"Philips, whose touch harmonious could remove
The pangs of guilty power or hapless love;
Rest here, distressed by poverty no more,
Here find that calm thou gav'st so oft before;
Sleep undisturbed within this peaceful shrine
Till angels wake thee with a note like thine!"

In classifying epitaphs various principles of division may be adopted. Arranged according to nationality they indicate distinctions of race less clearly perhaps than any other form of literature does,—and this obviously because when under the influence of the deepest feeling men think and speak very much in the same way whatever be their country. At the same time the influence of nationality may to some extent be traced in epitaphs. The characteristics of the French style, its grace, clearness, wit, and epigrammatic point, are all recognizable in French epitaphs. Instances such as "*La première au rendezvous*," inscribed on the grave of a mother, Piron's epitaph written for himself after his rejection by the French Academy—

"Ci-gît Piron, qui ne fut jamais rien,
Pas même Académicien—

and one by a relieved husband, to be seen at Père la Chaise—

"Ci-gît ma femme. Ah! qu'elle est bien
Pour son repos et pour le mien—"

might be multiplied indefinitely. One can hardly look through a collection of English epitaphs without being struck with the fact that these represent a greater variety of intellectual and emotional states than those of any other nation, ranging through every style of thought from the sublime to the commonplace, every mood of feeling from the most delicate and touching to the coarse and even brutal. Few subordinate illustrations of the wonderfully complex nature of the English nationality are more striking than this.

Epitaphs are sometimes classified according to their authorship and sometimes according to their subject, but neither division is so interesting as that which arranges them according to their characteristic features. What has just been said of English epitaphs is, of course, more true of epitaphs generally. They exemplify every variety of sentiment and taste, from lofty pathos and dignified eulogy to coarse buffoonery and the vilest scurrility. The extent to which the humorous and even the low comic element prevails among them is a noteworthy circumstance. It is curious that the most solemn of all subjects should have been frequently treated, intentionally or unintentionally, in a style so ludicrous that a collection of epitaphs is generally one of the most amusing books that can be picked up. In this as in other cases too it is to be observed that the unintended humour is generally of a much more entertaining kind than that which has been deliberately perpetrated. It would be out of place to give here any specimens of a class of epitaphs which—just because they are the most amusing—are the most abundantly represented in all the ordinary collections.

See Weever, *Ancient Funerall Monuments*, 1631, 1661 (Tooke's edit., 1767); Philip Labbe, *Thesaurus epitaphiorum*, Paris, 1666; *Theatrum Funebre extructum a* Dodone Richea seu Ottone Aicher, 1675; Hackett, *Select and Remarkable Epitaphs*, 1757; De la Place, *Recueil d'épitaphes*, 3 vols., Paris, 1782; Pulleyn, *Churchyard Gleanings*, c. 1830; L. Lewysohn, *Sechzig Epitaphien von Grabsteinen d. israelit. Friedhofes zu Worms*, 1855; Pettigrew, *Chronicles of the Tombs*, 1857; S. Tissington, *Epitaphs*, 1857; Robinson, *Epitaphs from Cemeteries in London, Edinburgh, &c.*, 1859; Le Blant, *Inscriptions Chrétiennes de la Gaule antérieures au VIII[e] siècle*, 1856, 1865; Blommaert, Gaillard, &c., *Inscriptions funeraires et monumentales de la prov. de Flandre Orient.* Ghent, 1857, 1860; *Inscriptions fun. et mon. de la prov. d'Anvers*, Antwerp, 1857-1860; Chwolson, *Achtzehn Hebraische Grabschriften aus der Krim*, 1859; J. Brown, *Epitaphs, &c., in Greyfriars Churchyard, Edinburgh*, 1867; H. J. Loaring, *Quaint, Curious, and Elegant Epitaphs*, 1872; Cansick, *Epitaphs in Cemeteries and Churches of St Pancras*, 1872; Northend, *Book of Epitaphs*, New York, 1873; J. R. Kippax, *Churchyard Literature: Choice coll. of American Epitaphs*, 1876. (W. B. S.)

EPITHALAMIUM (from ἐπί, and θάλαμος, a nuptial chamber), originally among the Greeks a song which was sung by a number of boys and girls at the door of the nuptial chamber. According to the scholiast on Theocritus, one form the κατακοιμητικόν, was employed at night, and another, the διεγερτικόν, to amuse the bride and bridegroom on the following morning. In either case, as was natural, the main burden of the song consisted of invocations of blessing and predictions of happiness, interrupted from time to time by the ancient chorus of *Hymen hymenæe.* Among the Romans, a similar custom was in vogue, but the song was sung by girls only, after the marriage guests had gone, and it contained much more of what modern morality would condemn as obscene. In the hands of the poets the epithalamium was developed into a special literary form, and received considerable cultivation. Sappho, Anacreon, Stesichorus, and Pindar are all regarded as masters of the species, but the finest example preserved in Greek literature is the 18th Idyll of Theocritus, which celebrates the marriage of Menelaus and Helena. Catullus, Statius, Ausonius, Sidonius Apollinaris, and Claudian are the authors of the best known epithalamia in classical Latin; and they have been imitated by Buchanan, Scaliger, Sannazarius, and a whole host of modern Latin poets, with whom, indeed, the form was at one time in great favour. The names of Ronsard, Malherbe, and Scarron are especially associated with the species in French literature, and Marini and Metastasio in Italian. Perhaps no poem of the class has been more universally admired than the epithalamium of Spenser, though he has found no unworthy rivals in Ben Jonson and one or two of his successors.

EPSOM, a market town in the county of Surrey, is situated about 14 miles S.W. of London, on a branch of the London and Brighton railway. The town is irregularly built, but contains some handsome new houses. The principal building is the parish church, a Gothic edifice, rebuilt in 1825, the interior of which contains some fine sculptures by Flaxman and Chantrey. Epsom has attained a wide celebrity on account of its mineral springs and its races. The former were discovered about 1618, and for some time after their discovery, the town enjoyed a wonderful degree of prosperity. After the Restoration, it was often visited by Charles II., and when Queen Anne came to the throne, her husband, Prince George of Denmark, made it his frequent resort. Epsom gradually lost its celebrity as a spa, but the annual races held on its downs have arrested the decay of the town. Races appear to have been established here as early as James I.'s residence at Nonsuch, but they did not assume a permanent character until 1730. The principal races—the Derby and Oaks—are named after one of the earls of Derby, and his seat, the Oaks, which is in the neighbourhood. The latter race was established in 1779, and the former in the following year. The spring races are held on a Thursday and Friday towards the close of April; and the great Epsom meeting takes place on the Tuesday and three following days immediately before Whitsuntide,—the Derby on the Wednesday and the Oaks on the Friday. The grand stand, erected in 1829, is 156 feet wide and 70 feet in depth, consists of three stories, accommodating nearly 5000 spectators, and includes a saloon 108 feet by 34. The population of the civil parish, in 1871 was 6276.

EPSOM SALTS, the *magnesiæ sulphas* of pharmacy, and the epsomite or hair-salt of mineralogical treatises, is an hydrated magnesium sulphate, of the chemical constitution $MgSO_4.7H_2O$, and isomorphous with zinc sulphate (see vol. vi. p. 527), which it resembles in appearance. The salt crystallizes in four-sided, right-rhombic, lustrous, colourless prisms, which in the commercial article are

usually acicular in shape. It can be obtained also in crystals of the monoclinic system. It is very soluble, one part dissolving in 0·79 parts of water at 18·75°C., and has a bitter, saline, and cooling taste. The salt is prepared on the large scale by several methods, *e.g.*, by the treatment of the bittern of salt works with sulphuric acid or ferrous sulphate, by which the magnesium chloride of the liquid is converted into sulphate; by acting on magnesite, the native magnesium carbonate, or on magnesian limestone, with sulphuric acid, preferably, in the case of the latter substance, after the removal of the calcium carbonate by means of hydrochloric acid; and, as in the neighbourhood of Genoa, by the roasting of pyritous serpentine, subsequent exposure to the air and lixiviation, peroxidization of ferrous salts by chlorine, precipitation of ferric oxide by burnt lime or dolomite, and evaporation of of the resultant solution of magnesium sulphate. The mineral waters of Seidlitz, Saidschütz, Püllna, and of other places besides Epsom owe their potency to magnesium sulphate. The salt occurs in fibrous crusts or botryoidal masses in some limestone caves; in gypsum quarries, as a result of the action of the gypsum on magnesian limestone; and in the old workings of mines, where it is produced by the oxidation of pyrites in the presence of magnesium compounds. As a hydragogue purgative, it is in common use; it is more especially valuable in febrile diseases, in congestion of the portal system, and in the obstinate constipation of painter's colic. To produce diuresis, the drug is far less frequently resorted to. It possesses the advantage of exercising but little irritant effect upon the bowels. In some cases, where full doses have failed, the repeated administration of small quantities has been found effectual. The chief application of Epsom salts or "Epsoms" is for weighting cotton-cloth. As a manure, magnesium sulphate has been chiefly employed as a top-dressing for clover-hay. The chlorides of magnesium and sodium and salts of iron and of calcium may occur as impurities in Epsom salts.

EQUATION. The present article includes DETERMINANT and THEORY OF EQUATIONS; and it may be proper to explain the relation to each other of the two subjects. Theory of Equations is used in its ordinary conventional sense to denote the theory of a single equation of any order in one unknown quantity; that is, it does not include the theory of a system or systems of equations of any order between any number of unknown quantities. Such systems occur very frequently in analytical geometry and other parts of mathematics, but they are hardly as yet the subject-matter of a distinct theory; and even Elimination, the transition-process for passing from a system of any number of equations involving the same number of unknown quantities to a single equation in one unknown quantity, hardly belongs to the Theory of Equations in the above restricted sense. But there is one case of a system of equations which precedes the Theory of Equations, and indeed presents itself at the outset of algebra, that of a system of simple (or linear) equations. Such a system gives rise to the function called a Determinant, and it is by means of these functions that the solution of the equations is effected. We have thus the subject Determinant as nearly equivalent to (but somewhat more extensive than) that of a system of linear equations; and we have the other subject, Theory of Equations, used in the restricted sense above referred to, and as not including Elimination.

DETERMINANT.

1. A sketch of the history of determinants is given under ALGEBRA; it thereby appears that the algebraical function called a determinant presents itself in the solution of a system of simple equations, and we have herein a natural source of the theory. Thus, considering the equations

$$\begin{aligned} ax + by + cz &= d, \\ a'x + b'y + c'z &= d', \\ a''x + b''y + c''z &= d''. \end{aligned}$$

and proceeding to solve them by the so-called method of cross multiplication, we multiply the equations by factors selected in such a manner that upon adding the results the whole coefficient of y becomes $=0$, and the whole coefficient of z becomes $=0$; the factors in question are $b'c''-b''c'$, $b''c-bc''$, $bc'-b'c$ (values which, as at once seen, have the desired property); we thus obtain an equation which contains on the left-hand side only a multiple of x, and on the right-hand side a constant term; the coefficient of x has the value

$$a(b'c''-b''c')+a'(b''c-bc'')+a''(bc'-b'c),$$

and this function, represented in the form

$$\begin{vmatrix} a, & b, & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{vmatrix},$$

is said to be a determinant; or, the number of elements being 3^2, it is called a determinant of the third order. It is to be noticed that the resulting equation is

$$\begin{vmatrix} a, & b, & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{vmatrix} x = \begin{vmatrix} d, & b, & c \\ d', & b', & c' \\ d'', & b'', & c'' \end{vmatrix}$$

where the expression on the right-hand side is the like function with d, d', d'' in place of a, a', a'' respectively, and is of course also a determinant. Moreover, the functions $b'c''-b''c'$, $b''c-bc''$, $bc'-b'c$ used in the process are themselves the determinants of the second order

$$\begin{vmatrix} b', & c' \\ b'', & c'' \end{vmatrix}, \quad \begin{vmatrix} b'', & c'' \\ b, & c \end{vmatrix}, \quad \begin{vmatrix} b, & c \\ b', & c' \end{vmatrix}.$$

We have herein the suggestion of the rule for the derivation of the determinants of the orders 1, 2, 3, 4, &c., each from the preceding one, viz., we have

$$|a| = a,$$

$$\begin{vmatrix} a, & b \\ a', & b' \end{vmatrix} = a|b'| - a'|b|,$$

$$\begin{vmatrix} a, & b, & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{vmatrix} = a\begin{vmatrix} b', & c' \\ b'', & c'' \end{vmatrix} + a'\begin{vmatrix} b'', & c'' \\ b, & c \end{vmatrix} + a''\begin{vmatrix} b, & c \\ b', & c' \end{vmatrix},$$

$$\begin{vmatrix} a, & b, & c, & d \\ a', & b', & c', & d' \\ a'', & b'', & c'', & d'' \\ a''', & b''', & c''', & d''' \end{vmatrix} = a\begin{vmatrix} b', & c', & d' \\ b'', & c'', & d'' \\ b''', & c''', & d''' \end{vmatrix} - a'\begin{vmatrix} b'', & c'', & d'' \\ b''', & c''', & d''' \\ b, & c, & d \end{vmatrix} + a''\begin{vmatrix} b''', & c''', & d''' \\ b, & c, & d \\ b', & c', & d' \end{vmatrix} - a'''\begin{vmatrix} b, & c, & d \\ b', & c', & d' \\ b'', & c'', & d'' \end{vmatrix},$$

and so on, the terms being all + for a determinant of an odd order, but alternately + and − for a determinant of an even order.

2. It is easy, by induction, to arrive at the general results:—

A determinant of the order n is the sum of the $1.2.3...n$ products which can be formed with n elements out of n^2 elements arranged in the form of a square, no two of the n elements being in the same line or in the same column, and each such product having the coefficient $\pm$ unity.

The products in question may be obtained by permuting in every possible manner the columns (or the lines) of the determinant, and then taking for the factors the n elements in the dexter diagonal. And we thence derive the rule for the signs, viz., considering the primitive arrangement of the columns as positive, then an arrangement obtained therefrom by a single interchange (inversion, or derangement) of two columns is regarded as negative; and so in general an arrangement is positive or negative according as it is derived from the primitive arrangement by an even or an odd number of interchanges. [This implies the theorem that a given arrangement can be derived from the primitive arrangement only by an odd number, or else only by an even number of interchanges,—a theorem the verification of which may be easily obtained from the theorem (in fact a particular case of the general one), an arrangement can be derived from itself only by an even number of interchanges.] And this being so, each product has the sign belonging to the corresponding arrangement of the columns; in particular, a determinant contains with the sign + the product of the elements in its dexter diagonal. It is to be observed that the rule gives as many positive as negative arrangements, the number of each being $=\frac{1}{2}1.2\ldots n$.

The rule of signs may be expressed in a different form. Giving to the columns in the primitive arrangement the numbers 1, 2, 3...n, to obtain the sign belonging to any other arrangement we take, as often as a lower number succeeds a higher one, the sign −, and, compounding together all these minus signs, obtain the proper sign, + or − as the case may be.

Thus, for three columns, it appears by either rule that 123, 231, 312 are positive; 213, 321, 213 are negative; and the developed expression of the foregoing determinant of the third order is

$$=ab'c''-ab''c'+a'b''c-a'bc''+a''bc'-a''b'c.$$

3. It further appears that a determinant is a linear function[1] of the elements of each column thereof, and also a linear function of the elements of each line thereof; moreover, that the determinant retains the same value, only its sign being altered, when any two columns are interchanged, or when any two lines are interchanged; more generally, when the columns are permuted in any manner, or when the lines are permuted in any manner, the determinant retains its original value, with the sign + or − according as the new arrangement (considered as derived from the primitive arrangement) is positive or negative according to the foregoing rule of signs. It at once follows that, if two columns are identical, or if two lines are identical, the value of the determinant is $=0$. It may be added, that if the lines are converted into columns, and the columns into lines, in such a way as to leave the dexter diagonal unaltered, the value of the determinant is unaltered: the determinant is in this case said to be *transposed*.

4. By what precedes it appears that there exists a function of the n^2 elements, linear as regards the terms of each column (or say, for shortness, linear as to each column), and such that only the sign is altered when any two columns are interchanged; these properties completely determine the function, except as to a common factor which may multiply all the terms. If, to get rid of this arbitrary common factor, we assume that the product of the elements in the dexter diagonal has the coefficient $+1$, we have a complete definition of the determinant, and it is interesting to show how from these properties, assumed for the definition of the determinant, it at once appears that the determinant is a function serving for the solution of a system of linear equations. Observe that the properties show at once that if any column is $=0$ (that is, if the elements in the column are each $=0$), then the determinant is $=0$; and further, that if any two columns are identical, then the determinant is $=0$.

5. Reverting to the system of linear equations written down at the beginning of this article, consider the determinant

$$\begin{vmatrix} ax+by+cz-d, & b, & c \\ a'x+b'y+c'z-d', & b', & c' \\ a''x+b''y+c''z-d'', & b'', & c'' \end{vmatrix};$$

it appears that this is

$$-x\begin{vmatrix} a, & b, & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{vmatrix}+y\begin{vmatrix} b, & b, & c \\ b', & b', & c' \\ b'', & b'', & c'' \end{vmatrix}+z\begin{vmatrix} c, & b, & c \\ c', & b', & c' \\ c'', & b'', & c'' \end{vmatrix}-\begin{vmatrix} d, & b, & c \\ d', & b', & c' \\ d'', & b'', & c'' \end{vmatrix},$$

viz., the second and third terms each vanishing, it is

$$-x\begin{vmatrix} a, & b, & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{vmatrix}-\begin{vmatrix} d, & b, & c \\ d', & b', & c' \\ d'', & b'', & c'' \end{vmatrix}$$

But if the linear equations hold good, then the first column of the original determinant is $=0$, and therefore the determinant itself is $=0$; that is, the linear equations give

$$x\begin{vmatrix} a, & b, & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{vmatrix}-\begin{vmatrix} d, & b, & c \\ d', & b', & c' \\ d'', & b'', & c'' \end{vmatrix}=0;$$

which is the result obtained above.

We might in a similar way find the values of y and z, but there is a more symmetrical process. Join to the original equations the new equation

$$\alpha x+\beta y+\gamma z=\delta;$$

a like process shows that, the equations being satisfied, we have

$$\begin{vmatrix} \alpha, & \beta, & \gamma, & \delta \\ a, & b, & c, & d \\ a', & b', & c', & d' \\ a'', & b'', & c'', & d'' \end{vmatrix}=0;$$

or, as this may be written,

$$\begin{vmatrix} \alpha, & \beta, & \gamma & \\ a & b & c, & d \\ a' & b' & c', & d' \\ a'' & b'' & c'', & d'' \end{vmatrix}-\delta\begin{vmatrix} a, & b, & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{vmatrix}=0;$$

which, considering δ as standing herein for its value $\alpha x+\beta y+\gamma z$, is a consequence of the original equations only: we have thus an expression for $\alpha x+\beta y+\gamma z$, an arbitrary linear function of the unknown quantities x, y, z; and by comparing the coefficients of α, β, γ on the two sides respectively, we have the values of x, y, z; in fact, these quantities, each multiplied by

$$\begin{vmatrix} a, & b, & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{vmatrix},$$

are in the first instance obtained in the forms

$$\begin{vmatrix} 1 & & & \\ a, & b, & c, & d \\ a', & b', & c', & d' \\ a'', & b'', & c'', & d'' \end{vmatrix}, \begin{vmatrix} & 1 & & \\ a, & b, & c, & d \\ a', & b', & c', & d' \\ a'', & b'', & c'', & d'' \end{vmatrix}, \begin{vmatrix} & & 1 & \\ a, & b, & c, & d \\ a', & b', & c', & d' \\ a'', & b'', & c'', & d'' \end{vmatrix};$$

but these are

$$-\begin{vmatrix} b, & c, & d \\ b', & c', & d' \\ b'', & c'', & d'' \end{vmatrix}, -\begin{vmatrix} c, & d, & a \\ c', & d', & a' \\ c'', & d'', & a'' \end{vmatrix}, \begin{vmatrix} d & a, & b \\ d' & a', & b' \\ d'' & a'', & b'' \end{vmatrix},$$

or, what is the same thing,

$$-\begin{vmatrix} b, & c, & d \\ b', & c', & d' \\ b'', & c'', & d'' \end{vmatrix}, \begin{vmatrix} c, & a, & d \\ c', & a', & d' \\ c'', & a'', & d'' \end{vmatrix}, \begin{vmatrix} a, & b, & d \\ a' & b', & d' \\ a'' & b'', & d'' \end{vmatrix}$$

respectively.

6. *Multiplication of two determinants of the same order.*—The theorem is obtained very easily from the last preceding definition of a determinant. It is most simply expressed thus—

$$\begin{array}{l|ccc|} & \underline{(\alpha,\alpha',\alpha''),} & \underline{(\beta,\beta',\beta''),} & \underline{(\gamma,\gamma',\gamma'')} \\ (a, b, c) & " & " & " \\ (a', b', c') & " & " & " \\ (a'', b'', c'') & " & " & " \end{array} = \begin{vmatrix} a, & b, & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{vmatrix} \cdot \begin{vmatrix} \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \\ \alpha'', & \beta'', & \gamma'' \end{vmatrix},$$

where the expression on the left side stands for a determinant, the terms of the first line being $(a, b, c)(\alpha, \alpha', \alpha'')$, that is, $a\alpha+b\alpha'+c\alpha''$, $(a, b, c)(\beta, \beta', \beta'')$, that is, $a\beta+b\beta'+c\beta''$, $(a, b, c)(\gamma, \gamma', \gamma'')$, that is $a\gamma+b\gamma'+c\gamma''$; and similarly the terms in the second and third lines are the like functions with (a', b', c') and (a'', b'', c'') respectively.

There is an apparently arbitrary transposition of lines and columns; the result would hold good if on the left-hand side we had written (α, β, γ), $(\alpha', \beta' \gamma')$, $(\alpha'', \beta'', \gamma'')$, or what is the same thing, if on the right-hand side we had transposed the second determinant; and either of these changes would, it might be thought, increase the elegance of the form, but, for a reason which need not be explained,[1] the form actually adopted is the preferable one.

To indicate the method of proof, observe that the determinant on the left-hand side, *qua* linear function of its columns, may be broken up into a sum of $(3^3=)\ 27$ determinants, each of which is either of some such form as

$$\pm\alpha\beta\gamma'\begin{vmatrix} a, & a, & b \\ a', & a', & b' \\ a'', & a'', & b'' \end{vmatrix},$$

where the term $\alpha\beta\gamma'$ is not a term of the $\alpha\beta\gamma$-determinant, and its coefficient (as a determinant with two identical columns) vanishes; or else it is of a form such as

$$\pm\alpha\beta'\gamma''\begin{vmatrix} a, & b, & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{vmatrix},$$

that is, every term which does not vanish contains as a factor the abc-determinant last written down; the sum of all other factors $\pm\alpha\beta'\gamma''$ is the $\alpha\beta\gamma$-determinant of the formula; and the final result then is, that the determinant on the left-hand side is equal to the product on the right-hand side of the formula.

7. *Decomposition of a determinant into complementary determinants.*—Consider, for simplicity, a determinant of the fifth order, $5=2+3$, and let the top two lines be

$$\begin{matrix} a, & b, & c, & d, & e \\ a', & b', & c', & d', & e' \end{matrix}$$

then, if we consider how these elements enter into the determinant, it is at once seen that they enter only through the determinants of the second order $\begin{vmatrix} a, & b \\ a', & b' \end{vmatrix}$, &c., which can be formed by selecting any two columns at pleasure. Moreover, representing the remaining three lines by

$$\begin{matrix} a'', & b'', & c'', & d'', & e'' \\ a''', & b''', & c''', & d''', & e''' \\ a'''', & b'''', & c'''', & d'''', & e'''' \end{matrix}$$

it is further seen that the factor which multiplies the determinant formed with any two columns of the first set is the determinant of the third order formed with the complementary three columns of the second set; and it thus appears that the determinant of the fifth order is a sum of all the products of the form

$$\pm\begin{vmatrix} a, & b \\ a', & b' \end{vmatrix}\begin{vmatrix} c'', & d'', & e'' \\ c''', & d''', & e''' \\ c'''', & d'''', & e'''' \end{vmatrix},$$

the sign $\pm$ being in each case such that the sign of the term $\pm ab'.c''d'''e''''$ obtained from the diagonal elements of the component determinants may be the actual sign of this term in the determinant of the fifth order; for the product written down the sign is obviously $+$.

Observe that for a determinant of the n-th order, taking the decomposition to be $1+(n-1)$, we fall back upon the equations given at the commencement, in order to show the genesis of a determinant.

8. Any determinant $\begin{vmatrix} a, & b \\ a', & b' \end{vmatrix}$ formed out of the elements of the original determinant, by selecting the lines and columns at pleasure, is termed a *minor* of the original determinant; and when the number of lines and columns, or order of the determinant, is $n-1$, then such determinant is called a *first minor*; the number of the first minors is $=n^2$, the first minors, in fact, corresponding to the several elements of the determinant—that is, the coefficient therein of any term whatever is the corresponding first minor. The first minors, each divided by the determinant itself, form a system of elements *inverse* to the elements of the determinant.

A determinant is *symmetrical* when every two elements symmetrically situated in regard to the dexter diagonal are equal to each other; if they are equal and opposite (that is, if the sum of the two elements be $=0$), this relation not extending to the diagonal elements themselves, which remain arbitrary, then the determinant is *skew*; but if the relation does extend to the diagonal terms (that is, if these are each $=0$), then the determinant is *skew symmetrical*; thus the determinants

$$\begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix}; \begin{vmatrix} a, & \nu, & -\mu \\ -\nu, & b, & \lambda \\ \mu, & -\lambda, & c \end{vmatrix}; \begin{vmatrix} 0, & \nu, & -\mu \\ -\nu, & 0, & \lambda \\ \mu, & -\lambda, & 0 \end{vmatrix}$$

are respectively symmetrical, skew, and skew symmetrical.

The theory admits of very extensive algebraic developments, and applications in algebraical geometry and other parts of mathematics; but the fundamental properties of the functions may fairly be considered as included in what precedes.

Theory of Equations.

9. In the subject "Theory of Equations" the term *equation* is used to denote an equation of the form $x^n-p_1x^{n-1}\ldots\pm p_n=0$, where $p_1, p_2\ldots p_n$ are regarded as known, and x as a quantity to be determined; for shortness the equation is written $f(x)=0$.

The equation may be *numerical*; that is, the coefficients $p_1, p_2^n, \ldots p_n$ are then numbers,—understanding by number a quantity of the form $\alpha+\beta i$ (α and β having any positive or negative real values whatever, or say each of these is regarded as susceptible of continuous variation from an indefinitely large negative to an indefinitely large positive value), and i denoting $\sqrt{-1}$.

Or the equation may be *algebraical*; that is, the coefficients are not then restricted to denote, or are not explicitly considered as denoting, numbers.

I. We consider first numerical equations. (Real theory, 10 to 14; Imaginary theory, 15 to 18.)

10. Postponing all consideration of imaginaries, we take in the first instance the coefficients to be real, and attend only to the real roots (if any); that is, $p_1, p_2,\ldots p_n$ are real positive or negative quantities, and a root a, if it exists, is a positive or negative quantity such that $a^n-p_1a^{n-1}\ldots\pm p_n=0$, or say, $f(a)=0$. The fundamental theorems are given under ALGEBRA, sections x., xiii., xiv.; but there are various points in the theory which require further development.

It is very useful to consider the curve $y=f(x)$,—or, what would come to the same, the curve $Ay=f(x)$,—but it is better to retain the first-mentioned form of equation, drawing, if need be, the ordinate y on a reduced scale. For instance, if the given equation be $x^3-6x^2+11x-6{\cdot}06=0$,[2] then the curve $y=x^3-6x^2+11x-6{\cdot}06$ is as shown in the figure at page 501, without any reduction of scale for the ordinate.

It is clear that in general y is a continuous one-valued function of x, finite for every finite value of x, but becoming infinite when x is infinite; *i.e.*, assuming throughout that the coefficient of x^n is $+1$, then when $x=\infty$, $y=+\infty$; but when $x=-\infty$, then $y=+\infty$ or $-\infty$,

[1] The expression, a linear function, is here used in its narrowest sense, a linear function without constant term; what is meant is, that the determinant is in regard to the elements $a, a', a'', \ldots$ of any column or line thereof, a function of the form $Aa+A'a'+A''a''+\ldots$, without any term independent of $a, a', a'', \ldots$

[1] The reason is the connexion with the corresponding theorem for the multiplication of two matrices.

[2] The coefficients were selected so that the roots might be nearly 1, 2, 3.

according as n is even or odd; the curve cuts any line whatever, and in particular it cuts the axis (of x), in at most n points; and the value of x, at any point of intersection with the axis, is a root of the equation $f(x)=0$.

If β, α are any two values of x ($\alpha>\beta$, that is, α nearer $+\infty$), then if $f(\beta)$, $f(\alpha)$ have opposite signs, the curve cuts the axis an odd number of times, and therefore at least once, between the points $x=\beta$, $x=\alpha$; but if $f(\beta)$, $f(\alpha)$ have the same sign, then between these points the curve cuts the axis an even number of times, or it may be not at all. That is, $f(\beta)$, $f(\alpha)$ having opposite signs, there are between the limits β, α an odd number of real roots, and therefore at least one real root; but $f(\beta)$, $f(\alpha)$ having the same sign, there are between these limits an even number of real roots, or it may be there is no real root. In particular, by giving to β, α the values $-\infty$, $+\infty$ (or, what is the same thing, any two values sufficiently near to these values respectively) it appears that an equation of an odd order has always an odd number of real roots, and therefore at least one real root; but that an equation of an even order has an even number of real roots, or it may be no real root.

If α be such that for $x=$ or $>\alpha$ (that is, x nearer to $+\infty$) $f(x)$ is always $+$, and β be such that for $x=$ or $<\beta$ (that is x nearer to $-\infty$) $f(x)$ is always $-$, then the real roots (if any) lie between these limits $x=\beta$, $x=\alpha$; and it is easy to find by trial such two limits including between them all the real roots (if any).

11. Suppose that the positive value δ is an inferior limit to the difference between two real roots of the equation; or rather (since the foregoing expression would imply the existence of real roots) suppose that there are not two real roots such that their difference taken positively is $=$ or $<\delta$; then, γ being any value whatever, there is clearly at most one real root between the limits γ and $\gamma+\delta$; and by what precedes there is such real root or there is not such real root, according as $f(\gamma)$, $f(\gamma+\delta)$ have opposite signs or have the same sign. And by dividing in this manner the interval β to α into intervals each of which is $=$ or $<\delta$, we should not only ascertain the number of the real roots (if any), but we should also *separate* the real roots, that is, find for each of them limits γ, $\gamma+\delta$ between which there lies this one, and only this one, real root.

In particular cases it is frequently possible to ascertain the number of the real roots, and to effect their separation by trial or otherwise, without much difficulty; but the foregoing was the general process as employed by Lagrange even in the second edition (1808) of the *Traité de la résolution des Equations Numériques*;[1] the determination of the limit δ had to be effected by means of the "equation of differences" or equation of the order $\frac{1}{2}n(n-1)$, the roots of which are the squares of the differences of the roots of the given equation, and the process is a cumbrous and unsatisfactory one.

12. The great step was effected by Sturm's theorem (1835)—viz., here starting from the function $f(x)$, and its first derived function $f'(x)$, we have (by a process which is a slight modification of that for obtaining the greatest common measure of these two functions) to form a series of functions

$$f(x), f'(x), f_2(x) \ldots . f_n(x)$$

of the degrees $n, n-1, n-2 \ldots . . 0$ respectively,—the last term $f_n(x)$ being thus an absolute constant. These lead to the immediate determination of the number of real roots (if any) between any two given limits β, α; viz., supposing $\alpha>\beta$ (that is, α nearer to $+\infty$), then substituting successively these two values in the series of functions, and attending only to the signs of the resulting values, the number of the changes of sign lost in passing from β to α is the required number of real roots between the two limits. In particular, taking β, $\alpha=-\infty$, $+\infty$ respectively, the signs of the several functions depend merely on the signs of the terms which contain the highest powers of x, and are seen by inspection, and the theorem thus gives at once the whole number of real roots.

And although theoretically, in order to complete by a finite number of operations the separation of the real roots, we still need to know the value of the before-mentioned limit δ; yet in any given case the separation may be effected by a limited number of repetitions of the process. The practical difficulty is when two or more roots are very near to each other. Suppose, for instance, that the theorem shows that there are two roots between 0 and 10; by giving to x the values 1, 2, 3, ... successively, it might appear that the two roots were between 5 and 6; then again that they were between 5·3 and 5·4, then between 5·34 and 5·35, and so on until we arrive at a separation; say it appears that between 5·346 and 5·347 there is one root, and between 5·348 and 5·349 the other root. But in the case in question δ would have a very small value, such as ·002, and even supposing this value known, the direct application of the first-mentioned process would be still more laborious.

13. Supposing the separation once effected, the determination of the single real root which lies between the two given limits may be effected to any required degree of approximation either by the processes of Horner and Lagrange (which are in principle a carrying out of the method of Sturm's theorem), or by the process of Newton, as perfected by Fourier (which requires to be separately considered).

First as to Horner and Lagrange. We know that between the limits β, α there lies one, and only one, real root of the equation; $f(\beta)$ and $f(\alpha)$ have therefore opposite signs. Suppose any intermediate value is θ; in order to determine by Sturm's theorem whether the root lies between β, θ, or between θ, α, it would be quite unnecessary to calculate the signs of $f(\theta), f'(\theta, f_2(\theta) \ldots$; only the sign of $f(\theta)$ is required; for, if this has the same sign as $f(\beta)$, then the root is between β, θ; if the same sign as $f(\alpha)$, then the root is between θ, α. We want to make θ increase from the inferior limit β, at which $f(\theta)$ has the sign of $f(\beta)$, so long as $f(\theta)$ retains this sign, and then to a value for which it assumes the opposite sign; we have thus two nearer limits of the required root, and the process may be repeated indefinitely.

Horner's method (1819) gives the root as a decimal, figure by figure; thus if the equation be known to have one real root between 0 and 10, it is in effect shown say that 5 is too small (that is, the root is between 5 and 6); next that 5·4 is too small (that is, the root is between 5·4 and 5·5); and so on to any number of decimals. Each figure is obtained, *not* by the successive trial of all the figures which precede it, but (as in the ordinary process of the extraction of a square root, which is in fact Horner's process applied to this particular case) it is given presumptively as the first figure of a quotient; such value may be too large, and then the next inferior integer must be tried instead of it, or it may require to be further diminished. And it is to be remarked that the process not only gives the approximate value α of the root, but (as in the extraction of a square root) it includes the calculation of the function $f(\alpha)$ which should be, and approximately is, $=0$. The arrangement of the calculations is very elegant, and forms an integral part of the actual method. It is to be observed that after a certain number of decimal places have

[1] The third edition (1826) is a reproduction of that of 1808; the first edition has the date 1798, but a large part of the contents is taken from memoirs of 1767–68 and 1770–71.

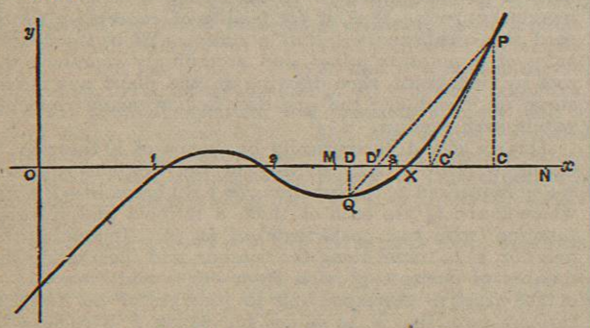
been obtained, a good many more can be found by a mere division. It is in the progress tacitly assumed that the roots have been first separated.

Lagrange's method (1767) gives the root as a continued fraction $a+\frac{1}{b+}\frac{1}{c+}\ldots$, where a is a positive or negative integer (which may be $=0$), but b, c, ... are positive integers. Suppose the roots have been separated; then (by trial if need be of consecutive integer values) the limits may be made to be consecutive integer numbers: say they are a, $a+1$; the value of x is therefore $=a+\frac{1}{y}$, where y is positive and greater than 1; from the given equation for x, writing therein $x=a+\frac{1}{y}$, we form an equation of the same order for y, and this equation will have one, and only one, positive root greater than 1; hence finding for it the limits b, $b+1$ (where b is $=$ or >1), we have $y=b+\frac{1}{z}$, where z is positive and greater than 1; and so on—that is, we thus obtain the successive denominators b, c, d .. of the continued fraction. The method is theoretically very elegant, but the disadvantage is that it gives the result in the form of a continued fraction, which for the most part must ultimately be converted into a decimal. There is one advantage in the method, that a commensurable root (that is, a root equal to a rational fraction) is found accurately, since, when such root exists, the continued fraction terminates.

14. Newton's method (1711), as perfected by Fourier (1831), may be roughly stated as follows. If $x=\gamma$ be an approximate value of any root, and $\gamma+h$ the correct value, then $f(\gamma+h)=0$, that is,

$$f(\gamma)+\frac{h}{1}f'(\gamma)+\frac{h^2}{1.2}f''(\gamma)+\ldots=0;$$

and then, if h be so small that the terms after the second may be neglected, $f(\gamma)+hf'(\gamma)=0$, that is, $h=-\frac{f(\gamma)}{f'(\gamma)}$, or the new approximate value is $x=\gamma-\frac{f(\gamma)}{f'(\gamma)}$; and so on as often as we please. It will be observed that so far nothing has been assumed as to the separation of the roots, or even as to the existence of a real root; γ has been taken as the approximate value of a root, but no precise meaning has been attached to this expression. The question arises, what are the conditions to be satisfied by γ in order that the process may by successive repetitions actually lead to a certain real root of the equation; or say that, γ being an approximate value of a certain real root, the new value $\gamma-\frac{f(\gamma)}{f'(\gamma)}$ may be a more approximate value.



Referring to the figure, it is easy to see that that if OC represent the assumed value γ, then, drawing the ordinate CP to meet the curve in P, and the tangent PC′ to meet the axis in C′, we shall have OC′ as the new approximate value of the root. But observe that there is here a real root OX, and that the curve beyond X is convex to the axis; under these conditions the point C′ is nearer to X than was C; and, starting with C′ instead of C, and proceeding in like manner to draw a new ordinate and tangent, and so on as often as we please, we approximate continually, and that with great rapidity, to the true value OX. But if C had been taken on the other side of X, where the curve is concave to the axis, the new point C′ might or might not be nearer to X than was the point C; and in this case the method, if it succeeds at all, does so by accident only, *i.e.*, it may happen that C′ or some subsequent point comes to be a point C, such that OC is a *proper* approximate value of the root, and then the subsequent approximations proceed in the same manner as if this value had been assumed in the first instance, all the preceding work being wasted. It thus appears that for the proper application of the method we require *more* than the mere separation of the roots. In order to be able to approximate to a certain root α, $=$ OX, we require to know that, between OX and some value ON, the curve is always convex to the axis (analytically, between the two values, $f(x)$ and $f''(x)$ must have always the same sign). When this is so, the point C may be taken anywhere on the proper side of X, and within the portion XN of the axis; and the process is then the one already explained. The approximation is in general a very rapid one. If we know for the required root OX the two limits OM, ON such that from M to X the curve is always *concave* to the axis, while from X to N it is always convex to the axis,—then, taking D anywhere in the portion MX and (as before) C in the portion XN, drawing the ordinates DQ, CP, and joining the points P, Q by a line which meets the axis in D′, also constructing the point C′ by means of the tangent at P as before, we have for the required root the new limits OD′, OC′; and proceeding in like manner with the points D′, C′, and so on as often as we please, we obtain at each step two limits approximating more and more nearly to the required root OX. The process as to the point D′, translated into analysis, is the ordinate process of interpolation. Suppose OD $=\beta$, OC $=\alpha$, we have approximately $f(\beta+h)=f(\beta)+\frac{h\{f(\alpha)-f(\beta)\}}{\alpha-\beta}$, whence if the root is $\beta+h$ then $h=-\frac{(\alpha-\beta)f(\beta)}{f(\alpha)-f(\beta)}$.

Returning for a moment to Horner's method, it may be remarked that the correction h, to an approximate value α, is therein found as a quotient the same or such as the quotient $f(\alpha)\div f'(\alpha)$ which presents itself in Newton's method. The difference is that with Horner the integer part of this quotient is taken as the presumptive value of h, and the figure is verified at each step. With Newton the quotient itself, developed to the proper number of decimal places, is taken as the value of h; if too many decimals are taken, there would be a waste of work; but the error would correct itself at the next step. Of course the calculation should be conducted without any such waste of work.

Next as to the theory of imaginaries.

15. It will be recollected that the expression *number* and the correlative epithet *numerical* were at the outset used in a wide sense, as extending to imaginaries. This extension arises out of the theory of equations by a process analogous to that by which number, in its original most restricted sense of positive integer number, was extended to have the meaning of a real positive or negative magnitude susceptible of continuous variation.

If for a moment number is understood in its most restricted sense as meaning positive integer number, the