and with the centres in the same horizontal plane, at distances of 300, 400, 500, and 600 millimetres, the fixed coil being, in one set of experiments, east or west from the suspended coil; in another set, north or south. In the present series of experiments the strength of the current was measured by means of a magnet acted on, not by the fixed coil, but by another coil in circuit with it. After proper corrections, the following results were arrived at :—

d	P	P′	Q	Q′
0	22960	22680	22960	22680
300	189·93	189·03	77·11	77·17
400	77·45	77·79	34·77	34·74
500	39·27	39·37	18·24	18·31
600	22·46	22·64	...	...

where d is the distance between the centres of the coils, P the couple[1] exerted on the movable coil when the direction of that distance is perpendicular to the meridian, Q the couple when it is in the meridian. P′ and Q′ are the values of the same couples calculated from the theory of Ampère. The agreement here again is as near as could be expected.

Weber further showed that the deflections (v, w) of the suspended coil, calculated by means of the formulæ

$$\tan v = \alpha d^{-3} + \beta d^{-5}$$
$$\tan w = \tfrac{1}{2}\alpha d^{-3} + \gamma d^{-5}$$

in the two cases where the centres of the coils were at a considerable distance apart, agreed with observation within the limits of experimental error. Now these formula are identical with those established by Gauss for two magnets with their axes placed like the axes of the coils. This agreement therefore is an experimental proof that the coils are replaceable by magnets.

On the whole, therefore, the experiments of Weber[2] confirm the theory of Ampère, as far as experiment can test it. They form, therefore, a sufficient basis for the proposition on which we founded our theory; for this proposition leads to the same result for closed circuits as the theory of Ampère.

The action of any current on a magnetic pole, and hence on any magnet, may be calculated either by replacing the circuit by an equivalent shell or by means of formula (19). We have already found this action in the particular case of an infinitely long straight current. This result was originally found experimentally by Biot and Savart, and Laplace showed that it followed from their result that the force exerted by an element of the current varies inversely as the square of the distance. The fact that a circular current acts on a magnetic pole at its centre in the same way as a zig-zag current which is everywhere very nearly coincident with it, leads, when properly interpreted, to the conclusion that the force varies as $\sin\theta$. In this way formula (16) was originally arrived at, independently of Ampère's theory.

A great variety of instances might be given of the action of a magnet on a current. The earth, for instance, acts on a circular current, hung up on Ampère's stand: the current, being movable about a vertical axis, will turn until the maximum number of the earth's lines of magnetic force pass through it—*i.e.*, it will set with its plane perpendicular to the magnetic meridian, in such a way that the current, looked at from the north side, goes round in the opposite direction to the hands of a watch.

A very simple way of showing the action between magnets and currents was devised by De la Rive. A small plate of copper and a small plate of zinc are connected together by a wire passing through a cork and making a circuit of several turns; the cork is placed in a vessel containing dilute sulphuric acid, and floats on the surface, carrying the little circuit about with it. Such a circuit will set under the earth's action, and may be chased and turned about, &c., by a magnet. After what has been already said, however, such experiments offer no new point of interest.

Electromagnetic Rotations.—It is obvious that no invariable system of electric currents can produce continuous rotation of a magnetized body. For, suppose an elementary magnet, whose action may be represented by two poles of strengths $\pm m$, to describe any path and to return exactly to its former position; either it has or has not embraced the circuit in its path; if it has not, no work has been done on either pole; if it has embraced the circuit n times, an amount of work $4nm\pi i$ has been done on the north pole, and an amount $-4nm\pi i$ on the south; on the whole, therefore, no work has been done on the magnet. As any magnetized body may be conceived to be made up of such elementary magnets, it is obvious that it is impossible for such a body to rotate continuously, doing work against friction,[3] &c. On rotations in general.

The same is obviously true if we replace the magnet by an invariable system of electric currents.

If, however, part of the electric circuit is movable with respect to the rest, and communicates therewith by means of sliding contacts or the like, continuous rotation of part of the circuit may occur. Again, if by any artifice the magnet can be transferred every revolution from one side of the current to the other, continuous rotation of the magnet may result. Lastly, if the direction of the current in some part of the apparatus be always reversed at a certain stage of the revolution, continuous motion may ensue.

Rotations of the first and second class were first discovered by Faraday, and the ground principle of most of the pieces of apparatus used in demonstrating them is that originally used by him. Faraday's apparatus.

One of the simplest cases is the rotation under the action of the vertical component of the earth's magnetic force. Let ABC (fig. 39) be a horizontal circular conductor, OP a conductor pivoted at O, having sliding contact at P with ABC. Let a current i enter ABC at A, and leave it at P, flowing through PO to O and thence to the battery again. The magnetic force at any element dr of OP is perpendicular to OP and to the plane of ABC, hence the electromagnetic force on the element will be in the plane of ABC, in the direction of the arrow p,[4] and will be equal to $iR\,dr$ (R = vertical component of earth's force). Hence the moment about O of the forces acting on OP is $\int iRr\,dr$, *i.e.* $\frac{1}{2}OP^2Ri$, which is independent of the position of OP. OP will therefore rotate about O, with an angular velocity which will go on increasing until the work lost by friction, &c., during each revolution is equal to $\pi OP^2 Ri$.

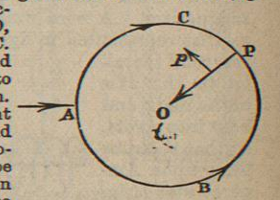


Fig. 39.

[1] Reduced to a standard current strength by means of the magnet deflections.

[2] For another verification by Cazin. see Wiedemann, *Galv.*, Bd. ii. § 44.

[3] Maxwell, vol. i., §§ 486 and 491

[4] We are here supposed to be in *southern* latitudes.

Ampère has given a general theory of the rotation of a circuit under the action of a magnet. Let $\overrightarrow{AB}$ (fig. 40) be any circuit, which we may suppose connected with the axis of the magnet, but free to rotate about it. We suppose the magnet replaced by quantities $\pm m$ of magnetism at its poles. Take the axis of the magnet for axis of z, and the other axes as in the figure, O being the centre of the magnet, and let $ON = OS = c$. Let PQ be any arc ds of AB, and let the coordinates of P be x, y, z; then if l, m, n be the direction cosines of NP, and $NP = D$, we have $Dl = x$, $Dm = y$, $Dn = z - c$; also the direction cosines of Pp, which is perpendicular to NP and PQ, and is the direction of the force exerted by the pole N on P, are $\left(n\frac{dy}{ds} - m\frac{dz}{ds}\right) \div \sin QPK$, &c.

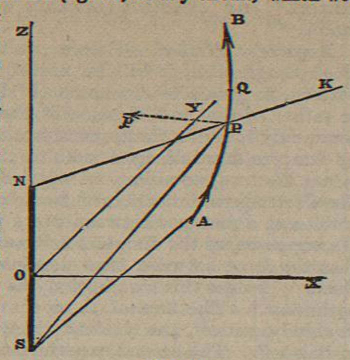


Fig. 40.

Hence by formula (6) the components of the force acting on PQ are

$$\frac{m}{D^2}\left(n\frac{dy}{ds} - m\frac{dz}{ds}\right)i\,ds, \text{ \&c.}$$

Hence, if K denote the moment of these forces about OZ, we have from the north pole alone

$$K = mi\int\frac{ds}{D^2}\left\{\left(l\frac{dz}{ds} - n\frac{dx}{ds}\right)x - \left(n\frac{dy}{ds} - m\frac{dz}{ds}\right)y\right\}.$$

If we substitute the values of l, m, n this reduces to

$$K = mi\int ds\frac{d}{ds}\left(\frac{z-c}{D}\right) = mi\int dn.$$

If therefore $\beta_1, \alpha_1, \beta_2, \alpha_2$ denote the angles BNZ, ANZ, BSZ, ASZ, we have, adding the results from both poles,

$$K = mi\,(\cos\beta_1 - \cos\alpha_1 - \cos\beta_2 + \cos\alpha_2) \quad . \quad . \quad (21).$$

It follows from this remarkable formula that the couple K tending to turn a part AB of an electric circuit about the axis of a magnet depends merely on the position of the ends A and B.

In particular, if A coincide with B, *i.e.* if AB form a closed circuit, or if A and B both lie on parts of the axis not included between N and S,[1] the couple will be *nil*, and there will be no rotation.

The application of this formula to cases where there are sliding contacts at A and B not lying on the axis presents no difficulty; we leave it to the reader.

Rotation apparatus. Several of these rotations may be exhibited by means of the apparatus represented in figure 41. ABC is a horizontal coil of wire terminating at the binding screws a, b. FG is a ring-shaped trough of mercury for the sliding contacts. A wire connects the mercury with the binding screw d. DE is an upright support screwed into a metal base D in connection with the binding screw c, and terminating above in a mercury cup E. When required, DE can be replaced by the shorter supports D′E′ and D″E″. HLK is a support for a screw L, which carries an adjustible centre.

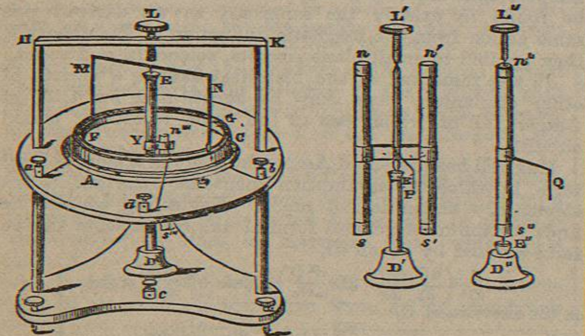


Fig. 41.

1. Poise in the cup E the wire stirrup MN, so that the ends just dip in the mercury trough. Then, if a strong current be sent from c to d, MN will rotate (in northern latitudes) in a direction opposite to the hands of a watch.

2. If we fix a vertical magnet $n'''s'''$ to DE by means of a clip at Y, then the rotation will take place with a weaker current in the same direction as before, if the north pole of the magnet be upwards (as shown in figure), but in the opposite direction if the magnet be reversed.

3. Reversing the current alone in either of the last two cases causes the direction of rotation to be reversed.

4. The magnet may be removed and a current sent from a to b round ABC in the direction opposite to the hands of a watch. The result is the same as for the magnet with its north pole upwards. If the current in ABC is reversed, the rotation is reversed; and so on.

5. The support D′E′ with the two magnets ns, $n's'$ may be screwed into D instead of DE, the wire P now dipping into the mercury. If the current be sent from c to d, the vertical current in D′E′ will act on s and s', and cause the magnet to rotate in the direction of the hands of a watch. This rotation is reversed if the current go from d to c.

6. We may consider any magnet of finite size as made up of a series of magnets like ns and $n's'$ arranged about an axis. Hence, if we replace D′E′ and the magnets D″E″ by the single magnet supported by means of the pivot L″, there will still be rotation.

Figure 42 represents a very elegant piece of apparatus devised by Faraday, to show the rotation at once of a magnet and of a movable conductor. The rotating pieces are the magnet sn, which is tied to the copper peg at the bottom of G by means of a piece of string, and swims round the vertical current buoyed up by the mercury in G, and the wire DE, which is hinged to D by a thin flexible wire, and swims round the pole of the vertical magnet $n's'$.

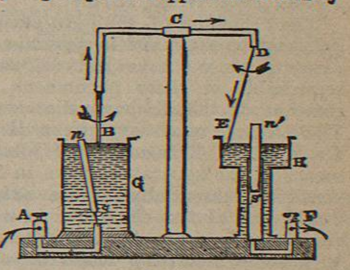


Fig. 42.

Another apparatus invented by Barlow, and known by the name of Barlow's wheel, is represented in figure 43. A current is caused to pass from the mercury trough C along the radius of the disc A through the field of magnetic force due to the horse-shoe magnet NO. The result is that the wheel rotates in the direction indicated by the arrow.

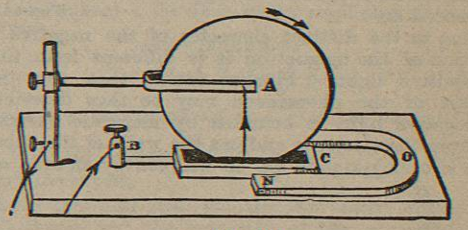


Fig. 43.

Fluid conductors may also be caused to rotate under the action of a magnet. We mentioned in our historical sketch the experiment by which Davy demonstrated this rotation in the case of mercury. A variety of such experiments have been since devised. The following is a simple one. Fill a small cylindrical copper vessel with dilute sulphuric acid and set it upon the north pole of a powerful electromagnet. If a thick zinc wire be connected by a piece of copper wire to the copper vessel, and then immersed in the acid so as to be in the axis of the vessel, a current is set up in the liquid which flows radially from the zinc to the copper across the lines of force. The

[1] We might consider what would happen if A or B lay on NS, but the case never arises in practice, for all magnets have a finite thickness (see on this subject Wiedemann, Bd. ii. § 119).