$$\sin^2 u\frac{d^2y}{du^2} + \sin u \cos u\frac{dy}{du} - y = u - \sin u,$$

$$\left(\frac{dy}{du}\right)_\beta = 0.$$

This method of development is due to Sir George Airy, whose original paper—the investigation is different in form from the above—will be found in the *Philosophical Magazine* for December 1861. The solution of the differential equation leads to this result—

$$\rho = 2\cot\frac{u}{2}\log_e\sec\frac{u}{2} + C\tan\frac{u}{2},$$

$$C = 2\cot^2\frac{\beta}{2}\log_e\sec\frac{\beta}{2}$$

The limiting radius of the map is $R = 2C\tan\frac{1}{2}\beta$. In this system, called by the Astronomer-Royal the "*Projection by balance of errors*," the total misrepresentation is an absolute minimum.

Returning to the general case where ρ is any function of u, let us consider the local misrepresentation of direction. Take any indefinitely small line, length $= i$, making an angle α with the meridian in co-latitude u. Its projections on a meridian and parallel are $i\cos\alpha$, $i\sin\alpha$, which in the map are represented by $i\sigma\cos\alpha$, $i\sigma'\sin\alpha$. If then α' be the angle in the map corresponding to α,

$$\tan\alpha' = \frac{\sigma'}{\sigma}\tan\alpha.$$

Put

$$\frac{\sigma'}{\sigma} = \frac{\rho du}{\sin u\, d\rho} = \Sigma,$$

and the error $\alpha' - \alpha$ of representation $= \epsilon$, then

$$\tan\epsilon = \frac{(\Sigma - 1)\tan\alpha}{1 + \Sigma\tan^2\alpha}.$$

Put $\Sigma = \cot^2\zeta$, then ϵ is a maximum when $\alpha = \zeta$, and the corresponding value of ϵ is

$$\epsilon = \frac{\pi}{2} - 2\zeta$$

For simplicity of explanation we have supposed this method of development so applied as to have the pole in the centre. There is, however, no necessity for this, and any point on the surface of the sphere may be taken as the centre. All that is necessary is to calculate by spherical trigonometry the azimuth and distance, with reference to the assumed centre, of all the points of intersection of meridians and parallels within the space which is to be represented in a plane. Then the azimuth is represented unaltered, and any spherical distance u is represented by ρ. Thus we get all the points of intersection transferred to the representation, and it remains merely to draw continuous lines through these points, which lines will be the meridians and parallels in the representation.

The exaggeration in such systems, it is important to remember, whether of linear scale, area, or angle, is the same for a given distance from the centre, whatever be the azimuth; that is, the exaggeration is a function of the distance from the centre only.

We shall now examine and exemplify some of the most important systems of projection and development, commencing with

Perspective Projections.

In perspective drawings of the sphere, the plane on which the representation is actually made may generally be any plane perpendicular to the line joining the centre of the sphere and the point of vision. If V be the point of vision, P any point on the spherical surface, then p, the point in which the straight line VP intersects the plane of the representation, is the projection of P.

In the orthographic projection, the point of vision is at an infinite distance and the rays consequently parallel; in this case the plane of the drawing may be supposed to pass through the centre of the sphere. Let the circle (fig. 8) represent the plane of the equator on which we propose to make an orthographic representation of meridians and parallels. The centre of this circle is clearly the projection of the pole, and the parallels are projected into circles having the pole for a common centre. The diameters aa', bb' being at right angles, let the semicircle bab' be divided into the required number of equal parts; the diameters drawn through these points are the projections of meridians. The distances of c, of d, and of e from the diameter aa' are the radii of the successive circles representing the parallels. It is clear that, when the points of division are very close, the parallels will be very much crowded towards the outside of the map: so much so, that this projection is not much used.

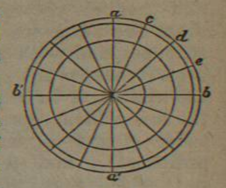

Fig. 8.

For an orthographic projection of the globe on a meridian plane, let $qnrs$ (fig. 9) be the meridian, ns the axis of rotation, then qr is the projection of the equator. The parallels will be represented by straight lines passing through the points of equal division; these lines are, like the equator, perpendicular to ns. The meridians will in this case be ellipses described on ns as a common major axis, the distances of c, of d, and of e from ns being the minor semiaxes.

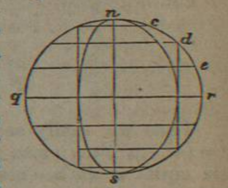

Fig. 9.

Let us next construct an orthographic projection of the sphere on the horizon of any place. Set off the angle aop (fig. 10) from the radius oa, equal to the latitude. Drop the perpendicular pP on oa, then P is the projection of the pole. On ao produced take $ob = p$P, then ob is the minor semi-axis of the ellipse representing the equator, its major axis being qr at right angles to ao. The points in which the meridians meet this elliptic equator are determined by lines drawn parallel to aob through the points of equal subdivision $cdefgh$. Take two points, as d and g, which are 90° apart, and let ik be their projections on the equator; then i is the pole of the meridian which passes through k. This meridian is of course an ellipse, and is described with reference to i exactly as the equator was described with reference to P. Produce io to l, and make lo equal to half the shortest chord that can be drawn through i; then lo is the semi-axis of the elliptic meridian, and the major axis is the diameter perpendicular to iol.

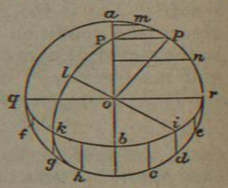

Fig. 10.

For the parallels: let it be required to describe the parallel whose co-latitude is u; take $pm = pn = u$, and let $m'n'$ be the projections of m and n on oPa; then $m'n'$ is the minor axis of the ellipse representing the parallel. Its centre is of course midway between m' and n', and the greater axis is equal to mn. Thus the construction is obvious. When pm is less than pa, the whole of the ellipse

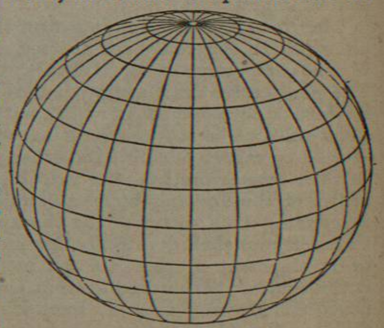
Fig. 11.—Orthographic Projection.

is to be drawn. When pm is greater than pa, the ellipse touches the circle in two points; these points divide the ellipse into two parts, one of which, being on the other side of the meridian plane aqr, is invisible.

Stereographic Projection.—In this case the point of vision is on the surface, and the projection is made on the plane of the great circle whose pole is V. Let kplV (fig. 12) be a great circle through the point of vision, and ors the trace of the plane of projection. Let c be the centre of a small circle whose radius is $cp = cl$; the straight line pl represents this small circle in orthographic projection.

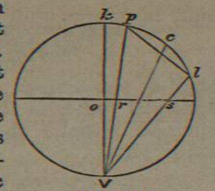

Fig. 12.

We have first to show that the stereographic projection of the small circle pl is itself a circle; that is to say, a straight line through V, moving along the circumference of pl, traces a circle on the plane of projection ors. This line generates an oblique cone standing on a circular base, its axis being cV (since the angle pVc = angle cVl); this cone is divided symmetrically by the plane of the great circle kpl, and also by the plane which passes through the axis Vc, perpendicular to the plane kpl. Now Vr·Vp, being = Vo sec kVp·Vk cos kVp = Vo·Vk, is equal to Vs·Vl; therefore the triangles Vrs, Vlp are similar, and it follows that the section of the cone by the plane rs is similar to the section by the plane pl. But the latter is a circle, hence also the projection is a circle; and since the representation of every infinitely small circle on the surface is itself a circle, it follows that in this projection the representation of small parts is (as we have before shown) strictly similar. Another inference is that the angle in which two lines on the sphere intersect is represented by the same angle in the projection. This may otherwise be proved by means of fig. 13, where Vok is the diameter of the sphere passing through the point of vision, fgh the plane of projection, kt a great circle, passing of course through V, and ouv the line of intersection of these two planes. A tangent plane to the surface at t cuts the plane of projection in the line rvs perpendicular to ov; tv is a tangent to the circle kt at t, tr and ts are any two tangents to the surface at t. Now the angle vtu (u being the projection of t) is $90° - ot$V $= 90° - o$V$t = ou$V $= tuv$, therefore tv is equal to uv; and since tvs and uvs are right angles, it follows that the angles vts and vus are equal. Hence the angle rts also is equal to its projection rus; that is, any angle formed by two intersecting lines on the surface is truly represented in the stereographic projection.

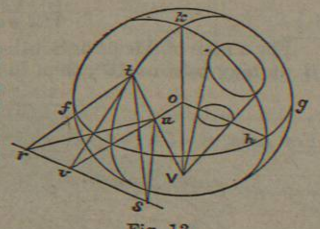

Fig. 13.

We have seen that the projection of any circle of the sphere is itself a circle. But in the case in which the circle to be projected passes through V, the projection becomes, for a great circle, a line through the centre of the sphere; otherwise, a line anywhere. It follows that meridians and parallels are represented in a projection on the horizon of any place by two systems of orthogonally cutting circles, one system passing through two fixed points, namely, the poles; and the projected meridians as they pass through the poles show the proper differences of longitude.

To construct a stereographic projection of the sphere on the horizon of a given place. Draw the circle $vlkr$ (fig. 14) with the diameters kv, lr at right angles; the latter is to represent the central meridian. Take koP equal to the co-latitude of the given place, say u; draw the diameter PoP′, and vP, vP′ cutting lr in pp': these are the projections of the poles, through which all the circles representing meridians have to pass. All their centres then will be in a line smn which crosses pp' at right angles through its middle point m. Now to describe the meridian whose west longitude is ω, draw pn making the angle $opn = 90° - \omega$, then n is the centre of the required circle, whose direction as it passes through p will make an angle $opg = \omega$ with pp'. The lengths of the several lines are

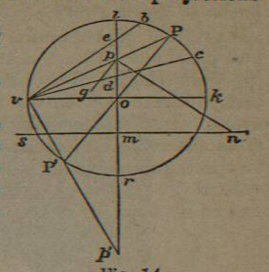

Fig. 14.

$$op = \tan\tfrac{1}{2}u;\quad op' = \cot\tfrac{1}{2}u;$$
$$om = \cot u;\quad mn = \operatorname{cosec} u\cot\omega.$$

Again, for the parallels, take Pb = Pc equal to the co-latitude, say c, of the parallel to be projected; join vb, vc cutting lr in e, d. Then ed is the diameter of the circle which is the required projection; its centre is of course the middle point of ed, and the lengths of the lines are

$$od = \tan\tfrac{1}{2}(u - c);\quad oe = \tan\tfrac{1}{2}(u + c).$$

The line sn itself is the projection of a parallel, namely, that of which the co-latitude $c = 180° - u$, a parallel which passes through the point of vision.

A very interesting connexion, noted by Professor Cayley, exists between the stereographic projection of the sphere on a meridian plane (*i.e.*, when a point on the equator occupies the centre of the drawing) and the projection on the horizon of any place whatever. The very same circles that represent parallels and meridians in the one case represent them in the other case also. In fig. 15, abs being a projection in which an equatorial point is in the centre, draw any chord ab perpendicular to the centre meridian cos, and on ab as diameter describe a circle, when the property referred to will be observed. This smaller circle is now the stereographic projection of the sphere on the horizon of some place whose co-latitude we may call u. The radius of the first circle being unity, let $ac = \sin x$, then by what has been proved above $co = \sin x \cot u = \cos x$; therefore $u = x$, and $ac = \sin u$. Although the meridian circles dividing the 360° at the pole into equal angles must be actually the same in both systems, yet a parallel circle whose co-latitude is c in the direct projection abs belongs in the oblique system to some other co-latitude as c'. To determine the connexion between c and c', consider the point t (not marked), in which one of the parallel circles crosses the line soc. In the direct system, p being the pole,

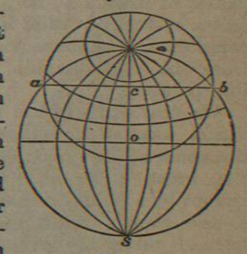

Fig. 15.

$$pt = 1 - \tan\tfrac{1}{2}(90° - c) = \frac{2}{1 + \cot\frac{1}{2}c}$$

and in the oblique,

$$pt = ac\left(\tan\tfrac{1}{2}u - \tan\tfrac{1}{2}(u - c')\right),$$

which, replacing ac by its value $\sin u$, becomes

$$\frac{2\sin\frac{1}{2}u\sin\frac{1}{2}c'}{\cos\frac{1}{2}(u - c')} = \frac{2}{1 + \cot\frac{1}{2}u\cot\frac{1}{2}c'},$$

therefore $\tan\frac{1}{2}c = \tan\frac{1}{2}c'\tan\frac{1}{2}u$ is the required relation.

Notwithstanding the facility of construction, the stereographic projection is not much used in map-making. But it may be made very useful as a means of graphical interpolation for drawing other projections in which points are represented in their true azimuths, but with an arbitrary law of distance, as $\rho=f(u)$. We may thus avoid the calculation of all the distances and azimuths (with reference to the selected centre point) of the intersections of meridians and parallels. Construct a stereographic projection of the globe on the horizon of the given place; then on this projection draw concentric circles (according to the stereographic law) representing the loci of points whose distances from the centre are consecutively 5°, 10°, 15°, 20°, &c., up to the required limit, and a system of radial lines at intervals of 5° Then to construct any other projection,—commence by drawing concentric circles, of which the radii are

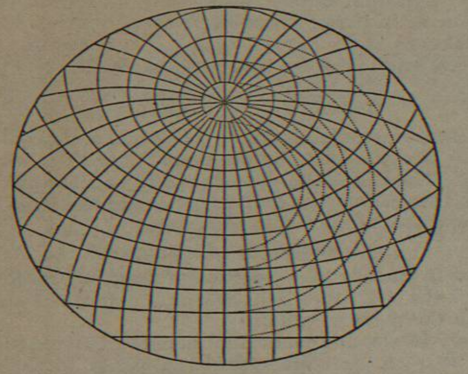

FIG. 16.—Stereographic Projection.

previously calculated by the law $\rho=f(u)$, for the successive values of u, 5°, 10°, 15°, 20°, &c., up to the limits as before, and a system of radial lines at intervals of 5°. This being completed, it remains to transfer the points of intersection from the stereographic to the new projection by graphic interpolation.

We now come to the general case in which the point of vision has any position outside the sphere. Let $abcd$ (fig. 17) be the great circle section of the sphere by a plane passing through c, the central point of the portion of surface to be represented, and V the point of vision. Let pj perpendicular to Vc be the plane of representation, join mV cutting pj in f, then f is the projection of any point m in the circl abc, and ef is the representation of cm. Let the angle $com=u$, V$e=k$. V$o=h$, $ef=\rho$; then, since $ef : e\mathrm{V} = mg : g\mathrm{V}$,

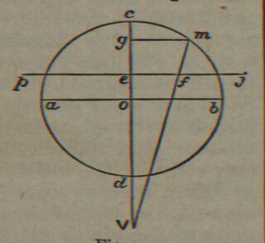

Fig.

$$\rho=\frac{k \sin u}{h+\cos u},$$

which gives the law connecting a spherical distance u with its rectilinear representation ρ. The relative scale at any point in this system of projection is given (keeping to our previously adopted notation) by

$$\sigma=k\frac{1+h\cos u}{(h+\cos u)^2}\ ;\ \sigma'=\frac{k}{h+\cos u},$$

the former applying to measurements made in a direction which passes through the centre of the map, the latter to the transverse direction. The product $\sigma\sigma'$ gives the exaggeration of areas. With respect to the alteration of angles we have

$$\Sigma=\frac{h+\cos u}{1+h\cos u},$$

and the greatest alteration of angle is

$$=\sin^{-1}\left(\frac{h-1}{h+1}\tan^2\frac{u}{2}\right).$$

This vanishes when $h=1$, that is, if the projection be stereographic; or for $u=0$, that is, at the centre of the map. At a distance of 90° from the centre, the greatest alteration is $90°-2\cot^{-1}\sqrt{h}$. (See *Philosoph. Mag.*, April 1862.)

The constants h and k can be determined, so that the total misrepresentation, viz.,

$$\mathrm{M}=\int_0^\beta \{(\sigma-1)^2+(\sigma'-1)^2\}\sin u\,du,$$

shall be a minimum, β being the greatest value of u, or the spherical radius of the map. On substituting the expressions for σ and σ' the integration is effected without difficulty. Put

$$\lambda=\frac{1-\cos\beta}{h+\cos\beta}\ ;\ \nu=(h-1)\lambda,$$
$$\mathrm{H}=\nu-(h+1)\log_e(\lambda+1),$$
$$\mathrm{H}'=\frac{\lambda}{h+1}(2-\nu+\tfrac{1}{3}\nu^2).$$

Then the value of M is

$$\mathrm{M}=4\sin^2\tfrac{1}{2}\beta+2k\mathrm{H}+k^2\mathrm{H}'.$$

When this is a minimum

$$\frac{d\mathrm{M}}{dh}=0;\quad \frac{d\mathrm{M}}{dk}=0.$$

$$\therefore\ k\mathrm{H}'+\mathrm{H}=0;\quad 2\frac{d\mathrm{H}}{dh}+k\frac{d\mathrm{H}'}{dh}=0.$$

Therefore $\mathrm{M}=4\sin^2\frac{1}{2}\beta-\frac{\mathrm{H}^2}{\mathrm{H}'}$, and h must be determined so as to make $\mathrm{H}^2 : \mathrm{H}'$ a maximum. In any particular case this maximum can only be ascertained by trial, that is to say, $\log \mathrm{H}^2-\log \mathrm{H}'$ must be calculated for certain equidistant values of h, and then the particular value of h which corresponds to the required maximum can be obtained by interpolation. Thus we find that if it be required to make the best possible perspective representation of a hemisphere, the values of h and k are $h=1·47$ and $k=2·034$; so that in this case

$$\rho=\frac{2·034\sin u}{1·47+\cos u}.$$

For a map of Africa or South America, the limiting radius β we may take as 40°; then in this case

$$\rho=\frac{2·543\sin u}{1·625+\cos u}.$$

For Asia, $\beta=54$, and the distance h of the point of sight

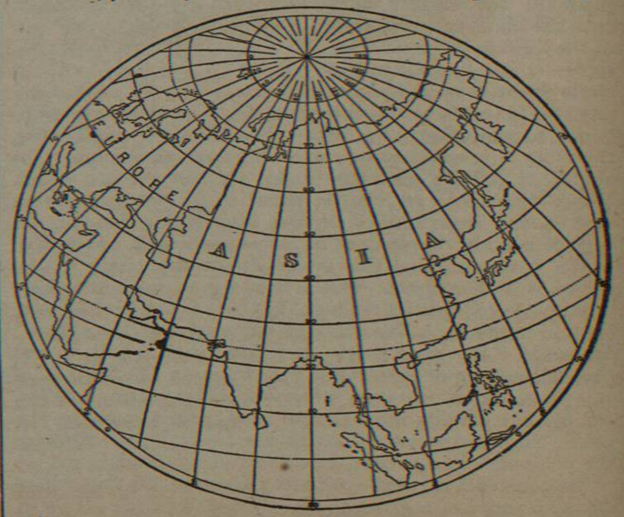

Fig. 18.

in this case is 1·61. Fig. 18 is a map of Asia having the meridians and parallels laid down on this system.

Figure 19 is a perspective representation of more than a hemisphere, the radius β being 108°, and the distance h of the point of vision, 1·40.

The co-ordinates xy of any point in this perspective may be expressed in terms of the latitude and longitude of the corresponding point on the sphere in the following manner. The co-ordinates originating at the centre, take the central meridian for the axis of y and a line perpendicular to it for the axis of x. Let the latitude of the point G, which is to occupy the centre of the map, be γ; if ϕ, ω be the latitude and longitude of any point P (the longitude being reckoned from the meridian of G), u the distance PG, and μ the

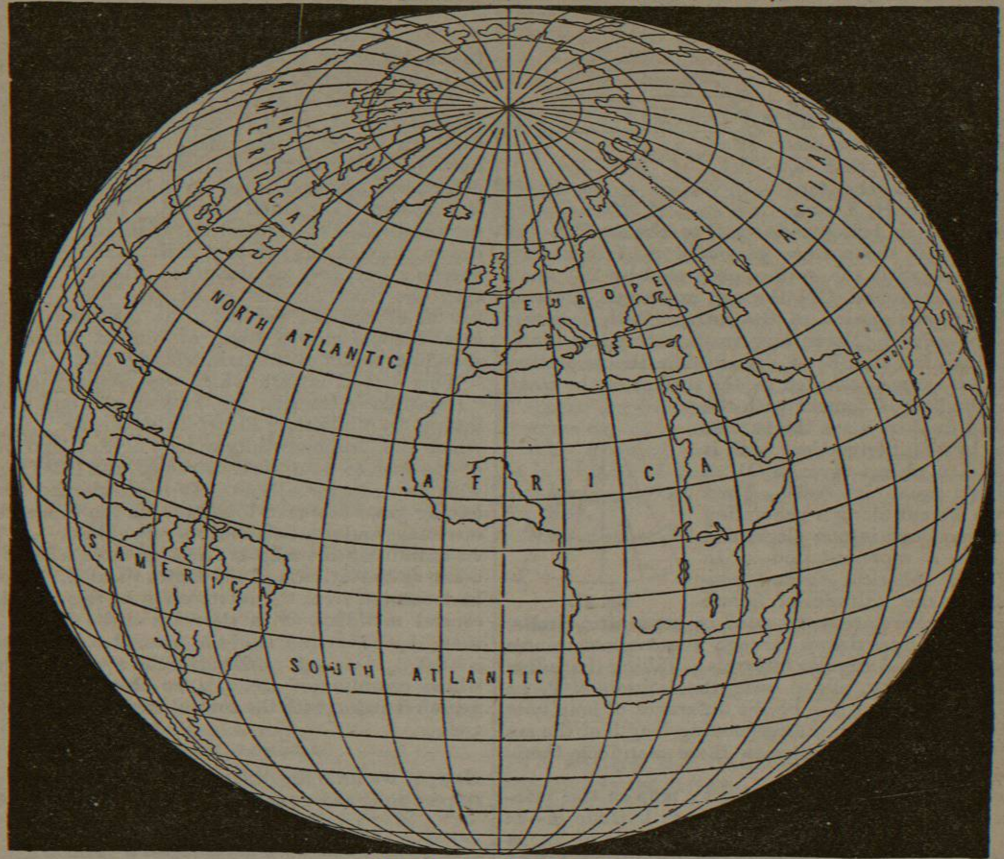

FIG. 19.—Twilight Projection.

azimuth of P at G, then the spherical triangle whose sides are $90°-\gamma$, $90°-\phi$, and u gives these relations—

$$\sin u\sin\mu=\cos\phi\sin\omega,$$
$$\sin u\cos\mu=\cos\gamma\sin\phi-\sin\gamma\cos\phi\cos\omega,$$
$$\cos u\quad=\sin\gamma\sin\phi+\cos\gamma\cos\phi\cos\omega.$$

Now $x=\rho\sin\mu$, $y=\rho\cos\mu$, that is,

$$\frac{x}{k}=\frac{\cos\phi\sin\omega}{h+\sin\gamma\sin\phi+\cos\gamma\cos\phi\cos\omega},$$
$$\frac{y}{k}=\frac{\cos\gamma\sin\phi-\sin\gamma\cos\phi\cos\omega}{h+\sin\gamma\sin\phi+\cos\gamma\cos\phi\cos\omega},$$

by which x and y can be computed for any point of the sphere. If from these equations we eliminate ω, we get the equation to the parallel whose latitude is ϕ; it is an ellipse whose centre is in the central meridian, and its greater axis perpendicular to the same. The radius of curvature of this ellipse at its intersection with the centre meridian is

$$\frac{k\cos\phi}{h\sin\gamma+\sin\phi}.$$

The elimination of ϕ between x and y gives the equation of the meridian whose longitude is ω, which also is an ellipse whose centre and axes may be determined.

The following table contains the computed co-ordinates for a map of Africa, which is included between latitudes 40° north and 40° south, and 40° of longitude east and west of a central meridian.

ϕ	Values of x and y.				
	$\omega=0°$	$\omega=10°$	$\omega=20°$	$\omega=30°$	$\omega=40°$
0°	$x=$ 0·00	9·69	19·43	29·25	39·17
	$y=$ 0·00	0·00	0·00	0·00	0·00
10°	$x=$ 0·00	9·60	19·24	28·95	38·76
	$y=$ 9·69	·9·75	9·92	10·21	10·63
20°	$x=$ 0·00	9·32	18·67	28·07	37·53
	$y=$19·43	19·54	19·87	20·43	21·25
30°	$x=$ 0·00	8·84	17·70	26·56	35·44
	$y=$29·25	29·40	29·87	30·67	31·83
40°	$x=$ 0·00	8·15	16·28	24·39	32·44
	$y=$39·17	39·36	39·94	40·93	42·34

Conical Development.

The conical development is adapted to the construction of maps of tracts of country of no great extent in latitude but any extent in the direction of a parallel. Selecting the mean parallel, or that which most nearly divides the area to be represented, we have to consider the cone which touches the sphere along that parallel. In fig. 20, which is an orthographic projection of the sphere on a meridian plane, let Pp be the parallel of contact with the cone. ON being the axis of revolution, the tangents at P and p will intersect ON produced in V. Let Qq be a parallel to the north of Pp, Rr another parallel the same distance to the south, that is, PQ = PR. Take on the tangent PV two points H, K such that PH = PK, each being made equal to the arc PQ. It is clear, then, that the surface generated by HK is very nearly coincident with the surface generated by RQ when the figure rotates round ON through any angle, great or small. The approximation of the surfaces will, however, be very close only if QR is very small. Suppose, now, that the paths of H and K, as described in the revolution round ON, are actually marked on the surface of the cone, as well as the line of contact with the sphere. And further, mark the surface of the cone by the intersections with it of the meridian planes through OV at the required equal intervals. Then let the cone be cut along a generating line and opened out into a plane, and we shall have a representation as in fig. 21 of the spherical surface contained between the latitudes of Q and R. The parallels here are represented by concentric circles, the meridians by lines drawn through the common centre of the circles at equal angular intervals. Taking the radius of the sphere as unity, and ϕ being the latitude of P, we see that $VP = \cot\phi$, and if ω be the difference of longitude between two meridians, the corresponding length of the arc Pp is $\omega\cos\phi$. The angle between these meridians themselves is $\omega\sin\phi$.

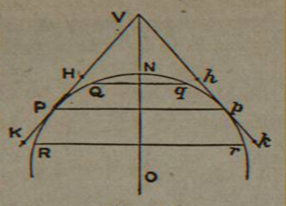

Fig. 20.

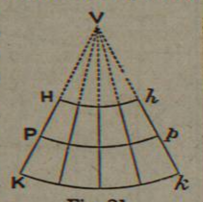

Fig. 21.

Suppose, now, we require to construct a map on this principle for a tract of country extending from latitude $\phi - m$ to $\phi + m$, and covering a breadth of longitude of $2n$, m and n being expressed in degrees. In fig. 21 let $HKkh$ be the quadrilateral formed by the extreme lines, so that $HK = hk = 2m$; then the angle HVh is $2n\sin\phi$ expressed in degrees. Now, taking the length of a degree as the unit, $VP = 57{\cdot}296\cot\phi$, and $VH = 57{\cdot}296\cot\phi - m$. It may be convenient in the first instance to calculate the chords Hh, Kk, and thus construct the rectilinear quadrilateral $HKkh$. The lengths of these chords are

$$Hh = 2(57{\cdot}296\cot\phi - m)\sin(n\sin\phi),$$
$$Kk = 2(57{\cdot}296\cot\phi + m)\sin(n\sin\phi),$$

and the distance between them is $2m\cos(n\sin\phi)$. The inclined sides of this trapezoid will then meet in a point at V, whose distance from P and p must correspond with the calculated length of VP. Now with this centre V describe the circular arcs representing the parallels through H, K, P. Also if the parallels are to be drawn at every degree of latitude, divide HK into $2m$ equal parts, and through each point of division describe a circular arc from the centre V. Then divide Pp into $2n$ equal parts, and draw the meridian lines through each of these points of division and the centre V.

If the centre V be inconveniently far off, it may be necessary to construct the centre parallel by points, that is, by calculating the coordinates of the various points of division. For this purpose, draw through the intersection of the centre meridian and centre parallel a line perpendicular to the meridian and therefore touching the parallel. Let the coordinate x be measured from the centre along this line, and y perpendicular to it. Then the coordinates of a point whose longitude measured from the centre meridian is ω are

$$x = \cot\phi\sin(\omega\sin\phi),$$
$$y = 2\cot\phi\sin^2\tfrac{1}{2}(\omega\sin\phi) = x\tan\tfrac{1}{2}(\omega\sin\phi),$$

the radius of the sphere being the unit; if a degree be the unit, these must be multiplied by 57·296.

The great defect of this projection is the exaggeration of the lengths of parallels towards either the northern or southern limits of the map. Various have been the devices to remedy this defect, and amongst these the following is a system very much adopted. Having subdivided the central meridian and drawn through the points of division the parallels precisely as described above, then the true lengths of degrees are set off along *each* parallel; the meridians, which in this case become curved lines, are drawn through the corresponding points of the parallels (fig. 22).

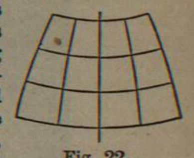

Fig. 22.

This system is that which was adopted in 1803 by the "Dépôt de la Guerre" for the map of France, and is there known by the title "*Projection de Bonne.*" It is that on which the Ordnance Survey map of Scotland on the scale of one inch to a mile is constructed, and it is frequently met with in ordinary atlases. It is ill-adapted for countries having great extent in longitude, as the intersections of the meridians and parallels become very oblique—as will be seen on examining the map of Asia in most atlases. If ϕ_0 be taken as the latitude of the centre parallel, and co-ordinates be measured from the intersection of this parallel with the central meridian, as in the case of the conical projection, then, if ρ be the radius of the parallel of latitude ϕ, we have $\rho = \cot\phi_0 + \phi_0 - \phi$. Also, if S be a point on this parallel whose co-ordinates are x, y, so that $VS = \rho$, and θ be the angle VS makes with the central meridian, then $\rho\theta = \omega\cos\phi$; and

$$x = \rho\sin\theta,\quad y = \cot\phi_0 - \rho\cos\theta.$$

Now, if we form the differential coefficients of x and y with respect to ϕ and ω, the latitude and longitude of S, we get

$$m'n - mn' = \cos\phi,$$
$$mn + m'n' = \frac{\omega}{\rho}\cos\phi(\cos\phi - \rho\sin\phi);$$

the first of which equations proves that the areas are truly represented. Moreover, if $90° \pm \psi$ be the angles of intersections of meridians and parallels,

$$\tan\psi = \theta - \omega\sin\phi,$$

which indeed might have been more easily obtained. In the case of Asia, the middle latitude $\phi_0 = 40°$, and the extreme northern latitude is 70°. Also the map extends 90° of longitude from the central meridian; hence, at the north-west and north-east corners of the map the angles of intersection of meridians and parallels are 90° ± 33°·54′. But for comparatively small tracts of country, as France or Scotland, this projection is very suitable.

Another modification of the conical projection consists in taking, not a tangent cone, but a cone which, having its vertex in the axis of revolution produced, intersects the sphere in two parallels,—these parallels being approximately midway between the centre parallel of the country and the extreme parallels. By this means part of the error is thrown on the centre parallel which is no longer represented by its true length, but is made too small, while the parallels forming the intersections of the cone are truly represented in length.

The exact position of these particular parallels may be

determined so as to give, upon the whole, the least amount of exaggeration for the entire map. This idea of a cutting cone seems to have originated[1] with the celebrated Gerard Mercator, who in 1554 made a map of Europe on this principle, selecting for the parallels of intersection those of 40° and 60°. The same system was adopted in 1745 by Delisle for the construction of a map of Russia. Euler in the *Acta Acad. Imp. Petrop.*, 1778, has discussed this projection and determined the conditions under which the errors at the northern extremity, at the centre, and at the southern extremity of a map so constructed shall be severally equal. Let c, c' be the co-latitudes of the extreme northern and southern parallels, γ, γ' those of two intermediate parallels, which are to be truly represented in the projection. Let OC′, Om' (fig. 23) be two consecutive meridians, as represented in the developed cone; the difference of longitude being ω, let the angle at O be $h\omega$. The degrees along the meridian being represented by their proper lengths, $CC' = c' - c$; and P corresponding to the pole, let $OP = z$, then $OC = z + c$; and so for G, G′, C′. The true lengths of $G'n'$ and Gn, namely, $\omega\sin\gamma'$ and $\omega\sin\gamma$, are equal to the represented lengths, namely, $h\omega(z+\gamma')$ and $h\omega(z+\gamma)$ respectively, whence γ and γ' are known when h and z are known. Comparing now the represented with the true lengths of parallel at the extremities and at the centre, if e be the common error that is to be allowed, then

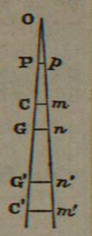

Fig. 23.

$$e = h\omega(z+c) - \omega\sin c,$$
$$e = -h\omega(z + \tfrac{1}{2}c + \tfrac{1}{2}c') + \omega\sin\tfrac{1}{2}(c+c'),$$
$$e = h\omega(z+c') - \omega\sin c'.$$

The difference of the first and third gives h, and then subtracting the second from the mean of the first and third, we get

$$z + \tfrac{1}{2}(c+c') = \tfrac{1}{4}(c'-c)\cot\tfrac{1}{4}(c'-c)\tan\tfrac{1}{2}(c'+c).$$

Thus z being known, the common centre of the circles representing the parallels is given. The value of h is given by the equation $h(c'-c) = \sin c' - \sin c$, and γ and γ' can be easily computed. But there is no necessity for doing this as we may construct the angles at O, which representing a difference of longitude ω are in reality equal to $h\omega$.

For instance, to construct a map of Asia on this system, having divided the central meridian into equal spaces for degrees, z must be calculated. Here we have $c = 20°$, $c' = 80°$, whence $z + 50° = 15° \tan 50° \cot 15° = 66°{\cdot}7$. Hence in this case the centre of the circles is 16°·7 beyond the north pole; also $h = {\cdot}6138$, so that a difference of longitude of 5° is represented at O by an angle of 3° 4′ 9″. The degrees of longitude in the parallel of 70° are in this map represented too large in the ratio of 1·150 : 1; those in the mid-latitude of 40° are too small in the ratio of 0·933 : 1; and those in 10° latitude are too large in the ratio of 1·05 to 1.

Gauss's Projection

may be considered as another variation of the conical system of development. Meridians are represented by lines drawn through a point, and a difference of longitude ω is represented by an angle $h\omega$, as in the preceding case. The parallels of latitude are circular arcs, all having as centre the point of divergence of the meridian lines, and the law of their formation is such that the representations of all small parts of the surface shall be precisely similar to the parts so represented. Let u be the co-latitude of a parallel, and ρ, a function of u, the radius of the circle representing this parallel. Consider the infinitely small space on the sphere contained by two consecutive meridians the difference of longitude of which is $d\mu$, and two consecutive parallels whose co-latitudes are u and $u + du$. The sides of this rectangle (fig. 24) are $pq = du$, $pr = \sin u\,d\mu$, whereas in the representation $p'q'r's'$, $p'q' = d\rho$, $p'r' = \rho h\,d\mu$, the angle at O being $= h\,d\mu$. Now, as the representation is to be similar to the original,

$$\frac{p'q'}{p'r'} = \frac{d\rho}{h\rho\,d\mu} = \frac{pq}{pr} = \frac{du}{\sin u\,d\mu},$$

whence $\frac{d\rho}{\rho} = h\frac{du}{\sin u}$, and integrating,

$$\rho = k\left(\tan\frac{u}{2}\right)^h,$$

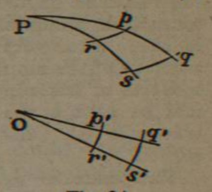

Fig. 24.

where the constant h is to be determined according to the requirements of each individual case. This investigation was first made in 1772 by the German mathematician J. H. Lambert,[2] but in 1825 it was again brought forward by Gauss in an essay written in answer to a prize question proposed by the Royal Society of Sciences at Copenhagen. A translation of this essay is to be found in the *Philosophical Magazine* for 1828 (see page 112), where Lambert's projection comes out as a particular solution of the general problem. Again, in a general investigation of the problem of "similar representation," Sir John Herschel, in the 30th volume of the *Journal of the Royal Geographical Society* (1860), deduced as a particular case this same projection. A large map of Russia was constructed and published on this system by the Geographical Society of St Petersburg in 1862.

The relative scale in this development is—

$$\frac{d\rho}{du} = \frac{hk}{a}\cdot\frac{\left(\tan\frac{u}{2}\right)^h}{\sin u},$$

where a is the radius of the sphere. It is a minimum when $u = \cos^{-1}h$. This minimum should occur in the vicinity of the central parallel of the map; if u_0 be the co-latitude of this parallel, we may put

$$\rho = k\left(\tan\frac{u}{2}\right)^{\cos u_0}$$

Or if we agree that the scale of the representation shall be the *same* at the extreme co-latitudes c, c', then

$$h = \frac{\log\sin c' - \log\sin c}{\log\tan\frac{1}{2}c' - \log\tan\frac{1}{2}c}.$$

To construct a map of North America extending from 10° latitude to 70°, we may take $h = \frac{2}{3}$, and k such as shall make the difference of radii of the extreme parallels = 60, namely $k = 104{\cdot}315$. The scales of the representation at the northern and southern limits are 1·116 and 1·096 respectively. The radii of the parallels are these—

70° . . .	32·801	30° . . .	72·328
60° . . .	43·356	20° . . .	82·255
50° . . .	53·177	10° . . .	92·801
40° . . .	62·728	0° . . .	104·315

Having drawn a line representing the central meridian, and selected a point on it as the centre of the concentric circles, let arcs be described with the above radii as parallels. For meridians, in this system a difference of longitude of 10° is represented by an angle of two-thirds that amount, or 6° 40′. The chord of this angle on the parallel of 10°, whose radius is 92·801, is easily found to be 10·792. Now stepping this quantity with a pair of compasses along the parallel, we have merely to draw lines through each of the points so found and the common centre of circles. The points of division of the parallel may be checked by taking the chord of 20°, or rather of 13° 20′,

[1] See page 178 of *Traité des Projections des Cartes Géographiques*, by A. Germain, Paris, an admirable and exhaustive essay. See also the work entitled *Coup d'œil historique sur la Projection des Cartes de Géographie*, by M. d'Avezac, Paris, 1863.

[2] *Beiträge zum Gebrauche der Mathematik und deren Anwendung*, vol. iii. p. 55, Berlin, 1772.