GEOMETRY

PART I.—PURE GEOMETRY.

GEOMETRY has been divided since the time of Euclid into an "elementary" and a "higher" part. The contents and limits of the former have been fixed by Euclid's *Elements*. The latter included at the time of the Greek mathematicians principally the properties of the conic sections and of a few other curves. The methods used in both were essentially the same. These began to be replaced during the 17th century by more powerful methods, invented by Roberval, Pascal, Desargues, and others. But the impetus which higher geometry received in their works was soon arrested, in consequence of the discoveries of Descartes,—the new calculus to which these gave rise absorbing the attention of mathematicians almost exclusively, until Monge, at the end of the 18th century, re-established "pure" as distinguished from Descartes's "coordinate" (or analytical) geometry. Since then the purely geometrical methods have been continuously extended, especially by Poncelet, Steiner, Von Staudt, and Cremona, and in England by Hirst and Henry Smith, to mention only a few of the leading names.

Whilst higher geometry thus made most rapid progress, the elementary part remained almost unaltered. It has been taught up to the present day on the basis of Euclid's *Elements*, the latter being either used directly as a text-book (in England), or being replaced (in most parts of the Continent) by text-books which are essentially Euclid's *Elements* rewritten, with a few additions about the mensuration of the circle, cone, cylinder, and sphere. Only within a very recent period have attempts been made to change the character of the elementary part by introducing some of the modern methods.

We shall give in this article—first, a survey of elementary geometry as contained in Euclid's *Elements*, and then, in form of an independent treatise, an introduction to higher geometry, based on modern methods. In the former part we shall suppose that a copy of Euclid's *Elements* is in the hands of the reader, so that we may dispense, as a rule, with giving proofs or drawing figures. We thus shall give only the contents of his propositions grouped together in such a way as to show their connexion, and often expressed in words which differ from the verbal translation in order to make their meaning clear. It will make little difference which of the many English editions of Euclid's *Elements* the reader takes. Of these we may mention Simson's, Potts's, and Todhunter's.

SECTION I.—ELEMENTARY OR EUCLIDIAN GEOMETRY.

The Axioms.

§ 1. The object of geometry is to investigate the properties of space. The first step must consist in establishing those fundamental properties from which all others follow by processes of deductive reasoning. They are laid down in the Axioms, and these ought to form such a system that nothing need be added to them in order fully to characterize space, and that nothing may be left out without making the system incomplete. They must, in fact, completely "define" space. Several such systems are conceivable. Euclid has given one, others have been put forward in recent times by Riemann (*Abhandl. der königl. Gesellsch. zu Göttingen*, vol. xiii.), by Helmholz (*Göttinger Nachrichten*, June 1868), and by Grassmann (*Ausdehnungslehre von* 1844). How many axioms the system ought to contain, and which system is the simplest, may be said to be still an open question. We shall consider only Euclid's system.

§ 2. The axioms are obtained from inspection of space and of solids in space,—hence from experience. The same source gives us the notions of the geometrical entities to which the axioms relate, viz., solids, surfaces, lines or curves, and points. A solid is directly given by experience; we have only to abstract all material from it in order to gain the notion of a geometrical solid. This has shape, size, position, and may be moved. Its boundary or boundaries are called surfaces. They separate one part of space from another, and are said to have no thickness. Their boundaries are curves or lines, and these have length only. Their boundaries, again, are points, which have no magnitude but only position. We thus come in three steps from solids to points which have no magnitude; in each step we lose one extension. Hence we say a solid has three dimensions, a surface two, a line one, and a point none. Space itself, of which a solid forms only a part, is also said to be of three dimensions. The same thing is intended to be expressed by saying that a solid has length, breadth, and thickness, a surface length and breadth, a line length only, and a point no extension whatsoever.

Euclid gives the essence of these statements as definitions:—

Def. 1, I. *A point is that which has no parts, or which has no magnitude.*
Def. 2, I. *A line is length without breadth.*
Def. 5, I. *A superficies is that which has only length and breadth.*
Def. 1, XI. *A solid is that which has length, breadth, and thickness.*

If we allow motion in geometry,—and it seems impossible to avoid it,—we may generate these entities by moving a point, a line, or a surface, thus:—

The path of a moving point is a line.
The path of a moving line is, in general, a surface.
The path of a moving surface is, in general, a solid.

And we may then assume that the lines, surfaces, and solids, as defined before, can all be generated in this manner. From this generation of the entities it follows again that the boundaries—the first and last position of the moving element—of a line are points, and so on; and thus we come back to the considerations with which we started.

Euclid points this out in his definitions,—Def. 3, I., Def. 6, I., and Def. 2, XI. He does not, however, show the connexion which these definitions have with those mentioned before. When points and lines have been defined, a statement like Def. 3, I., "The extremities of a line are points," is a proposition which either has to be proved, and then it is a theorem, or which has to be taken for granted, in which case it is an axiom. And so with Def. 6, I., and Def. 2, XI.

§ 3. Euclid's definitions mentioned above are attempts to describe, in a few words, notions which we have obtained by inspection of and abstraction from solids. A few more notions have to be added to these, principally those of the simplest line—the straight line, and of the simplest surface—the flat surface or plane. These notions we possess, but to define them accurately is difficult. Euclid's Definition 4, I., "A straight line is that which lies evenly between its extreme points," must be meaningless to any one who has not the notion of straightness in his mind. Neither does it state a property of the straight line which can be used in any further investigation. Such a property is given in Axiom 10, I. It is really this axiom, together with Postulates 2 and 3, which characterizes the straight line.

Whilst for the straight line the verbal definition and axiom are kept apart, Euclid mixes them up in the case of the plane. Here the Definition 7, I., includes an axiom. It defines a plane as a surface which has the property that every straight line which joins any two points in it lies altogether in the surface. But if we take a straight line and a point in such a surface, and draw all straight lines which join the latter to all points in the first line, the surface will be fully determined. This construction is therefore sufficient as a definition. That every other straight line which joins any two points in this surface lies altogether in it is a further property, and to assume it gives another axiom.

Thus a number of Euclid's axioms are hidden among his first definitions. A still greater confusion exists in the present editions of Euclid between the postulates and axioms so-called, but this is due to later editors and not to Euclid himself. The latter had the last three axioms put together with the postulates (αἰτήματα), so that these were meant to include all assumptions relating to space. The remaining assumptions which relate to magnitudes in general, viz., the first eight "axioms" in modern editions, were called "common notions" (κοιναὶ ἔννοιαι). Of the latter a few may be said to be definitions. Thus the eighth might be taken as a definition of "equal," and the seventh of halves. If we wish to collect the axioms used in Euclid's *Elements*, we have therefore to take the three postulates, the last three axioms as generally given, a few axioms hidden in the definitions, and an axiom used by Euclid in the proof of Prop. 4 and on a few other occasions, viz., that figures may be moved in space without change of shape or size.

We shall not enter into the investigation how far the assumptions which would be included in such a list are sufficient, and how far they are necessary. It may be sufficient here to state that from the beginning of a geometrical science to the present century attempts without end have been made to prove the last of Euclid's axioms, that only at the beginning of the present century the futility of this attempt was shown, and that only within the last twenty years the true nature of the connexion between the axioms has become known through the researches of Riemann and Helmholz, although Grassmann had published already, in 1844, his classical but long-neglected *Ausdehnungslehre*.

§ 4. The assumptions actually made by Euclid may be stated as follows:—

1. Straight lines exist which have the property that any one of them may be produced both ways without limit, that through any two points in space such a line may be drawn, and that any two of them coincide throughout their indefinite extensions as soon as two points in the one coincide with two points in the other. (This gives the contents of Def. 4, part of Def. 35, the first two Postulates, and Axiom 10.)
2. Plane surfaces or planes exist having the property laid down in Def. 7, that every straight line joining any two points in such a surface lies altogether in it.
3. Right angles, as defined in Def. 10, are possible, and all right angles are equal; that is to say, wherever in space we take a plane, and wherever in that plane we construct a right angle, all angles thus constructed will be equal, so that any one of them may be made to coincide with any other. (Axiom 11.)
4. The 12th Axiom of Euclid. This we shall not state now, but only introduce it when we cannot proceed any further without it.
5. Figures may be freely moved in space without change of shape or size. This is assumed by Euclid, but not stated as an axiom.
6. In any plane a circle may be described, having any point in that plane as centre, and its distance from any other point in that plane as radius. (Postulate 3.)

The definitions which have not been mentioned are all "nominal definitions," that is to say, they fix a name for a thing described. Many of them overdetermine a figure. (Compare notes to definitions in Simson's or Todhunter's edition.)

§ 5. Euclid's *Elements* are contained in thirteen books. Of these the first four and the sixth are devoted to "plane geometry," as the investigation of figures in a plane is generally called. The 5th book contains the theory of proportion which is used in Book VI. The 7th, 8th, and 9th books are purely arithmetical, whilst the 10th contains a most ingenious treatment of geometrical irrational quantities. These four books will be excluded from our survey. The remaining three books relate to figures in space, or, as it is generally called, to "solid geometry." The 7th, 8th, 9th, 10th, 13th, and part of the 11th and 12th books are now generally omitted from the school editions of the *Elements*. In the first four and in the 6th book it is to be understood that all figures are drawn in a plane.

BOOK I. OF EUCLID'S "ELEMENTS."

§ 6. According to the third postulate it is possible to draw in any plane a circle which has its centre at any given point, and its radius equal to the distance of this point from any other point given in the plane. This makes it possible (Prop. 1) to construct on a given line AB an equilateral triangle, by drawing first a circle with A as centre and AB as radius, and then a circle with B as centre and BA as radius. The point where these circles intersect—that they intersect Euclid quietly assumes—is the vertex of the required triangle. Euclid does not suppose, however, that a circle may be drawn which has its radius equal to the distance between any two points unless one of the points be the centre. This implies also that we are not supposed to be able to make any straight line equal to any other straight line, or to carry a distance about in space. Euclid therefore next solves the problem: It is required along a given straight line from a point in it to set off a distance equal to the length of another straight line given anywhere in the plane. This is done in two steps. It is shown in Prop. 2 how a straight line may be drawn from a given point equal in length to another given straight line not drawn from that point. And then the problem itself is solved in Prop. 3, by drawing first through the given point some straight line of the required length, and then about the same point as centre a circle having this length as radius. This circle will cut off from the given straight line a length equal to the required one. Now-a-days, instead of going through this long process, we take a pair of compasses and set off the given length by its aid. This assumes that we may move a length about without changing it. But Euclid has not assumed it, and this proceeding would be fully justified by his desire not to take for granted more than was necessary, if he were not obliged at his very next step actually to make this assumption, though without stating it.

§ 7. We now come (in Prop. 4) to the first theorem. It is the fundamental theorem of Euclid's whole system, there being only a very few propositions (like Props. 13, 14, 15, I.), except those in the 5th book and the first half of the 11th, which do not depend upon it. It is stated very accurately, though somewhat clumsily, as follows:—

If two triangles have two sides of the one equal to two sides of the other, each to each, and have also the angles contained by those sides equal to one another, they shall also have their bases or third sides equal; and the two triangles shall be equal; and their other angles shall be equal, each to each, namely, those to which the equal sides are opposite.

That is to say, the triangles are "identically" equal, and one may be considered as a copy of the other. The proof is very simple. The first triangle is taken up and placed on the second, so that the parts of the triangles which are known to be equal fall upon each other. It is then easily seen that also the remaining parts of one coincide with those of the other, and that they are therefore equal. This process of applying one figure to another Euclid scarcely uses again, though many proofs would be simplified by doing so. The process introduces motion into geometry, and includes, as already stated, the axiom that figures may be moved without change of shape or size.

If the last proposition be applied to an isosceles triangle, which has two sides equal, we obtain the theorem (Prop. 5), *if two sides of a triangle are equal, then the angles opposite these sides are equal.*

Euclid's proof is somewhat complicated, and a stumbling-block to many schoolboys. The proof becomes much simpler if we consider the isosceles triangle ABC (AB=AC) twice over, once as a triangle BAC, and once as a triangle CAB; and now remember that

AB, AC in the first are equal respectively to AC, AB in the second, and the angles included by these sides are equal. Hence the triangles are equal, and the angles in the one are equal to those in the other, viz., those which are opposite equal sides, *i.e.*, angle ABC in the first equals angle ACB in the second, as they are opposite the equal sides AC, and AB in the two triangles.

There follows the converse theorem (Prop. 6). *If two angles in a triangle are equal, then the sides opposite them are equal,*—*i.e.*, the triangle is isosceles. The proof given consists in what is called a *reductio ad absurdum*, a kind of proof often used by Euclid, and principally in proving the converse of a previous theorem. It assumes that the theorem to be proved is wrong, and then shows that this assumption leads to an absurdity, *i.e.*, to a conclusion which is in contradiction to a proposition proved before—that therefore the assumption made cannot be true, and hence that the theorem is true. It is often stated that Euclid invented this kind of proof, but the method is most likely much older.

§ 8. It is next proved *that two triangles which have the three sides of the one equal respectively to those of the other are identically equal, hence that the angles of the one are equal respectively to those of the other, those being equal which are opposite equal sides.* This is Prop. 8, Prop. 7 containing only a first step towards its proof.

These theorems allow now of the solution of a number of problems, viz.:—

To bisect a given angle (Prop. 9).

To bisect a given finite straight line (Prop. 10).

To draw a straight line perpendicularly to a given straight line through a given point in it (Prop. 11), *and also through a given point not in it* (Prop. 12).

The solutions all depend upon properties of isosceles triangles.

§ 9. The next three theorems relate to angles only, and might have been proved before Prop. 4, or even at the very beginning. The first (Prop. 13) says, *The angles which one straight line makes with another straight line on one side of it either are two right angles or are together equal to two right angles.* This theorem would have been unnecessary if Euclid had admitted the notion of an angle such that its two limits are in the same straight line, and had besides defined the sum of two angles.

Its converse (Prop. 14) is of great use, inasmuch as it enables us in many cases to prove that two straight lines drawn from the same point are one the continuation of the other. So also is

Prop. 15. *If two straight lines cut one another, the vertical or opposite angles shall be equal.*

§ 10. Euclid returns now to properties of triangles. Of great importance for the next steps (though afterwards superseded by a more complete theorem) is

Prop. 16. *If one side of a triangle be produced, the exterior angle shall be greater than either of the interior opposite angles.*

Prop. 17, *Any two angles of a triangle are together less than two right angles*, is an immediate consequence of it. By the aid of these two, the following fundamental properties of triangles are easily proved:—

Prop. 18. *The greater side of every triangle has the greater angle opposite to it*;

Its converse, Prop. 19. *The greater angle of every triangle is subtended by the greater side, or has the greater side opposite to it*;

Prop. 20. *Any two sides of a triangle are together greater than the third side*;

And also Prop. 21. *If from the ends of the side of a triangle there be drawn two straight lines to a point within the triangle, these shall be less than the other two sides of the triangle, but shall contain a greater angle.*

§ 11. Having solved two problems (Props. 22, 23), he returns to two triangles which have two sides of the one equal respectively to two sides of the other. It is known (Prop. 4) that if the included angles are equal then the third sides are equal; and conversely (Prop. 8), if the third sides are equal, then the angles included by the first sides are equal. From this it follows that if the included angles are not equal, the third sides are not equal, and conversely, that if the third sides are not equal, the included angles are not equal. Euclid now completes this knowledge by proving, that "*if the included angles are not equal, then the third side in that triangle is the greater which contains the greater angle*;" and conversely, that "*if the third sides are unequal, that triangle contains the greater angle which contains the greater side.*" These are Prop. 24 and Prop. 25.

§ 12. The next theorem (Prop. 26) says that *if two triangles have one side and two angles of the one equal respectively to one side and two angles of the other, viz., in both triangles either the angles adjacent to the equal side, or one angle adjacent and one angle opposite it, then the two triangles are identically equal.*

This theorem belongs to a group with Prop. 4 and Prop. 8. Its first case might have been given immediately after Prop. 4, but the second case requires Prop. 16 for its proof.

§ 13. We come now to the investigation of parallel straight lines, *i.e.*, of straight lines which lie in the same plane, and cannot be made to meet however far they be produced either way. The investigation, which starts from Prop. 16, will become clearer if a few names be explained which are not all used by Euclid. If two straight lines be cut by a third, the latter is now generally called a "transversal" of the figure. It forms at the two points where it cuts the given lines four angles with each. Those of the angles which lie between the given lines are called interior angles, and of these, again, any two which lie on opposite sides of the transversal but one at each of the two points are called "alternate angles."

We may now state Prop. 16 thus:—*If two straight lines which meet are cut by a transversal, their alternate angles are unequal.* For the lines will form a triangle, and one of the alternate angles will be an exterior angle to the triangle, the other interior and opposite to it.

From this follows at once the theorem contained in Prop. 27. *If two straight lines which are cut by a transversal make alternate angles equal, the lines cannot meet, however far they be produced, hence they are parallel.* This proves the existence of parallel lines.

Prop. 28 states the same fact in different forms. *If a straight line, falling on two other straight lines, make the exterior angle equal to the interior and opposite angle on the same side of the line, or make the interior angles on the same side together equal to two right angles, the two straight lines shall be parallel to one another.*

Hence we know that, "if two straight lines which are cut by a transversal meet, their alternate angles are not equal"; and hence that, "if alternate angles are equal, then the lines are parallel."

The question now arises, Are the propositions converse to these true or not? That is to say, "If alternate angles are unequal, do the lines meet?" And "if the lines are parallel, are alternate angles necessarily equal?"

The answer to either of these two questions implies the answer to the other. But it has been found impossible to prove that the negation or the affirmation of either is true.

The difficulty which thus arises is overcome by Euclid assuming that the first question has to be answered in the affirmative. This gives his last axiom (12), which we quote in his own words.

Axiom 12.—*If a straight line meet two straight lines, so as to make the two interior angles on the same side of it taken together less than two right angles, these straight lines, being continually produced, shall at length meet on that side on which are the angles which are less than two right angles.*

The answer to the second of the above questions follows from this, and gives the theorem Prop. 29. *If a straight line fall on two parallel straight lines, it makes the alternate angles equal to one another, and the exterior angle equal to the interior and opposite angle on the same side, and also the two interior angles on the same side together equal to two right angles.*

§ 14. With this a new part of elementary geometry begins. The earlier propositions are independent of this axiom, and would be true even if a wrong assumption had been made in it. They all relate to figures in a plane. But a plane is only one among an infinite number of conceivable surfaces. We may draw figures on any one of them and study their properties. We may, for instance, take a sphere instead of the plane, and obtain "spherical" in the place of "plane" geometry. If on one of these surfaces lines and figures could be drawn, answering to all the definitions of our plane figures, and if the axioms with the exception of the last all hold, then all propositions up to the 28th will be true for these figures. This is the case in spherical geometry if we substitute "shortest line" or "great circle" for "straight line," "small circle" for "circle," and if, besides, we limit all figures to a part of the sphere which is less than a hemisphere, so that two points on it cannot be opposite ends of a diameter, and therefore determine always one and only one great circle.

For spherical triangles, therefore, all the important propositions 4, 8, 26; 5 and 6; and 18, 19, and 20 will hold good.

This remark will be sufficient to show the impossibility of proving Euclid's last axiom, which would mean proving that this axiom is a consequence of the others, and hence that the theory of parallels would hold on a spherical surface, where the other axioms do hold, whilst parallels do not even exist.

It follows that the axiom in question states an inherent difference between the plane and other surfaces, and that the plane is only fully characterized when this axiom is added to the other assumptions.

§ 15. The introduction of the new axiom and of parallel lines leads to a new class of propositions.

After proving (Prop. 30) that "*two lines which are each parallel to a third are parallel to each other*," we obtain the new properties of triangles contained in Prop. 32. Of these the second part is the most important, viz., the theorem, *The three interior angles of every triangle are together equal to two right angles.*

As easy deductions not given by Euclid but added by Simson follow the propositions about the angles in polygons, they are given in English editions as corollaries to Prop. 32.

These theorems do not hold for spherical figures. The sum of the interior angles of a spherical triangle is always greater than two right angles, and increases with the area.

§ 16. The theory of parallels as such may be said to be finished with Props. 33 and 34, which state properties of the parallelogram, *i.e.*, of a quadrilateral formed by two pairs of parallels. They are—

Prop. 33. *The straight lines which join the extremities of two equal and parallel straight lines towards the same parts are themselves equal and parallel*; and

Prop. 34. *The opposite sides and angles of a parallelogram are equal to one another, and the diameter (diagonal) bisects the parallelogram, that is, divides it into two equal parts.*

§ 17. The rest of the first book relates to areas of figures. The theory is made to depend upon the theorems—

Prop. 35. *Parallelograms on the same base and between the same parallels are equal to one another*; and

Prop. 36. *Parallelograms on equal bases, and between the same parallels, are equal to one another.*

As each parallelogram is bisected by a diagonal, the last theorems hold also if the word parallelogram be replaced by "triangle," as is done in Props. 37 and 38.

It is to be remarked that Euclid proves these propositions only in the case when the parallelograms or triangles have their bases in the same straight line.

The theorems converse to the last form the contents of the next three propositions, viz.:—

Theorem (Props. 40 and 41).—*Equal triangles, on the same or on equal bases, in the same straight line, and on the same side of it, are between the same parallels.*

That the two cases here stated are given by Euclid in two separate propositions proved separately is characteristic of his method.

§ 18. To compare areas of other figures, Euclid shows first, in Prop. 42, how *to draw a parallelogram which is equal in area to a given triangle, and has one of its angles equal to a given angle.* If the given angle is right, then the problem is solved *to draw a "rectangle" equal in area to a given triangle.*

Next this parallelogram is transformed into another parallelogram, *which has one of its sides equal to a given straight line*, whilst its angles remain unaltered. This may be done by aid of the theorem in

Prop. 43. *The complements of the parallelograms which are about the diameter of any parallelogram are equal to one another.*

Thus the problem (Prop. 44) is solved *to construct a parallelogram on a given line, which is equal in area to a given triangle, and which has one angle equal to a given angle* (generally a right angle).

As every polygon can be divided into a number of triangles, we can now construct a parallelogram having a given angle, say a right angle, and being equal in area to a given polygon. For each of the triangles into which the polygon has been divided, a parallelogram may be constructed, having one side equal to a given straight line, and one angle equal to a given angle. If these parallelograms be placed side by side, they may be added together to form a single parallelogram, having still one side of the given length. This is done in Prop. 45.

Herewith a means is found to compare areas of different polygons. We need only construct two rectangles equal in area to the given polygons, and having each one side of given length. By comparing the unequal sides we are enabled to judge whether the areas are equal, or which is the greater. Euclid does not state this consequence, but the problem is taken up again at the end of the second book, where it is shown how to construct a square equal in area to a given polygon.

§ 19. The first book concludes with one of the most important theorems in the whole of geometry, and one which has been celebrated since the earliest times. It is stated, but on doubtful authority, that Pythagoras discovered it, and it has been called by his name. If we call that side in a right-angled triangle which is opposite the right angle the hypotenuse, we may state it as follows:—

Theorem of Pythagoras (Prop. 47).—*In every right-angled triangle the square on the hypotenuse is equal to the sum of the squares of the other sides.*

And conversely—

Prop. 48. *If the square described on one of the sides of a triangle be equal to the squares described on the other sides, then the angle contained by these two sides is a right angle.*

On this theorem (Prop. 47) almost all geometrical measurement depends, which cannot be directly obtained.

Book II.

§ 20. The propositions in the second book are very different in character from those in the first; they all relate to areas of rectangles and squares. Their true significance is best seen by stating them in an algebraic form. This is often done by expressing the lengths of lines by aid of numbers, which tell how many times a chosen unit is contained in the lines. If there is a unit to be found which is contained an exact number of times in each side of a rectangle, it is easily seen, and generally shown in the teaching of arithmetic, that the rectangle contains a number of unit squares equal to the product of the numbers which measure the sides, a unit square being the square on the unit line. If, however, no such unit can be found, this process requires that connexion between lines and numbers which is only established by aid of ratios of lines, and which is therefore at this stage altogether inadmissible. But there exists another way of connecting these propositions with algebra, based on modern notions which seem destined greatly to change and to simplify mathematics. We shall introduce here as much of it as is required for our present purpose.

At the beginning of the second book we find a definition according to which "a rectangle is said to be 'contained' by the two sides which contain one of its right angles"; in the text this phraseology is extended by speaking of rectangles contained by any two straight lines, meaning the rectangle which has two adjacent sides equal to the two straight lines.

We shall denote a finite straight line by a single small letter, $a, b, c, \ldots x$, and the area of the rectangle contained by two lines a and b by ab, and this we shall call the product of the two lines a and b. It will be understood that this definition has nothing to do with the definition of a product of numbers.

We define as follows:—

The *sum* of two straight lines a and b means a straight line c which may be divided in two parts equal respectively to a and b. This sum is denoted by $a+b$.

The *difference* of two lines a and b (in symbols, $a-b$) means a line c which when added to b gives a; that is,

$$a-b=c \text{ if } b+c=a.$$

The *product* of two lines a and b (in symbols, ab) means the area of the rectangle contained by the lines a and b. For aa, which means the square on the line a, we write a^2.

§ 21. The first ten of the fourteen propositions of the second book may then be written in the form of formulæ as follows:—

Prop. 1. $a(b+c+d+\ldots)=ab+ac+ad+\ldots$
" 2. $ab+ac=a^2$ if $b+c=a$.
" 3. $a(a+b)=a^2+ab$.
" 4. $(a+b)^2=a^2+2ab+b^2$.
" 5. $(a+b)(a-b)+b^2=a^2$.
" 6. $(a+b)(a-b)+b^2=a^2$.
" 7. $a^2+(a-b)^2=2a(a-b)+b^2$.
" 8. $4(a+b)a+b^2=(2a+b)^2$.
" 9. $(a+b)^2+(a-b)^2=2a^2+2b^2$.
" 10. $(a+b)^2+(a-b)^2=2a^2+2b^2$.

It will be seen that 5 and 6, and also 9 and 10, are identical. In Euclid's statement they do not look the same, the figures being arranged differently.

If the letters $a, b, c, \ldots$ denoted numbers, it follows from algebra that each of these formulæ is true. But this does not prove them in our case, where the letters denote lines, and their products areas without any reference to numbers. To prove them we have to discover the laws which rule the operations introduced, viz., addition and multiplication of segments. This we shall do now; and we shall find that these laws are the same with those which hold in algebraical addition and multiplication.

§ 22. In a sum of numbers we may change the order in which the numbers are added, and we may also add the numbers together in groups, and then add these groups. But this also holds for the sum of segments and for the sum of rectangles, as a little consideration shows. That the sum of rectangles has always a meaning follows from the Props. 43–45 in the first book. These laws about addition are reducible to the two—

$$a+b=b+a \quad . \quad . \quad . \quad (1),$$
$$a+(b+c)=a \quad +c \quad . \quad . \quad (2);$$

or, when expressed for rectangles,

$$ab+cd=cd+ab \quad . \quad . \quad . \quad (3),$$
$$ab+(cd+ef)=ab+cd+ef \quad . \quad . \quad (4).$$

The brackets mean that the terms in the bracket have been added together before they are added to another term. The more general cases for more terms may be deduced from the above.

For the product of two numbers we have the law that it remains unaltered if the factors be interchanged. This also holds for our geometrical product. For if ab denotes the area of the rectangle which has a as base and b as altitude, then ba will denote the area of the rectangle which has b as base and a as altitude. But in a rectangle we may take either of the two lines which contain it as base, and then the other will be the altitude. This gives

$$ab=ba \quad . \quad . \quad . \quad . \quad (5).$$

In order further to multiply a sum by a number, we have in algebra the rule:—Multiply each term of the sum, and add the products thus obtained. That this holds for our geometrical products is shown by Euclid in his first proposition of the second book, where he proves that the area of a rectangle whose base is the sum of a number of segments is equal to the sum of rectangles which have these segments separately as bases. In symbols this gives, in the simplest case,

$$\left.\begin{aligned} a(b+c)&=ab+ac \\ (b+c)a&=ba+ca \end{aligned}\right\} \quad . \quad . \quad . \quad (6).$$

and

To these laws which have been investigated by Sir William Hamilton and by Hermann Grassmann, the former has given special names. He calls the laws expressed in

(1) and (3) the commutative law for addition;
(5) „ „ multiplication;
(2) and (4) the associative laws for addition;
(6) the distributive law.

§ 23. Having proved that these six laws hold, we can at once prove every one of the above propositions in their algebraical form.

The first is proved geometrically, it being one of the fundamental laws. The next two propositions are only special cases of the first. Of the others we shall prove one, viz., the fourth:—

$$(a+b)^2=(a+b)\,(a+b)=(a+b)a+(a+b)b \qquad \text{by (6).}$$

But

$$(a+b)a=aa+ba \qquad \text{by (6),}$$
$$=aa+ab \qquad \text{by (5);}$$

and

$$(a+b)b=ab+bb \qquad \text{by (6).}$$

Therefore

$$\left.\begin{aligned}(a+b)^2&=aa+ab+(ab+bb)\\ &=aa+(ab+ab)+bb\\ &=aa+2ab+bb\end{aligned}\right\} \qquad \text{by (4).}$$

This gives the theorem in question.

In the same manner every one of the first ten propositions is proved.

It will be seen that the operations performed are exactly the same as if the letters denoted numbers.

Props. 5 and 6 may also be written thus—

$$(a+b)\,(a-b)=a^2-b^2.$$

Prop. 7, which is an easy consequence of Prop. 4, may be transformed. If we denote by c the line $a+b$, so that

$$c=a+b\,.\;a=c-b,$$

we get

$$c^2+(c-b)^2=2c(c-b)+b^2$$
$$=2c^2-2bc+b^2.$$

Subtracting c^2 from both sides, and writing a for c, we get

$$(a-b)^2=a^2-2ab+b^2.$$

In Euclid's *Elements* this form of the theorem does not appear, all propositions being so stated that the notion of subtraction does not enter into them.

§ 24. The remaining two theorems (Props. 12 and 13) connect the square on one side of a triangle with the sum of the squares on the other sides, in case that the angle between the latter is acute or obtuse. They are important theorems in trigonometry, where it is possible to include them in a single theorem.

§ 25. There are in the second book two problems, Props. 11 and 14.

If written in the above symbolic language, the former requires to find a line x such that $a(a-x)=x^2$. Prop. 11 contains, therefore, the solution of a quadratic equation, which we may write $x^2+ax=a^2$. The solution is required later on in the construction of a regular decagon.

More important is the problem in the last proposition (Prop. 14). It requires the construction of a square equal in area to a given rectangle, hence a solution of the equation

$$x^2=ab.$$

In Book I., 42–45, it has been shown how a rectangle may be constructed equal in area to a given figure bounded by straight lines. By aid of the new proposition we may therefore now determine a line such that the square on that line is equal in area to any given rectilinear figure, or we can *square* any such figure.

As of two squares that is the greater which has the greater side, it follows that now the comparison of two areas has been reduced to the comparison of two lines.

The problem of reducing other areas to squares is frequently met with among Greek mathematicians. We need only mention the problem of squaring the circle.

In the present day the comparison of areas is performed in a simpler way by reducing all areas to rectangles having a common base. Their altitudes give then a measure of their areas.

The construction of a rectangle having the base u, and being equal in area to a given rectangle, depends upon Prop. 43, I. This therefore gives a solution of the equation

$$ab=ux,$$

where x denotes the unknown altitude.

Book III.

§ 26. The third book of the *Elements* relates exclusively to properties of the circle. A circle and its circumference have been defined in Book I., Def. 15. We restate it here in slightly different words:—

Definition.—The circumference of a circle is a plane curve such that all points in it have the same distance from a fixed point in the plane. This point is called the "centre" of the circle.

Of the new definitions, of which eleven are given at the beginning of the third book, a few only require special mention. The first, which says that circles with equal radii are equal, is in part a theorem, but easily proved by applying the one circle to the other. Or it may be considered proved by aid of Prop. 24. equal circles not being used till after this theorem.

In the second definition is explained what is meant by a line which "touches" a circle. Such a line is now generally called a tangent to the circle. The introduction of this name allows us to state many of Euclid's propositions in a much shorter form.

For the same reason we shall call a straight line joining two points on the circumference of a circle a "chord."

Definitions 4 and 5 may be replaced with a slight generalization by the following:—

Definition.—By the distance of a point from a line is meant the length of the perpendicular drawn from the point to the line.

§ 27. From the definition of a circle it follows that every circle has a centre. Prop. 1 requires to find it when the circle is given, *i.e.*, when its circumference is drawn.

To solve this problem a chord is drawn (that is, any two points in the circumference are joined), and through the point where this is bisected a perpendicular to it is erected. Euclid then proves, first, that no point off this perpendicular can be the centre, hence that the centre must lie in this line; and, secondly, that of the points on the perpendicular one only can be the centre, viz., the one which bisects the part of the perpendicular bounded by the circle. In the second part Euclid silently assumes that the perpendicular there used does cut the circumference in two, and only in two points. The proof therefore is incomplete. The proof of the first part, however, is exact. By drawing two non-parallel chords, and the perpendiculars which bisect them, the centre will be found as the point where these perpendiculars intersect.

§ 28. In Prop. 2 it is proved that a chord of a circle lies altogether within the circle.

What we have called the first part of Euclid's solution of Prop. 1 may be stated as a theorem:—

Theorem.—*Every straight line which bisects a chord, and is at right angles to it, passes through the centre of the circle.*

The converse to this gives Prop. 3, which may be stated thus:—

If a straight line through the centre of a circle bisect a chord, then it is perpendicular to the chord, and if it be perpendicular to the chord it bisects it.

An easy consequence of this is the following theorem, which is essentially the same as Prop. 4:—

Theorem (Prop. 4).—*Two chords of a circle, of which neither passes through the centre, cannot bisect each other.*

These last three theorems are fundamental for the theory of the circle. It is to be remarked that Euclid never proves that a straight line cannot have more than two points in common with a circumference.

§ 29. The next two propositions (5 and 6) might be replaced by a single and a simpler theorem, viz.:—

Theorem.—*Two circles which have a common centre, and whose circumferences have one point in common, coincide.*

Or, more in agreement with Euclid's form:—

Theorem.—*Two different circles, whose circumferences have a point in common, cannot have the same centre.*

That Euclid treats of two cases is characteristic of Greek mathematics.

The next two propositions (7 and 8) again belong together. They may be combined thus:—

Theorem.—*If from a point in a plane of a circle, which is not the centre, straight lines be drawn to the different points of the circumference, then of all these lines one is the shortest, and one the longest, and these lie both in that straight line which joins the given point to the centre. Of all the remaining lines each is equal to one and only one other, and these equal lines lie on opposite sides of the shortest or longest, and make equal angles with them.*

Euclid distinguishes the two cases where the given point lies within or without the circle, omitting the case where it lies in the circumference.

From the last proposition it follows that if from a point more than two equal straight lines can be drawn to the circumference, this point must be the centre. This is Prop. 9.

As a consequence of this we get

Theorem.—*If the circumferences of the two circles have three points in common they coincide.*

For in this case the two circles have a common centre, because from the centre of the one three equal lines can be drawn to points on the circumference of the other. But two circles which have common centre, and whose circumferences have a point in common, coincide. (Compare above statement of Props. 5 and 6.)

This theorem may also be stated thus:—

Through three points only one circumference may be drawn; or, *Three points determine a circle.*

Euclid does not give the theorem in this form. He proves, however, *that the two circles cannot cut another in more than two points* (Prop. 10), and *that two circles cannot touch one another in more points than one* (Prop. 13).

§ 30. Propositions 11 and 12 assert that *if two circles touch, then the point of contact lies on the line joining their centres.* This gives two propositions, because the circles may touch either internally or externally.

§ 31. Propositions 14 and 15 relate to the length of chords. The first says; *that equal chords are equidistant from the centre, and that chords which are equidistant from the centre are equal*;

Whilst Prop. 15 compares unequal chords, viz., *Of all chords the diameter is the greatest, and of other chords that is the greater which is nearer to the centre*; and conversely, *the greater chord is nearer to the centre.*

§ 32. In Prop. 16 the tangent to a circle is for the first time introduced. The proposition is meant to show that the straight line at the end point of the diameter, and at right angles to it is a tangent. The proposition itself does not state this. It runs thus:—

Prop. 16. *The straight line drawn at right angles to the diameter of a circle, from the extremity of it, falls without the circle; and no straight line can be drawn from the extremity, between that straight line and the circumference, so as not to cut the circle.*

Corollary.—The straight line at right angles to a diameter drawn through the end point of it touches the circle.

The statement of the proposition and its whole treatment show the difficulties which the tangents presented to Euclid.

Prop. 17 solves the problem *through a given point, either in the circumference or without it, to draw a tangent to a given circle.*

Closely connected with Prop. 16 are Props. 18 and 19, which state (Prop. 18), *that the line joining the centre of a circle to the point of contact of a tangent is perpendicular to the tangent*; and conversely (Prop. 19), *that the straight line through the point of contact of, and perpendicular to, a tangent to a circle passes through the centre of the circle.*

§ 33. The rest of the book relates to angles connected with a circle, viz., angles which have the vertex either at the centre or on the circumference, and which are called respectively angles at the centre and angles at the circumference. Between these two kinds of angles exists the important relation expressed as follows:—

Prop. 20. *The angle at the centre of a circle is double of the angle at the circumference on the same base, that is, on the same arc.*

This is of great importance for its consequences, of which the two following are the principal:—

Prop. 21. *The angles in the same segment of a circle are equal to one another*;

And Prop. 22. *The opposite angles of any quadrilateral figure inscribed in a circle are together equal to two right angles.*

Further consequences are:—

Prop. 23. *On the same straight line, and on the same side of it, there cannot be two similar segments of circles, not coinciding with one another*;

And Prop. 24. *Similar segments of circles on equal straight lines are equal to one another.*

The problem Prop. 25, *A segment of a circle being given to describe the circle of which it is a segment*, may be solved much more easily by aid of the construction described in relation to Prop. 1, III., in § 27.

§ 34. There follow four theorems connecting the angles at the centre, the arcs into which they divide the circumference, and the chords subtending these arcs. They are expressed for angles, arcs, and chords in equal circles, but they hold also for angles, arcs, and chords in the same circle.

The theorems are:—

Prop. 26. *In equal circles equal angles stand on equal arcs, whether they be at the centres or circumferences*;

Prop. 27 (converse to Prop. 26). *In equal circles the angles which stand on equal arcs are equal to one another, whether they be at the centres or the circumferences*;

Prop. 28. *In equal circles equal straight lines* (equal chords) *cut off equal arcs, the greater equal to the greater, and the less equal to the less*;

Prop. 29 (converse to Prop. 28). *In equal circles equal arcs are subtended by equal straight lines.*

§ 35. Other important consequences of Props. 20–22 are:—

Prop. 31. *In a circle the angle in a semicircle is a right angle; but the angle in a segment greater than a semicircle is less than a right angle; and the angle in a segment less than a semicircle is greater than a right angle*;

Prop. 32. *If a straight line touch a circle, and from the point of contact a straight line be drawn cutting the circle, the angles which this line makes with the line touching the circle shall be equal to the angles which are in the alternate segments of the circle.*

§ 36. Propositions 30, 33, 34, contain problems which are solved by aid of the propositions preceding them:—

Prop. 30. *To bisect a given arc, that is, to divide it into two equal parts*;

Prop. 33. *On a given straight line to describe a segment of a circle containing an angle equal to a given rectilineal angle*;

Prop. 34. *From a given circle to cut off a segment containing an angle equal to a given rectilineal angle.*

§ 37. If we draw chords through a point A within a circle, they will each be divided by A into two segments. Between these segments the law holds that the rectangle contained by them has the same area on whatever chord through A the segments are taken. The value of this rectangle changes, of course, with the position of A.

A similar theorem holds if the point A be taken without the circle. On every straight line through A, which cuts the circle in two points B and C, we have two segments AB and AC, and the rectangles contained by them are again equal to one another, and equal to the square on a tangent drawn from A to the circle.

The first of these theorems gives Prop. 35, and the second Prop. 36, with its corollary, whilst Prop. 37, the last of Book III., gives the converse to Prop. 36. The first two theorems may be combined in one:—

Theorem.—*If through a point A in the plane of a circle a straight line be drawn cutting the circle in B and C, then the rectangle AB.AC has a constant value so long as the point A be fixed; and if from A a tangent AD can be drawn to the circle, touching at D, then the above rectangle equals the square on AD.*

Prop. 37 may be stated thus:—

Theorem.—*If from a point A without a circle a line be drawn cutting the circle in B and C, and another line to a point D on the circle, and if $AB.AC=AD^2$, then the line AD touches the circle at D.*

It is not difficult to prove also the converse to the general proposition as above stated. It may be expressed as follows:—

If four points ABCD be taken on the circumference of a circle, and if the lines AB, CD, produced if necessary, meet at E, then

$$EA\,.\,EB=EC\,.\,ED;$$

and conversely, if this relation holds then the four points lie on a circle, that is, the circle drawn through three of them passes through the fourth.

That a circle may always be drawn through three points, provided that they do not lie in a straight line, is proved only later on in Book IV.

Book IV.

§ 38. The fourth book contains only problems, all relating to the construction of triangles and polygons inscribed in and circumscribed about circles, and of circles inscribed in or circumscribed about triangles and polygons. They are nearly all given for their own sake, and not for future use in the construction of figures, as are most of those in the former books. In seven definitions at the beginning of the book it is explained what is understood by figures inscribed in or described about other figures, with special reference to the case where one figure is a circle. Instead, however, of saying that one figure is described about another, it is now generally said that the one figure is circumscribed about the other. We may then state the definitions 3 or 4 thus:—

Definition.—A polygon is said to be inscribed in a circle, and the circle is said to be circumscribed about the polygon, if the vertices of the polygon lie in the circumference of the circle.

And definitions 5 and 6 thus:—

Definition.—A polygon is said to be circumscribed about a circle, and a circle is said to be inscribed in a polygon, if the sides of the polygon are tangents to the circle.

§ 39. The first problem is merely constructive. It requires to draw in a given circle a chord equal to a given straight line, which is not greater than the diameter of the circle. The problem is not a determinate one, inasmuch as the chord may be drawn from any point in the circumference. This may be said of almost all problems in this book, especially of the next two. They are:—

Prop. 2. *In a given circle to inscribe a triangle equiangular to a given triangle*;

Prop. 3. *About a given circle to circumscribe a triangle equiangular to a given triangle.*

§ 40. Of somewhat greater interest are the next problems, where the triangles are given and the circles to be found.

Prop. 4. *To inscribe a circle in a given triangle.*

The result is that the problem has always a solution, viz., the centre of the circle is the point where the bisectors of two of the interior angles of the triangle meet. The solution shows, though Euclid does not state this, that the problem has but one solution; and also,

Theorem.—*The three bisectors of the interior angles of any triangle* *meet in a point, and this is the centre of the circle inscribed in the triangle.*

The solutions of most of the other problems contain also theorems. Of these we shall state those which are of special interest; Euclid does not state any one of them.

§ 41. Prop. 5. *To circumscribe a circle about a given triangle.*

The one solution which always exists contains the following:—

THEOREM.—*The three straight lines which bisect the sides of a triangle at right angles meet in a point, and this point is the centre of the circle circumscribed about the triangle.*

Euclid adds in a corollary the following property:—

The centre of the circle circumscribed about a triangle lies within, on a side of, or without the triangle, according as the triangle is acute-angled, right-angled, or obtuse-angled.

§ 42. Whilst it is always possible to draw a circle which is inscribed in or circumscribed about a given triangle, this is not the case with quadrilaterals or polygons of more sides. Of those for which this is possible the regular polygons are the most interesting. In each of them a circle may be inscribed. and another may be circumscribed about it.

Euclid does not use the word regular, but he describes the polygons in question as *equiangular* and *equilateral*. We shall use the name regular polygon. The regular triangle is equilateral, the regular quadrilateral is the square.

Euclid considers the regular polygons of 4, 5, 6, and 15 sides. For each of the first three he solves the problems—(1) to inscribe such a polygon in a given circle; (2) to circumscribe it about a given circle; (3) to inscribe a circle in, and (4) to circumscribe a circle about, such a polygon.

For the regular triangle the problems are not repeated, because more general problems have been solved.

Props. 6, 7, 8, and 9 solve these problems for the square.

The general problem of inscribing in a given circle a regular polygon of n sides depends upon the problem of dividing the circumference of a circle into n equal parts, or what comes to the same thing, of drawing from the centre of the circle n radii such that the angles between consecutive radii are equal, that is, to divide the space about the centre into n equal angles. Thus, if it is required to inscribe a square in a circle, we have to draw four lines from the centre, making the four angles equal. This is done by drawing two diameters at right angles to one another. The ends of these diameters are the vertices of the required square. If, on the other hand, tangents be drawn at these ends, we obtain a square circumscribed about the circle.

§ 43. To construct a *regular pentagon*, we find it convenient first to construct a *regular decagon*. This requires to divide the space about the centre into ten equal angles. Each will be $\frac{1}{10}$th of a right angle, or $\frac{1}{5}$th of two right angles. If we suppose the decagon constructed, and if we join the centre to the end of one side, we get an isoceles triangle, where the angle at the centre equals $\frac{1}{5}$th of two right angles; hence each of the angles at the base will be $\frac{2}{5}$ths of two right angles, as all three angles together equal two right angles. Thus we have to construct an isoceles triangle, having the angle at the vertex equal to half an angle at the base. This is solved in Prop. 10, by aid of the problem in Prop. 11 of the second book. If we make the sides of this triangle equal to the radius of the given circle, then the base will be the side of the regular decagon inscribed in the circle. This side being known the decagon can be constructed, and if the vertices are joined alternately, leaving out half their number, we obtain the regular pentagon.

Euclid does not proceed thus. He wants the pentagon before the decagon. This, however, does not change the real nature of his solution, nor does his solution become simpler by not mentioning the decagon.

Once the regular pentagon is inscribed, it is easy to circumscribe another by drawing tangents at the vertices of the inscribed pentagon. This is shown in Prop. 12.

Prop. 13 and 14 teach how a circle may be inscribed in or circumscribed about any given regular pentagon.

§ 44. The *regular hexagon* is more easily constructed, as shown in Prop. 15. The result is that the side of the regular hexagon inscribed in a circle is equal to the radius of the circle.

For this polygon the other three problems mentioned are not solved.

§ 45. The book closes with Prop. 16. To inscribe a regular quindecagon in a given circle. That this may be done is easily seen. If we inscribe a regular pentagon and a regular hexagon in the circle, having one vertex in common, then the arc from the common vertex to the next vertex of the pentagon is $\frac{1}{5}$th of the circumference, and to the next vertex of the hexagon is $\frac{1}{6}$th of the circumference. The difference between these arcs is, therefore, $\frac{1}{5}-\frac{1}{6}=\frac{1}{30}$th of the circumference. The latter may, therefore, be divided into thirty, and hence also in fifteen equal parts, and the regular quindecagon be described.

§ 46. We conclude with a few theorems about regular polygons which are not given by Euclid.

THEOREM.—*The straight lines perpendicular to and bisecting the sides of any regular polygon meet in a point. The straight lines bisecting the angles in the regular polygon meet in the same point. This point is the centre of the circles circumscribed about and inscribed in the regular polygon.* The proof, which is easy, is left to the reader.

We can bisect any given arc (Prop. 30, III.). Hence we can divide a circumference into $2n$ equal parts as soon as it has been divided into n equal parts, or as soon as a regular polygon of n sides has been constructed. Hence—

THEOREM.—*If a regular polygon of* n *sides has been constructed, then a regular polygon of* 2n *sides, of* 4n, *of* 8n *sides, &c., may also be constructed.* Euclid shows how to construct regular polygons of 3, 4, 5, and 15 sides. It follows that we can construct regular polygons of

3, 6, 12, 24 . . . sides
4, 8, 16, 32 . . . „
5, 10, 20, 40 . . . „
15, 30, 60, 120 . . . „

The construction of any new regular polygon not included in one of these series will give rise to a new series. Till the beginning of this century nothing was added to the knowledge of regular polygons as given by Euclid. Then Gauss, in his celebrated *Arithmetic*, proved that every regular polgon of 2^n+1 sides may be constructed if this number 2^n+1 be prime, and that no others can be constructed by elementary methods. This shows that regular polygons of 7, 9, 13 sides cannot thus be constructed, but that a regular polygon of 17 sides is possible; for $17=2^4+1$. The next polygon is one of 257 sides. The construction becomes already rather complicated for 17 sides.

BOOK V.

§ 47. The fifth book of the *Elements* is not exclusively geometrical. It contains the theory of ratios and proportion of quantities in general. The treatment, as here given, is admirable, and in every respect superior to the algebraical method by which Euclid's theory is now generally replaced. It has, however, the reputation of being too difficult for schools, and is therefore very seldom read. We shall try to make the subject clear, and to show why the usual algebraical treatment of proportion is not really sound. We begin by quoting those definitions at the beginning of Book V. which are most important. These definitions have given rise to much discussion.

The only definitions which are essential for the fifth book are Defs. 1, 2, 4, 5, 6, and 7. Of the remainder 3, 8, and 9 are more than useless, and probably not Euclid's, but additions of later editors, of whom Theon of Alexandria was the most prominent. Defs. 10 and 11 belong rather to the sixth book, whilst all the others are merely nominal. The really important ones are 4. 5. 6, and 7.

§ 48. To define a magnitude is not attempted by Euclid. The first two definitions state what is meant by a "part," that is, a submultiple or measure, and by a "multiple" of a given magnitude. The meaning of Def. 4 is that two given quantities can have a ratio to one another only in case that they are comparable as to their magnitude, that is, if they are of the same kind.

Def. 3, which is probably due to Theon, professes to define a ratio, but is as meaningless as it is uncalled for, for all that is wanted is given in Defs. 5 and 7.

In Def. 5 it is explained what is meant by saying that two magnitudes have the same ratio to one another as two other magnitudes, and in Def. 7 what we have to understand by a greater or a less ratio. The 6th definition is only nominal, explaining the meaning of the word *proportional*.

Euclid represents magnitudes by lines, and often denotes them either by single letters or, like lines, by two letters. We shall use only single letters for the purpose. If a and b denote two magnitudes of the same kind, their ratio will be denoted by $a:b$; if c and d are two other magnitudes of the same kind, but possibly of a different kind from a and b, then if c and d have the same ratio to one another as a and b, this will be expressed by writing—

$$a:b::c:d.$$

Further, if m is a (whole) number, ma shall denote the multiple of a which is obtained by taking it m times.

§ 49. The whole theory of ratios is based on Def. 5.

Def. 5. *The first of four magnitudes is said to have the same ratio to the second that the third has to the fourth when, any equimultiples whatever of the first and the third being taken, and any equimultiples whatever of the second and the fourth, if the multiple of the first be less than that of the second, the multiple of the third is also less than that of the fourth; and if the multiple of the first is equal to that of the second, the multiple of the third is also equal to that of the fourth; and if the multiple of the first is greater than that of the second, the multiple of the third is also greater than that of the fourth.*

It will be well to show at once in an example how this definition can be used, by proving the first part of the first proposition in the sixth book. *Triangles of the same altitude are to one another as their bases*, or if a and b are the bases, and α and β the areas, of two triangles which have the same altitude, then $a:b::\alpha:\beta$.

To prove this, we have, according to Definition 5, to show—

$$\text{if } ma>nb, \text{ then } m\alpha>n\beta,$$
$$\text{if } ma=nb, \text{ then } m\alpha=n\beta,$$
$$\text{if } ma<nb, \text{ then } m\alpha<n\beta.$$

That this is true is in our case easily seen. We may suppose that the triangles have a common vertex, and their bases in the same line. We set off the base a along the line containing the bases m times; we then join the different parts of division to the vertex, and get m triangles all equal to α. The triangle on ma as base equals, therefore, $m\alpha$. If we proceed in the same manner with the base b, setting it off n times, we find that the area of the triangle on the base nb equals $n\beta$, the vertex of all triangles being the same. But if two triangles have the same altitude, then their areas are equal if the bases are equal; hence $m\alpha=n\beta$ if $ma=nb$, and if their bases are unequal, then that has the greater area which is on the greater base; in other words, $m\alpha$ is greater than, equal to, or less than $n\beta$, according as ma is greater than. equal to, or less than nb, which was to be proved.

§ 50. It will be seen that even in this example it does not become evident what a ratio really is. It is still an open question whether ratios are magnitudes which we can compare. We do not know whether the ratio of two lines is a magnitude of the same kind as the ratio of two areas. Though we might say that Def. 5 defines *equal* ratios, still we do not know whether they are equal in the sense of the axiom, that two things which are equal to a third are equal to one another. That this is the case requires a proof, and until this proof is given we shall use the :: instead of the sign =, which, however, we shall afterwards introduce.

As soon as it has been established that all ratios are like magnitudes, it becomes easy to show that, in some cases at least, they are numbers. This step was never made by Greek mathematicians. They distinguished always most carefully between continuous magnitudes and the discrete series of numbers. In modern times it has become the custom to ignore this difference.

If, in determining the ratio of two lines, a common measure can be found, which is contained m times in the first, and n times in the second, then the ratio of the two lines equals the ratio of the two numbers $m:n$. This is shown by Euclid in Prop, 5, X. But the ratio of two numbers is, as a rule, a fraction, and the Greeks did not, as we do, consider fractions as numbers. Far less had they any notion of introducing irrational numbers, which are neither whole nor fractional, as we are obliged to do if we wish to say that all ratios are numbers. The incommensurable numbers which are thus introduced as ratios of incommensurable quantities are now-a-days as familiar to us as fractions; but a proof is generally omitted that we may apply to them the rules which have been established for rational numbers only. Euclid's treatment of ratios avoids this difficulty. His definitions holds for commensurable as well as for incommensurable quantities. Even the notion of incommensurable quantities is avoided in Book V. But he proves that the more elementary rules of algebra hold for ratios. We shall state all his propositions in that algebraical form to which we are now accustomed. This may, of course, be done without changing the character of Euclid's method.

§ 51. Using the notation explained above we express the first propositions as follows:—

Prop. 1. If $a=ma',\ b=mb',\ c=mc'$,
then $a+b+c=m(a'+b'+c')$.

Prop. 2. If $a=mb$, and $c=md$,
$e=nb$, and $f=nd$,
then $a+e$ is the same multiple of b as $c+f$ is of d, viz.:—

$$a+e=(m+n)b, \text{ and } c+f=(m+n)d.$$

Prop. 3. If $a=mb,\ c=md$, then is na the same multiple of b that nc is of d, viz., $na=nmb,\ nc=nmd$.

Prop. 4. If $a:b::c:d$,
then $ma:nb::mc:nd$.

Prop. 5. If $a=mb$, and $c=md$
then $a-c=m(b-d)$.

Prop. 6. If $a=mb,\ c=md$,
then are $a-nb$ and $c-nd$ either equal to, or equimultiples of, b and d, viz., $a-nb=(m-n)b$ and $c-nd=(m-n)d$, where $m-n$ may be unity.

All these propositions relate to *equimultiples*. Now follow propositions about ratios which are compared as to their magnitude.

§ 52. Prop. 7. If $a=b$, then $a:c::b:c$ and $c:a::c:b$.

The proof is simply this. As $a=b$ we know that $ma=mb$:
therefore if $ma>nc$, then $mb>nc$,
if $ma=nc$, then $mb=nc$,
if $ma<nc$, then $mb<nc$,
therefore the first proportion holds by Definition 5.

Prop. 8. If $a>b$, then $a:c>b:c$,
and $c:a<c:b$.
The proof depends on Definition 7.

Prop. 9 (converse to Prop. 7). If
$a:c::b:c$,
or if $c:a::c:b$, then $a=b$.

Prop. 10 (converse to Prop. 8). If
$a:c>b:c$, then $a>b$
and if $c:a<c:b$, then $a<b$.

Prop. 11. If $a:b::c:d$,
and $a:b::e:f$,
then $c:d::e:f$.

In words, *if two ratios are equal to a third, they are equal to one another.* After these propositions have been proved, we have a right to consider a ratio as a *magnitude*, for only now can we consider a ratio as something for which the axiom about magnitudes holds: things which are equal to a third are equal to one another.

We shall indicate this by writing in future the sign = instead of ::. The remaining propositions. which explain themselves, may then be stated as follows:—

§ 53. Prop. 12. If $a:b=c:d=e:f$,
then $a+c+e:b+d+f=a:b$.

Prop. 13. If $a:b=c:d$ and $c:d>e:f$,
then $a:b>e:f$.

Prop. 14. If $a:b=c:d$, and $a>c$, then $b>d$.

Prop. 15. Magnitudes have the same ratio to one another that their equimultiples have—

$$ma:mb=a:b.$$

Prop. 16. If a b, c, d are magnitudes of the same kind, and if
$a:b=c:d$,
then $a:c=b:d$.

Prop. 17. If $a+b:b=c+d:d$,
then $a:b=c:d$.

Prop. 18 (converse to 17). If
$a:b=c:d$
then $a+b:b=c+d:d$.

Prop. 19. If a, b, c, d are quantities of the same kind, and if
$a:b=c:d$,
then $a-c:b-d=a:b$.

§ 54. Prop. 20. *If there be three magnitudes, and other three, which have the same ratio, taken two and two, then if the first be greater than the third, the fourth shall be greater than the sixth; and if equal, equal; and if less. less.*

If we understand by

$$a:b:c:d:e:\ldots=a':b':c':d':e':\ldots$$

that the ratio of any two consecutive magnitudes on the first side equals that of the corresponding magnitudes on the second side. we may write this theorem in symbols, thus:—

If a, b, c be quantities of one, and d, e, f magnitudes of the same or any other kind, such that
$a:b:c=d:e:f$,
and if $a>c$, then $d>f$,
but if $a=c$, then $d=f$,
and if $a<c$, then $d<f$.

Prop. 21. If $a:b=e:f$ and $b:c=d:e$,
or if $a:b:c=\frac{1}{f}:\frac{1}{e}:\frac{1}{d}$,
and if $a>c$, then $d>f$,
but if $a=c$, then $d=f$,
and if $a<c$, then $d<f$.

By aid of these two propositions the following two are proved.

§ 55. Prop. 22. *If there be any number of magnitudes, and as many others, which have the same ratio, taken two and two in order, the first shall have to the last of the first magnitudes the same ratio which the first of the others has to the last.*

We may state it more generally, thus:—

If $a:b:c:d:e:\ldots=a':b':c':d':e':\ldots$,

then not only have two consecutive, but any two magnitudes on the first side, the same ratio as the corresponding magnitudes on the other. For instance—

$$a:c=a':c';\quad b:e=b':e', \&c.$$

Prop. 23 we state only in symbols, viz.:—

If $a:b:c:d:e:\ldots=\frac{1}{a'}:\frac{1}{b'}:\frac{1}{c'}:\frac{1}{d'}:\frac{1}{e'}\ldots$