then

$$a : c = c' : a',$$
$$b : e = e' : b',$$

and so on.

Prop. 24 comes to this : If $a : b = c : d$ and $e : b = f : d$, then

$$a + e : b = c + f : d.$$

Some of the proportions which are considered in the above propositions have special names. These we have omitted, as being of no use, since algebra has enabled us to bring the different operations contained in the propositions under a common point of view.

§ 56. The last proposition in the fifth book is of a different character.

Prop. 25. *If four magnitudes of the same kind be proportional, the greatest and least of them together shall be greater than the other two together.* In symbols—

If a, b, c, d be magnitudes of the same kind, and if $a : b = c : d$, and if a is the greatest, hence d the least, then $a + d > b + c$.

§ 57. We return once again to the question, What is a ratio? We have seen that we may treat ratios as magnitudes, and that all ratios are magnitudes of the same kind, for we may compare any two as to their magnitude. It will presently be shown that ratios of lines may be considered as *quotients* of lines, so that a ratio appears as answer to the question, How often is one line contained in another? But the answer to this question is given by a number, at least in some cases, and in all cases if we admit incommensurable numbers. Considered from this point of view, we may say the fifth book of the *Elements* shows that some of the simpler algebraical operations hold for incommensurable numbers. In the ordinary algebraical treatment of numbers this proof is altogether omitted, or given by a process of limits which does not seem to be natural to the subject.

Book VI.

§ 58. The sixth book contains the theory of similar figures. After a few definitions explaining terms, the first proposition gives the first application of the theory of proportion.

Prop. 1. *Triangles and parallelograms of the same altitude are to one another as their bases.*

The proof has already been considered in § 49.

From this follows easily the important theorem

Prop. 2. *If a straight line be drawn parallel to one of the sides of a triangle, it shall cut the other sides, or those sides produced, proportionally; and if the sides or the sides produced be cut proportionally, the straight line which joins the points of section, shall be parallel to the remaining side of the triangle.*

§ 59. The next proposition, together with one added by Simson as Prop. A, may be expressed more conveniently if we introduce a modern phraseology, viz., if in a line AB we assume a point C between A and B, we shall say that C divides AB internally in the ratio AC : CB; but if C be taken in the line AB produced, we shall say that AB is divided externally in the ratio AC : CB.

The two propositions then come to this :

Theorem (Prop. 3).—*The bisector of an angle in a triangle divides the opposite side internally in a ratio equal to the ratio of the two sides including that angle;* and conversely, *if a line through the vertex of a triangle divide the base internally in the ratio of the two other sides, then that line bisects the angle at the vertex.*

Theorem (Simson's Prop. A).—*The line which bisects an exterior angle of a triangle divides the opposite side externally in the ratio of the other sides;* and conversely, *if a line through the vertex of a triangle divide the base externally in the ratio of the sides, then it bisects an exterior angle at the vertex of the triangle.*

If we combine both we have—

Theorem.—*The two lines which bisect the interior and exterior angles at one vertex of a triangle divide the opposite side internally and externally in the same ratio, viz.. in the ratio of the other two sides.*

§ 60. The next four propositions contain the theory of similar triangles; of which four cases are considered. They may be stated together.

Theorem.—*Two triangles are similar,—*

1. (Prop. 4). *If the triangles are equiangular;*
2. (Prop. 5). *If the sides of the one are proportional to those of the other;*
3. (Prop. 6). *If two sides in one are proportional to two sides in the other, and if the angles contained by these sides are equal;*
4. (Prop. 7). *If two sides in one are proportional to two sides in the other, if the angles opposite homologous sides are equal, and if the angles opposite the other homologous sides are both acute, both right, or both obtuse; homologous sides being in each case those which are opposite equal angles.*

An important application of these theorems is at once made to a right-angled triangle, viz.:—

Prop. 8. *In a right-angled triangle, if a perpendicular be drawn from the right angle to the base, the triangles on each side of it are similar to the whole triangle, and to one another.*

Corollary.—From this it is manifest that the perpendicular drawn from the right angle of a right-angled triangle to the base is a mean proportional between the segments of the base, and also that each of the sides is a mean proportional between the base and the segment of the base adjacent to that side.

§ 61. There follow four propositions containing problems, viz., in language slightly different from Euclid's:—

Prop. 9. *To divide a straight line into a given number of equal parts.*

Prop. 10. *To divide a straight line in a given ratio.*

Prop. 11. *To find a third proportional to two given straight lines.*

Prop. 12. *To find a fourth proportional to three given straight lines.*

Prop. 13. *To find a mean proportional between two given straight lines.*

The last three may be written as equations with one unknown quantity,—viz., if we call the given straight lines a, b, c, and the required line x, we have to find a line x so that

Prop. 11. $a : b = b : x$;
Prop. 12. $a : b = c : x$;
Prop. 13. $a : x = x : b$.

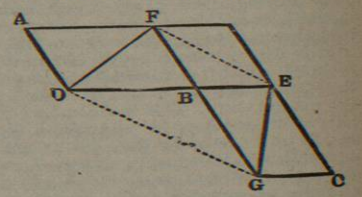
We shall see presently how these may be written without the signs of ratios.

§ 62. Euclid considers next proportions connected with parallelograms and triangles which are equal in area.

Prop. 14. *Equal parallelograms which have one angle of the one equal to one angle of the other have their sides about the equal angles reciprocally proportional; and parallelograms which have one angle of the one equal to one angle of the other, and their sides about the equal angles reciprocally proportional, are equal to one another.*

Prop. 15. *Equal triangles which have one angle of the one equal to one angle of the other, have their sides about the equal angles reciprocally proportional; and triangles which have one angle of the one equal to one angle of the other, and their sides about the equal angles reciprocally proportional, are equal to one another.*

The latter proposition is really the same as the former, for if, as in the accompanying diagram, in the figure belonging to the former the two equal parallelograms AB and BC be bisected by the lines DF and EG, and if EF be drawn, we get the figure belonging to the latter.



It is worth noticing that the lines FE and DG are parallel. We may state therefore the theorem—

Theorem.—*If two triangles are equal in area, and have one angle in the one vertically opposite to one angle in the other, then the two straight lines which join the remaining two vertices of the one to those of the other triangle are parallel.*

§ 63. A most important theorem is

Prop. 16. *If four straight lines be proportionals, the rectangle contained by the extremes is equal to the rectangle contained by the means; and if the rectangle contained by the extremes be equal to the rectangle contained by the means, the four straight lines are proportionals.*

In symbols, if a, b, c, d are the four lines, and

if $a : b = c : d$,
then $ad = bc$;
and conversely, if $ad = bc$,
then $a : b = c \cdot d$,

where ad and bc denote (as in § 20), the areas of the rectangles contained by a and d and by b and c respectively.

This allows us to transform every proportion between four lines into an equation between two products.

It shows further that the operation of forming a product of two lines, and the operation of forming their ratio are each the inverse of the other.

If we now define a quotient $\frac{a}{b}$ of two lines as the *number* which multiplied into b gives a, so that

$$\frac{a}{b} b = a,$$

we see that from the equality of two quotients

$$\frac{a}{b} = \frac{c}{d}$$

follows, if we multiply both sides by bd,

$$\frac{a}{b} b \, . \, d = \frac{c}{d} d \, . \, b,$$
$$ad = cb.$$

But from this it follows according to the last theorem that

$$a : b = c : d.$$

Hence we conclude that the quotient $\frac{a}{b}$ and the ratio $a : b$ are different forms of the same magnitude, only with this important difference that the quotient $\frac{a}{b}$ would have a meaning only if a and b have a common measure, until we introduce incommensurable numbers, while the ratio $a : b$ has always a meaning, and thus gives rise to the introduction of incommensurable numbers.

Thus it is really the theory of ratios in the fifth book which enables us to extend the geometrical calculus given before in connexion with Book II. It will also be seen that if we write the ratios in Book V. as quotients, or rather as fractions, then most of the theorems state properties of quotients or of fractions.

§ 64. Prop. 17 contains only a special case of 16. After the problem, Prop. 18. *On a given straight line to describe a rectilineal figure similar and similarly situated to a given rectilineal figure,* there follows another fundamental theorem:

Prop. 19. *Similar triangles are to one another in the duplicate ratio of their homologous sides.* In other words, the areas of similar triangles are to one another as the squares on homologous sides. This is generalized in

Prop. 20. *Similar polygons may be divided into the same number of similar triangles, having the same ratio to one another that the polygons have; and the polygons are to one another in the duplicate ratio of their homologous sides.*

§ 65. Prop. 21. *Rectilineal figures which are similar to the same rectilineal figure are also similar to each other,* is an immediate consequence of the definition of similar figures. As similar figures may be said to be equal in "shape" but not in "size," we may state it also thus:

"Figures which are equal in shape to a third are equal in shape to each other."

Prop. 22. *If four straight lines be proportionals, the similar rectilineal figures similarly described on them shall also be proportionals; and if the similar rectilineal figures similarly described on four straight lines be proportionals, those straight lines shall be proportionals.*

This is essentially the same as the following :—

If $a : b = c : d$,
then $a^2 : b^2 = c^2 : d^2$.

§ 66. Now follows a proposition which has been much discussed with regard to Euclid's exact meaning in saying that a ratio is *compounded* of two other ratios, viz. :

Prop. 23. *Parallelograms which are equiangular to one another, have to one another the ratio which is compounded of the ratios of their sides.*

The proof of the proposition makes its meaning clear. In symbols the ratio $a : c$ is compounded of the two ratios $a : b$ and $b : c$, and if $a : b = a' : b'$, $b : c = b'' : c''$, then $a : c$ is compounded of $a' : b'$ and $a'' : b''$.

If we consider the ratios as numbers, we may say that the one ratio is the product of those of which it is compounded, or in symbols.

$$\frac{a}{c} = \frac{a}{b} \cdot \frac{b}{c} = \frac{a'}{b'} \cdot \frac{b''}{c''}, \text{ if } \frac{a}{b} = \frac{a'}{b'} \text{ and } \frac{b}{c} = \frac{b''}{c''}.$$

The theorem in Prop. 23 is the foundation of all mensuration of areas. From it we see at once that two rectangles have the ratio of their areas compounded of the ratios of their sides.

If A is the area of a rectangle contained by a and b, and B that of a rectangle contained by c and d, so that $A = ab$, $B = cd$, then $A : B = ab : cd$, and this is, the theorem says, compounded of the ratios $a : c$ and $b : d$. In forms of quotients,

$$\frac{a}{c} \cdot \frac{b}{d} = \frac{ab}{cd}.$$

This shows how to multiply quotients in our geometrical calculus.

Further, *Two triangles have the ratios of their areas compounded of the ratios of their bases and their altitude.* For a triangle is equal in area to half a parallelogram which has the same base and the same altitude.

To bring these theorems to the form in which they are usually given, we assume a straight line u as our unit of length (generally an inch, a foot, a mile, &c.), and determine the number α which expresses how often u is contained in a line a, so that α denotes the ratio $a : u$ whether commensurable or not, and that $a = \alpha u$. We call this number α the numerical value of a. If in the same manner β be the numerical value of a line b we have

$$a : b = \alpha : \beta ;$$

in words: *The ratio of two lines (and of two like quantities in general) is equal to that of their numerical values.*

This is easily proved by observing that $a = \alpha u$, $b = \beta u$, therefore $a : b = \alpha u : \beta u$, and this may without difficulty be shown to equal $\alpha : \beta$.

If now a, b be base and altitude of one, a', b' those of another parallelogram, α, β and α', β' their numerical values respectively, and A, A' their areas, then

$$\frac{A}{A'} = \frac{a}{a'} \cdot \frac{b}{b'} = \frac{\alpha}{\alpha'} \cdot \frac{\beta}{\beta'} = \frac{\alpha\beta}{\alpha'\beta'}.$$

In words: *The areas of two parallelograms are to each other as the products of the numerical values of their bases and altitudes.*

If especially the second parallelogram is the unit square, *i.e.*, a square on the unit of length, then $\alpha' = \beta' = 1$, $A' = u^2$, and we have

$$\frac{A}{A'} = \alpha\beta \, . \text{ or } A = \alpha\beta \, . \, u^2.$$

This gives the theorem: The number of unit squares contained in a parallelogram equals the product of the numerical values of base and altitude, and similarly the number of unit squares contained in a triangle equals half the product of the numerical values of base and altitude.

This is often stated by saying that the area of a parallelogram is equal to the product of the base and the altitude, meaning by this product the product of the numerical values, and not the product as defined above in § 20.

§ 68. Propositions 24 and 26 relate to parallelograms about diagonals, such as are considered in Book I., 43. They are—

Prop. 24. *Parallelograms about the diameter of any parallelogram are similar to the whole parallelogram and to one another;* and its converse (Prop. 26), *If two similar parallelograms have a common angle, and be similarly situated, they are about the same diameter.*

Between these is inserted a problem.

Prop. 25. *To describe a rectilineal figure which shall be similar to one given rectilinear figure, and equal to another given rectilineal figure.*

§ 69. Prop. 27 contains a theorem relating to the theory of maxima and minima. We may state it thus:

Prop 27. *If a parallelogram be divided into two by a straight line cutting the base, and if on half the base another parallelogram be constructed similar to one of those parts, then this third parallelogram is greater than the other part.*

Of far greater interest than this general theorem is a special case of it, where the parallelograms are changed into rectangles, and where one of the parts into which the parallelogram is divided is made a square; for then the theorem changes into one which is easily recognized to be identical with the following:—

Theorem.—*Of all rectangles which have the same perimeter the square has the greatest area.*

This may also be stated thus:—

Theorem.—*Of all rectangles which have the same area the square has the least perimeter.*

§ 70. The next three propositions contain problems which may be said to be solutions of quadratic equations. The first two are, like the last, involved in somewhat obscure language. We transcribe them as follows:—

Problem.—To describe on a given base a parallelogram, and to divide it either internally (Prop. 28) or externally (Prop. 29) from a point on the base into two parallelograms, of which the one has a given size (is equal in area to a given figure), whilst the other has a given shape (is similar to a given parallelogram).

If we express this again in symbols, calling the given base a, the one part x, and the altitude y, we have to determine x and y in the first case from the equations

$$(a - x)y = k^2,$$
$$\frac{x}{y} = \frac{p}{q},$$

k^2 being the given size of the first, and p and q the base and altitude of the parallelogram which determine the shape of the second of the required parallelograms.

If we substitute the value of y, we get

$$(a - x)x = \frac{pk^2}{q},$$

or,

$$ax - x^2 = b^2,$$

where a and b are known quantities, taking $b^2 = \frac{rk^2}{a}$.

The second case (Prop. 29) gives rise, in the same manner, to the quadratic

$$ax + x^2 = b^2.$$

The next problem—

Prop. 30. *To cut a given straight line in extreme and mean ratio,* leads to the equation

$$ax + x^2 = a^2.$$

This is, therefore, only a special case of the last, and is, besides, an old acquaintance, being essentially the same problem as that proposed in II. 11.

Prop. 30 may therefore be solved in two ways, either by aid of Prop. 29 or by aid of II. 11. Euclid **gives both solutions.**

§ 71. Prop. 31 (Theorem). *In any right-angled triangle, any rectilineal figure described on the side subtending the right angle is equal to the similar and similarly-described figures on the sides containing the right angle,*—is a pretty generalization of the theorem of Pythagoras (I. 47).

Leaving out the next proposition, which is of little interest, we come to the last in this book.

Prop. 33. *In equal circles angles, whether at the centres or the circumferences, have the same ratio which the arcs on which they stand have to one another; so also have the sectors.*

Of this, the part relating to angles at the centre is of special importance; it enables us to measure angles by arcs.

With this closes that part of the *Elements* which is devoted to the study of figures in a plane.

Book XI.

§ 72. In this book figures are considered which are not confined to a plane, viz., first relations between lines and planes in space, and afterwards properties of solids.

Of new definitions we mention those which relate to the perpendicularity and the inclination of lines and planes.

Def. 3. *A straight line is perpendicular, or at right angles, to a plane when it makes right angles with every straight line meeting it in that plane.*

The definition of perpendicular planes (Def. 4) offers no difficulty. Euclid defines the inclination of lines to planes and of planes to planes (Defs. 5 and 6) by aid of plane angles, included by straight lines, with which we have been made familiar in the first books.

The other important definitions are those of parallel planes, which never meet (Def. 8), and of solid angles formed by three or more planes meeting in a point (Def. 9).

To these we add the definition of a line parallel to a plane as a line which does not meet the plane.

§ 73. Before we investigate the contents of Book XI., it will be well to recapitulate shortly what we know of planes and lines from the definitions and axioms of the first book. There a plane has been defined as a surface which has the property that every straight line which joins two points in it lies altogether in it. This is equivalent to saying that a straight line which has two points in a plane has all points in the plane. Hence, a straight line which does not lie in the plane cannot have more than one point in common with the plane. This is virtually the same as Euclid's Prop. 1, viz.:—

Prop. 1. *One part of a straight line cannot be in a plane and another part without it.*

It also follows, as was pointed out in § 3, in discussing the definitions of Book I., that a plane is determined already by one straight line and a point without it, viz., if all lines be drawn through the point, and cutting the line, they will form a plane.

This may be stated thus:—

A plane is determined—

1st, *By a straight line and a point which does not lie on it*;

2d, *By three points which do not lie in a straight line*; for if two of these points be joined by a straight line we have case 1;

3d, *By two intersecting straight lines*; for the point of intersection and two other points, one in each line, give case 2;

4th, *By two parallel lines* (Def. 35, I.).

The third case of this theorem is Euclid's

Prop. 2. *Two straight lines which cut one another are in one plane, and three straight lines which meet one another are in one plane.*

And the fourth is Euclid's

Prop. 7. *If two straight lines be parallel, the straight line drawn from any point in one to any point in the other is in the same plane with the parallels.* From the definition of a plane further follows:

Prop. 3. *If two planes cut one another, their common section is a straight line.*

§ 74. Whilst these propositions are virtually contained in the definition of a plane, the next gives us a new and fundamental property of space, showing at the same time that it is possible to have a straight line perpendicular to a plane, according to Def. 3. It states—

Theorem (Prop. 4).—*If a straight line is perpendicular to two straight lines in a plane which it meets, then it is perpendicular to all lines in the plane which it meets, and hence it is perpendicular to the plane.*

Def. 3 may be stated thus: If a straight line is perpendicular to a plane, then it is perpendicular to every line in the plane which it meets. The converse to this would be

Theorem.—*All straight lines which meet a given straight line in the same point, and are perpendicular to it, lie in a plane which is perpendicular to that line.*

This Euclid states thus:

Prop. 5. *If three straight lines meet all at one point, and a straight line stands at right angles to each of them at that point, the three straight lines shall be in one and the same plane.*

§ 75. There follow theorems relating to the theory of parallel lines in space, viz.:—

Theorem (Prop. 6).—*Any two lines which are perpendicular to the same plane are parallel to each other*; and conversely

Theorem (Prop. 8).—*If of two parallel straight lines one is perpendicular to a plane, the other is so also.*

Further, the important theorems

Prop. 9. *Two straight lines which are each of them parallel to the same straight line, and not in the same plane with it, are parallel to one another*; where the words, "and not in the same plane with it," may be left out, for they exclude the case of three parallels in a plane, which has been proved before; and

Theorem (Prop. 10).—*If two angles in different planes have the two limits of the one parallel to those of the other, then the angles are equal.* That their planes are parallel is shown later on in Prop. 15.

This theorem is not necessarily true, for the angles in question may be supplementary; but then the one angle will be equal to that which is adjacent and supplementary to the other, and this latter angle will also have its limits parallel to those of the first.

From this theorem it follows that if we take any two straight lines in space which do not meet, and if we draw through any point P in space two lines parallel to them, then the angle included by these lines will always be the same, whatever the position of the point P may be. This angle has in modern times been called the angle between the given lines:—

Definition.—*By the angles between two not intersecting lines we understand the angles which two intersecting lines include that are parallel respectively to the two given lines.*

§ 76. It is now possible to solve the following two problems:—

Problem.—*To draw a straight line perpendicular to a given plane from a given point which lies*

1. *Not in the plane* (Prop. 11).
2. *In the plane* (Prop. 12).

The second case is easily reduced to the first—viz., if by aid of the first we have drawn any perpendicular to the plane from some point without it, we need only draw through the given point in the plane a line parallel to it, in order to have the required perpendicular given. The solution of the first part is of interest in itself. It depends upon a construction which may be expressed as a theorem.

Theorem.—*If from a point A without a plane a perpendicular AB be drawn to the plane, and if from the foot B of this perpendicular another perpendicular BC be drawn to any straight line in the plane, then the straight line joining A to the foot C of this second perpendicular will also be perpendicular to the line in the plane.*

The theory of perpendiculars to a plane is concluded by the theorem—

Prop. 13. *Through any point in space, whether in or without a plane, only one straight line can be drawn perpendicular to the plane.*

§ 77. The next four propositions treat of parallel planes. It is shown *that planes which have a common perpendicular are parallel* (Prop. 14); *that two planes are parallel if two intersecting straight lines in the one are parallel respectively to two straight lines in the other plane* (Prop. 15); *that parallel planes are cut by any plane in parallel straight lines* (Prop. 16); and lastly, *that any two straight lines are cut proportionally by a series of parallel planes* (Prop. 17).

This theory is made more complete by adding the following theorems, which are easy deductions from the last:—*Two parallel planes have common perpendiculars* (converse to 14); and *Two planes which are parallel to a third plane are parallel to each other.*

It will be noted that Prop. 15 at once allows of the solution of the problem: "Through a given point to draw a plane parallel to a given plane." And it is also easily proved that this problem allows always of one, and only of one, solution.

§ 78. We come now to planes which are perpendicular to one another. Two theorems relate to them.

Prop. 18. *If a straight line be at right angles to a plane, every plane which passes through it shall be at right angles to that plane.*

Prop. 19. *If two planes which cut one another be each of them perpendicular to a third plane, their common section shall be perpendicular to the same plane.*

§ 79. If three planes pass through a common point, and if they bound each other, a solid angle of three faces, or a *trihedral* angle, is formed, and similarly by more planes a solid angle of more faces, or a *polyhedral* angle. These have many properties which are quite analogous to those of triangles and polygons in a plane. Euclid states some, viz:—

Prop. 20. *If a solid angle be contained by three plane angles, any two of them are together greater than the third.*

But the next—

Prop. 21. *Every solid angle is contained by plane angles, which are together less than four right angles*—has no analogous theorem in the plane.

We may mention, however, that the theorems about triangles contained in the propositions of Book I., which do not depend upon the theory of parallels (that is all up to Prop. 27), have their corresponding theorems about trihedral angles. The latter are formed, if for "side of a triangle" we write "plane angle" or "face" of trihedral angle, and for "angle of triangle" we substitute "angle between two faces" where the planes containing the solid angle are called its *faces.* We get, for instance, from I. 4, the theorem, *If two trihedral angles have the angles of two faces in the one equal to the angles of two faces in the other, and have likewise the angles included by these faces equal, then the angles in the remaining faces are equal, and the angles between the other faces are equal each to each, viz., those which are opposite equal faces.* The solid angles themselves are not necessarily equal, for they may be only symmetrical like the right hand and the left.

The connexion indicated between triangles and trihedral angles will also be recognized in

Prop. 22. *If every two of three plane angles be greater than the third, and if the straight lines which contain them be all equal, a triangle may be made of the straight lines that join the extremities of those equal straight lines.*

And Prop. 23. solves the problem, *To construct a trihedral angle having the angles of its faces equal to three given plane angles, any two of them being greater than the third.* It is, of course, analogous to the problem of constructing a triangle having its sides of given length.

Two other theorems of this kind are added by Simson in his edition of Euclid's *Elements.*

§ 80. These are the principal properties of lines and planes in space, but before we go on to their applications it will be well to define the word *distance.* In geometry distance means always "shortest distance"; viz., the distance of a point from a straight line, or from a plane, is the length of the perpendicular from the point to the line or plane. The distance between two non-intersecting lines is the length of their common perpendicular, there being but one. The distance between two parallel lines or between two parallel planes is the length of the common perpendicular between the lines or the planes.

§ 81. *Parallelepipeds.*—The rest of the book is devoted to the study of the parallelepiped. In Prop. 24 the possibility of such a solid is proved, viz.:—

Prop. 24. *If a solid be contained by six planes two and two of which are parallel, the opposite planes are similar and equal parallelograms.*

Euclid calls this solid henceforth a parallelepiped, though he never defines the word. Either face of it may be taken as *base,* and its distance from the opposite face as *altitude.*

Prop 25. *If a solid parallelepiped be cut by a plane parallel to two of its opposite planes, it divides the whole into two solids, the base of one of which shall be to the base of the other as the one solid is to the other.*

This theorem corresponds to the theorem (VI. 1) that parallelograms between the same parallels are to one another as their bases. A similar analogy is to be observed among a number of the remaining propositions.

§ 82. After solving a few problems we come to

Prop. 28. *If a solid parallelepiped be cut by a plane passing through the diagonals of two of the opposite planes, it shall be cut in two equal parts.*

In the proof of this, as of several other propositions, Euclid neglects the difference between solids which are symmetrical like the right hand and the left.

Prop. 31. *Solid parallelepipeds, which are upon equal bases, and of the same altitude, are equal to one another.*

Props. 29 and 30 contain special cases of this theorem leading up to the proof of the general theorem.

As consequences of this fundamental theorem we get

Prop. 32. *Solid parallelepipeds, which have the same altitude, are to one another as their bases*; and Prop. 33, *Similar solid parallelepipeds are to one another in the triplicate ratio of their homologous sides.*

If we consider, as in § 67, the ratios of lines as numbers, we may also say—

The ratio of the volumes of similar parallelepipeds is equal to the ratio of the third powers of homologous sides.

Parallelepipeds which are not similar but equal are compared by aid of the theorem

Prop. 34. *The bases and altitudes of equal solid parallelepipeds are reciprocally proportional; and if the bases and altitudes be reciprocally proportional, the solid parallelepipeds are equal.*

§ 83. Of the following propositions the 37th and 40th are of special interest.

Prop. 37. *If four straight lines be proportionals, the similar solid parallelepipeds, similarly described from them, shall also be proportionals; and if the similar parallelepipeds similarly described from four straight lines be proportionals, the straight lines shall be proportionals.*

In symbols it says—

$$\text{If } a : b = c : d, \text{ then } a^3 . b^3 = c^3 : d^3.$$

Prop. 40 teaches how to compare the volumes of triangular prisms with those of parallelepipeds, by proving *that a triangular prism is equal in volume to a parallelepiped, which has its altitude and its base equal to the altitude and the base of the triangular prism.*

§ 84. From these propositions follow all results relating to the mensuration of volumes. We shall state these as we did in the case of areas. The starting point is the "rectangular" parallelepiped, which has every edge perpendicular to the planes it meets, and which takes the place of the rectangle in the plane. If this has all its edges equal we obtain the "cube."

If we take a certain line u as unit length, then the square on u is the unit of area, and the cube on u the unit of volume, that is to say, if we wish to measure a volume we have to determine how many unit cubes it contains.

A rectangular parallelepiped has, as a rule, the three edges unequal, which meet at a point. Every other edge is equal to one of them. If a, b, c be the three edges meeting at a point, then we may take the rectangle contained by two of them, say by b and c, as base and the third as altitude. Let V be its volume, V′ that of another rectangular parallelepiped which has the edges a', b, c, hence the same base as the first. It follows then easily, from Prop. 25 or 32, that $V : V' = a : a'$; or in words,

Rectangular parallelepipeds on equal bases are proportional to their altitudes.

If we have two rectangular parallelepipeds, of which the first has the volume V and the edges a, b, c, and the second, the volume V′ and the edges a', b', c', we may compare them by aid of two new ones which have respectively the edges a', b, c, and a', b', c, and the volumes V_1 and V_2. We then have

$$V : V_1 = a : a'; \quad V_1 : V_2 = b : b' \quad V_2 : V' = c \cdot c'.$$

Compounding these, we have

$$V : V' = (a : a')\,(b : b')\,(c : c'),$$

or

$$\frac{V}{V'} = \frac{a}{a'} \cdot \frac{b}{b'} \cdot \frac{c}{c'}.$$

Hence, as a special case, making V′ equal to the unit cube U on u we get

$$\frac{V}{U} = \frac{a}{u} \cdot \frac{b}{u} \cdot \frac{c}{u} = \alpha . \beta . \gamma,$$

where α, β, γ are the numerical values of a, b, c; that is, *The number of unit cubes in a rectangular parallelepiped* is equal to the product of the numerical values of its three edges. This is generally expressed by saying the volume of a rectangular parallelepiped is measured by the product of its sides, or by the product of its base into its altitude, which in this case is the same.

Prop. 31 allows us to extend this to any parallelepipeds, and Props. 28 or 40, to triangular prisms.

Theorem.—*The volume of any parallelepiped, or of any triangular prism, is measured by the product of base and altitude.*

The consideration that any polygonal prism may be divided into a number of triangular prisms, which have the same altitude and the sum of their bases equal to the base of the polygonal prism, shows further that the same holds for any prism whatever.

Book XII.

§ 85. In the last part of Book XI. we have learnt how to compare the volumes of parallelepipeds and of prisms. In order to determine the volume of any solid bounded by plane faces we must determine the volume of pyramids, for every such solid may be decomposed into a number of pyramids.

As every pyramid may again be decomposed into triangular pyramids, it becomes only necessary to determine their volume. This is done by the

Theorem.—Every triangular pyramid is equal in volume to one third of a triangular prism having the same base and the same altitude as the pyramid.

This is an immediate consequence of Euclid's

Prop. 7. *Every prism having a triangular base may be divided into three pyramids that have triangular bases, and are equal to one another.*

The proof of this theorem is difficult, because the three triangular pyramids into which the prism is divided are by no means equal in shape, and cannot be made to coincide. It has first to be proved that two triangular pyramids have equal volumes, if they have equal bases and equal altitudes. This Euclid does in the following manner. He first shows (Prop. 3) that a triangular pyramid may be divided into four parts, of which two are equal triangular pyramids similar to the whole pyramid, whilst the other two are equal triangular prisms, and further, that these two prisms together are greater than the two pyramids, hence more than half the given pyramid. He next shows (Prop. 4) that if two triangular pyramids are given, having equal bases and equal altitudes, and if each be divided as above, then the two triangular prisms in the one are equal to those in the other, and each of the remaining pyramids in the one has its base and altitude equal to the base and altitude of the remaining pyramids in the other. Hence to these pyramids the same process is again applicable. We are thus enabled to cut out of the two given pyramids equal parts, each greater than half the original pyramid. Of the remainder we can again cut out equal parts greater than half these remainders, and so on as far as we like. This process may be continued till the last remainder is smaller than any assignable quantity, however small. It follows, so we should conclude at present, that the two volumes must be equal, for they cannot differ by any assignable quantity.

To Greek mathematicians this conclusion offers far greater difficulties. They prove elaborately, by a *reductio ad absurdum*, that the volumes cannot be unequal. This proof must be read in the *Elements*. A further discussion of this method of exhaustion, as it is called, would belong to a treatise on the history of geometry. We refer readers to Hankel, *Geschichte der Mathematik* (p. 115 *sq*). We must, however, state that we have in the above not proved Euclid's Prop. 5, but only a special case of it. Euclid does not suppose that the bases of the two pyramids to be compared are equal, and hence he proves that the volumes are as the bases. The reasoning of the proof becomes clearer in the special case, from which the general one may be easily deduced.

§ 86. Prop. 6 extends the result to pyramids with polygonal bases. From these results follow again the rules at present given for the mensuration of solids, viz., a pyramid is the third part of a triangular prism having the same base and the same altitude. But a triangular prism is equal in volume to a parallelepiped which has the same base and altitude. Hence if B is the base and h the altitude, we have

$$\text{Volume of prism} = Bh,$$
$$\text{Volume of pyramid} = \tfrac{1}{3}Bh,$$

statements which have to be taken in the sense that B means the number of square units in the base, h the number of units of length in the altitude, or that B and h denote the numerical values of base and altitude.

§ 87. A method similar to that used in proving Prop. 5 leads to the following results relating to solids bounded by simple curved surfaces:—

Prop. 10. *Every cone is the third part of a cylinder which has the same base, and is of an equal altitude with it.*

Prop. 11. *Cones and cylinders of the same altitude are to one another as their bases.*

Prop. 12. *Similar cones and cylinders have to one another the triplicate ratio of that which the diameters of their bases have.*

Prop. 13. *If a cylinder be cut by a plane parallel to its opposite planes or bases, it divides the cylinder into two cylinders, one of which is to the other as the axis of the first to the axis of the other*; which may also be stated thus:—

Cylinders on the same base are proportional to their altitudes.

Prop. 14. *Cones and cylinders upon equal bases are to one another as their altitudes.*

Prop. 15. *The bases and altitudes of equal cones and cylinders are reciprocally proportional, and if the bases and altitudes be reciprocally proportional, the cones and cylinders are equal to one another.*

These theorems again lead to formulæ in mensuration, if we compare a cylinder with a prism having its base and altitude equal to the base and altitude of the cylinder. This may be done by the method of exhaustion. We get, then, the result that their bases are equal, and have, if B denotes the numerical value of the base, and h that of the altitude,

$$\text{Volume of cylinder} = Bh.$$
$$\text{Volume of cone} = \tfrac{1}{3}Bh.$$

§ 88. The remaining propositions relate to circles and spheres. Of the sphere only one property is proved, viz.:—

Prop. 18. *Spheres have to one another the triplicate ratio of that which their diameters have.* The mensuration of the sphere, like that of the circle, the cylinder, and the cone, had not been settled in the time of Euclid. It was done by Archimedes.

Book XIII.

§ 89. The 13th and last book of Euclid's *Elements* is devoted to the regular solids. It is shown that there are five of them, viz.:—

1. The regular *tetrahedron*, with 4 triangular faces and 4 vertices;
2. The *cube*, with 8 vertices and 6 square faces;
3. The *octohedron*, with 6 vertices and 8 triangular faces;
4. The *dodecahedron*, with 12 pentagonal faces, 3 at each of the 20 vertices;
5. The *icosahedron*, with 20 triangular faces, 5 at each of the 12 vertices.

It is shown how to inscribe these solids in a given sphere, and how to determine the lengths of their edges.

These results are—if r denotes the radius of the circumscribed sphere, and a the side of the regular solid—

For tetrahedron	$a^2 = \frac{8}{3}r^2$,
,, octahedron	$a^2 = 2r^2$,
,, hexahedron or cube	$a^2 = \frac{4}{3}r^2$,
,, icosahedron	$a^2 = 2\left(1 - \frac{\sqrt{5}}{5}\right)r^2$,
,, dodecahedron	$a^2 = 2\left(1 - \frac{\sqrt{5}}{3}\right)r^2$.

§ 90. The 13th book, and therefore the *Elements*, conclude with the scholium, "that no other regular solid exists besides the five ones enumerated."

The proof is very simple. Each face is a regular polygon, hence the angles of the faces at any vertex must be angles in equal regular polygons, must be together less than four right angles (XI. 21), and must be three or more in number. Each angle in a regular triangle equals two-thirds of one right angle. Hence it is possible to form a solid angle with three, four, or five regular triangles or faces. These give the solid angles of the tetrahedron, the octohedron, and the icosahedron. The angle in a square (the regular quadrilateral) equals one right angle. Hence three will form a solid angle, that of the cube, and four will not. The angle in the regular pentagon equals $\frac{6}{5}$ of a right angle. Hence three of them equal $\frac{18}{5}$ (*i.e.*, less than 4) right angles, and form the solid angle of the dodecahedron. Three regular polygons of six or more sides cannot form a solid angle. Therefore no other regular solids are possible.

Section II.—Higher or Projective Geometry.

It is difficult, at the outset, to characterize Projective Geometry as compared with Euclidian. But a few examples will at least indicate the difference between the two.

In Euclid's *Elements* almost all propositions refer to the *magnitude* of lines, angles, areas, or volumes, and therefore to measurement. The statement that an angle is right, or that two straight lines are parallel, refers to measurement. On the other hand, the fact that a straight line does or does not cut a circle is independent of measurement, it being dependent only upon the mutual "position" of the line and the circle. This difference becomes clearer if we project any figure from one plane to another. By this the length of lines, the magnitude of angles and areas, is altered, so that the projection, or shadow, of a square on a plane will not be a square; it will, however, be some quadrilateral. Again, the projection of a circle will not be a circle, but some other curve more or less resembling a circle. But one property may be stated at once,—no straight line can cut the projection of a circle in more than two points, because no straight line can cut a circle in more than two points. There are, then, some properties of figures which do not alter by projection, whilst others do. To the latter belong nearly all properties relating to measurement, at least in the form in which they are generally given. The others are said to be projective properties, and their investigation forms the subject of Projective Geometry.

Different as are the kinds of properties investigated in the old and the new sciences, the methods followed differ in a still greater degree. In Euclid each proposition stands by itself; its connexion with others is never indicated; the leading ideas contained in its proof are not stated; general principles do not exist. In the modern methods, on the other hand, the greatest importance is attached to the leading thoughts which pervade the whole; and general principles, which bring whole groups of theorems under one aspect, are given rather than separate propositions. The whole tendency is towards generalization. A straight line is considered as given in its entirety, extending both ways to infinity, while Euclid

is very careful never to admit anything but finite quantities. The treatment of the infinite is in fact another fundamental difference between the two methods. Euclid avoids it. In modern geometry it is systematically introduced, for only thus is generality obtained.

Of the different modern methods of geometry, we shall treat principally of the methods of projection and correspondence which have proved to be the most powerful. These have become independent of Euclidian Geometry, especially through the *Geometrie der Lage* of V. Standt, and the *Ausdehnungslehre* of Grassmann.

For the sake of brevity we shall presuppose a knowledge of Euclid's *Elements*, although we shall use only a few of his propositions.

§ 1. We consider space as filled with points, lines, and planes, and these we call the elements out of which our figures are to be formed, calling any combination of these elements a "figure."

By a line we mean a straight line in its entirety, extending both ways to infinity; and by a plane, a plane surface, extending in all directions to infinity. We suppose

That through any two points in space one and only one line may be drawn (Eucl. I., Def. 4, Ax. 10, Post. 2);

That through any three points which are not in a line, one and only one plane may be placed (compare p. 386, § 73, above);

That the intersection of two planes is a line (Eucl. XI. 13);

That a line which has two points in common with a plane lies in the plane (Eucl. I., Def. 7), hence that the intersection of a line and a plane is a single point; and

That three planes which do not meet in a line have one single point in common.

These results may be stated differently in the following form:—

I. A plane is determined—	A point is determined—
1. By three points which do not lie in a line;	1. By three planes which do not pass through a line;
2. By two intersecting lines;	2. By two intersecting lines;
3. By a line and a point which does not lie in it.	3. By a line and a plane which does not pass through it.
II. A line is determined—	
1. By two points;	2. By two planes.

The reader will observe that not only are planes determined by points, but also points by planes; that therefore we have a right to consider the planes as elements, like points; and also that in any one of the above statements we may interchange the words point and plane, and we obtain again a correct statement, provided that these statements themselves are true. As they stand, we ought, in several cases, to add "if they are not parallel," or some such words, parallel lines and planes being evidently left altogether out of consideration. To correct this we have to reconsider the theory of parallels.

§ 2. Let us take in a plane a line p (fig. 1), a point S not in this line, and a line q drawn through S. Then this line q will meet the line p in a point A. If we turn the line q about S towards q', its point of intersection with p will move along p towards B, passing, on continued turning, to a greater and greater distance, until it is moved out of our reach. If we turn q still farther, its continuation will meet p, but now at the other side of A. The point of intersection has disappeared to the right and reappeared to the left. There is one intermediate position where q is parallel to p—that is where it does not cut p. In every other position it cuts p in some finite point. If, on the other hand, we move the point A to an infinite distance in p, then the line q which passes through A will be a line which does not cut p at any finite point. Thus we are led to say: *Every* line through S which joins it to any point at an infinite distance in p is parallel to p. But by Euclid's 12th axiom there is but one line parallel to p through S. The difficulty in which we are thus involved is due to the fact that we try to reason about infinity as if we, with our finite capabilities, could comprehend the infinite. To overcome this difficulty, we may say that all points at infinity in a line *appear* to us as one, and may be replaced by a single "ideal" point, just as all points in a fixed star—which is not at an infinite, only at a great distance—cannot be distinguished by us and to beings on the earth count as one. We may therefore now give the following definitions and axiom:—

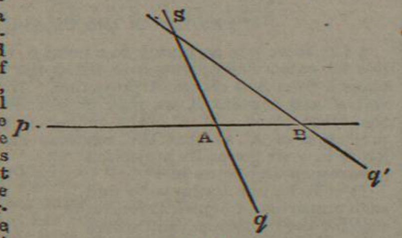


Fig. 1.

Definition.—Lines which meet at infinity are called parallel.

Axiom.—All points at an infinite distance in a line may be considered as one single point.

Definition.—This ideal point is called the point at infinity in the line.

The axiom is equivalent to Euclid's Axiom 12, for it follows from either that through any point only one line may be drawn parallel to a given line.

This point at infinity in a line is reached whether we move a point in the one or in the opposite direction of a line to infinity. A line thus appears closed by this point, and we speak as if we could move a point along the line from one position A to another B in two ways, either through the point at infinity or through finite points only.

It must never be forgotten that this point at infinity is ideal, that the results based on this assumption are true for that finite region of space which is within our reach, and that beyond this region they may or may not be true,—we do not know.

The advantage of this view of parallels will become apparent at every step as we go on.

§ 3. Having thus arrived at the notion of replacing all points at infinity in a line by one ideal point, there is no difficulty in replacing all points at infinity in a plane by one ideal line.

To make this clear, let us suppose that a line p, which cuts two fixed lines a and b in the points A and B, moves parallel to itself to a greater and greater distance. It will at last cut both a and b at their points at infinity, so that a line which joins the two points at infinity in two intersecting lines lies altogether at infinity. Every other line in the plane will meet it therefore at infinity, and thus it contains all points at infinity in the plane.

All points at infinity in a plane lie in a line, which is called the line at infinity in the plane.

It follows that parallel planes must be considered as planes having a common line at infinity, for any other plane cuts them in parallel lines (Eucl. XI. 16), which have a point at infinity in common.

If we next take two intersecting planes, then the point at infinity in their line of intersection lies in both planes, so that their lines at infinity meet. Hence every line at infinity meets every other line at infinity, and they are therefore all in one plane.

All points at infinity in space may be considered as lying in one ideal plane, which is called the plane at infinity.

§ 4. We have now the following definitions:—

Parallel lines are lines which meet at infinity;

Parallel planes are planes which meet at infinity;

A line is parallel to a plane if it meets it at infinity.

Theorems like this—Lines (or planes) which are parallel to a third are parallel to each other—follow at once.

This view of parallels leads therefore to no contradiction of Euclid's *Elements*.

As immediate consequences we get the propositions:—

Every line meets a plane in one point, or it lies in it;

Every plane meets every other plane in a line;

Any two lines in the same plane meet.

§ 5. We have called points, lines, and planes the elements of geometrical figures. We also say that an element of one kind *contains* one of the other if it lies in it or passes through it.

All the elements of one kind which are contained in one or two elements of a different kind form aggregates which have to be enumerated. They are the following:—

I. Of one dimension.
   1. The *row*, or range, *of points* formed by all points in a line, which is called its base.
   2. The *flat pencil* formed by all the lines through a point in a plane. Its base is the point in the plane.
   3. The *axial pencil* formed by all planes through a line which is called its base or axis.

II. Of two dimensions.
   1. The field of points and lines—that is, a plane with all its points and all its lines.
   2. The pencil of lines and planes—that is, a point in space with all lines and all planes through it.

III. Of three dimensions.
   The space of points—that is, all points in space.
   The space of planes—that is, all planes in space.

IV. Of four dimensions.
   The space of lines, or all lines in space.

§ 6. The word dimension in the above needs explanation.

If in a plane we take a row p and a pencil with centre Q, then through every point in p one line in the pencil will pass, and every ray in Q will cut p in one point, so that we are entitled to say a row contains as many points as a flat pencil lines, and, we may add, as an axial pencil planes, because an axial pencil is cut by a plane in a flat pencil.

The number of elements in the row, in the flat pencil, and in the axial pencil is, of course, infinite and indefinite too, but the same in all. For a moment we shall treat it as being a definite number which we denote by α. Then a plane contains α^2 points and as many lines. To see this, take a flat pencil in a plane. It contains α lines, and each line contains α points, whilst each point in the plane lies on one of these lines. Similarly, in a plane each line cuts a fixed line in a point. But this line is cut at each point by α lines and contains α points; hence there are α^2 lines in a plane.

A pencil in space contains as many lines as a plane contains points and as many planes as a plane contains lines, for any plane cuts the pencil in a field of points and lines. Hence a pencil contains α^2 lines and α^2 planes. *The field and the pencil are of two dimensions.*

To count the number of points in space we observe that each point lies on some line in a pencil. But the pencil contains α^2 lines, and each line α points; hence space contains α^3 points. Each plane cuts any fixed plane in a line. But a plane contains α^2 lines, and through each pass α planes; therefore space contains α^3 planes.

Hence space contains as many planes as points, but it contains an infinite number of times more lines than points or planes. To count them, notice that every line cuts a fixed plane in one point. But α^2 lines pass through each point, and there are α^2 points in the plane. Hence there are α^4 lines in space. *The space of points and planes is of three dimensions, but the space of lines is of four dimensions.*

A field of points or lines contains an infinite number of rows and flat pencils; a pencil contains an infinite number of flat pencils and of axial pencils; space contains a triple infinite number of pencils and of fields, α^4 rows and axial pencils, and α^5 flat pencils—or, in other words, each point is a centre of α^2 flat pencils.

§ 7. The above enumeration allows a classification of figures.

Figures in a row consist of groups of points only, and figures in the flat or axial pencil consist of groups of lines or planes. In the plane we may draw polygons; and in the pencil or in the point, solid angles, and so on.

We may also distinguish the different measurements. We have—

In the row, length of segment;
In the flat pencil, angles;
In the axial pencil, dihedral angles between two planes;
In the plane, areas;
In the pencil, solid angles;
In the space of points or planes, volumes.

Segments of a Line.

§ 8. Any two points A and B in space determine on the line through them a finite part, which may be considered as being described by a point moving from A to B. This we shall denote by AB, and distinguish it from BA, which is supposed as being described by a point moving from B to A, and hence in a direction or in a "sense" opposite to AB. Such a finite line, which has a definite sense, we shall call a "segment," so that AB and BA denote different segments, which are said to be equal in length but of opposite sense. The one sense is often called positive and the other negative.

In introducing the word "sense" for direction in a line, we have the word direction reserved for direction of the line itself, so that different lines have different directions, unless they be parallel, whilst in each line we have a positive and negative sense.

We may also say, with Professor Clifford, that AB denotes the "step" of going from A to B.

§ 9. If we now have three points A, B, C in a line (fig. 2), the step AB will bring us from A to B, and the step BC from B to C. Hence both steps are equivalent to the one step AC. This is expressed by saying that AC is the "sum" of AB and BC: in symbols—

$$AB+BC=AC,$$

where account is to be taken of the sense.

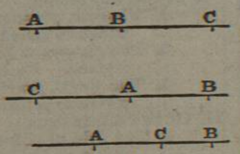


Fig. 2.

This equation is true whatever be the position of the three points on the line. As a special case we have

$$AB+BA=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

and similarly

$$AB+BC+CA=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2),$$

which again is true for any three points in a line.

We further write

$$AB=-BA,$$

where — denotes negative sense.

We can then, just as in algebra, change subtraction of segments into addition by changing the sense, so that AB—CB is the same as AB+(—CB) or AB+BC. A figure will at once show the truth of this. The sense is, in fact, in every respect equivalent to the "sign" of a number in algebra.

§ 10. Of the many formulæ which exist between points in a line we shall have to use only one more, which connects the segments between any four points A, B, C, D in a line. We have—

$$BC=BD+DC,$$
$$CA=CD+DA,$$
$$AB=AD+DB;$$

or multiplying these by AD, BD, CD respectively, we get—

$$BC \, . \, AD = BD \, . \, AD + DC \, . \, AD = BD \, . \, AD - CD \, . \, AD$$
$$CA \, . \, BD = CD \, . \, BD + DA \, . \, BD = CD \, . \, BD - AD \, . \, BD$$
$$AB \, . \, CD = AD \, . \, CD + DB \, . \, CD = AD \, . \, CD - BD \, . \, CD.$$

It will be seen that the sum of the right hand sides vanishes, hence that

$$BC \, . \, AD + CA \, . \, BD + AB \, . \, CD = 0 \quad . \quad . \quad . \quad (3)$$

for any four points on a line.

§ 11. If C is any point in the line AB, then we say that C divides the segment AB in the ratio $\frac{AC}{CB}$, account being taken of the sense of the two segments AC and CB. If C lies between A and C the ratio is positive, as AC and CB have the same sense. But if C lies without the segment AB, *i.e.*, if C divides AB externally, then the ratio is negative. To see how the value of this ratio changes with C, we will move C along the whole line (fig. 3), whilst A and B remain fixed.

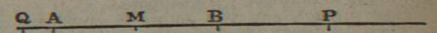


Fig. 3.

If C lies at the point A, then AC=0, hence the ratio AC : CB vanishes. As C moves towards B, AC increases and CB decreases, so that our ratio increases. At the middle point M of AB it assumes the value +1, and then increases till it reaches an infinitely large value, when C arrives at B. On passing beyond B the ratio becomes negative. If C is at P we have AC=AP=AB+BP, hence

$$\frac{AC}{CB}=\frac{AB}{PB}+\frac{BP}{PB}=-\frac{AB}{BP}-1.$$

In the last expression the ratio AB : BP is positive, has its greatest value ∞ when C coincides with B, and vanishes when BC becomes infinite. Hence, as C moves from B to the right to the point at infinity, the ratio AC : CB varies from —∞ to —1.

If on the other hand C is to the left of A, say at Q, we have

$$AC=AQ=AB+BQ=AB-QB, \text{ hence } \frac{AC}{CB}=\frac{AB}{QB}-1$$

Here AB <QB, hence the ratio AB : QB is positive and always less than one, so that the whole is negative and <1. If C is at the point at infinity it is —1, and then increases as C moves to the right, till for C at A we get the ratio =0. Hence—

"As C moves along the line from an infinite distance to the left to an infinite distance at the right, the ratio always increases; it starts with the value —1, reaches 0 at A, +1 at M, ∞ at B, changes now sign to —∞, and increases till at an infinite distance it reaches again the value —1. *It assumes therefore all possible values from —∞ to +∞, and each value only once, so that not only does every position of* C *determine a definite value of the ratio* AC : CB, *but also, conversely, to every positive or negative value of this ratio belongs one single point in the line* AB.

Projection and Cross-Ratios.

§ 12. If we join a point A to a point S, then the point where the line SA cuts a fixed plane π is called the projection of A on the plane π from S as centre of projection. If we have two planes π and π' and a point S, we may project every point A in π to the other plane. If A′ is the projection of A, then A is also the projection of A′, so that the relations are reciprocal. To every figure in π we get as its projection a corresponding figure in π'.

It will be our business to find such properties of figures as remain true for the projection, and which are called projective properties. For this purpose it will be sufficient to consider at first only constructions in one plane.

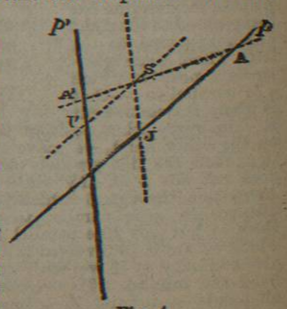


Fig. 4.

Let us suppose we have given in a plane two lines p and p' and a centre S (fig. 4); we may then project the points in p from S to p'. Let A′, B′... be the projections of A, B..., the point at infinity in p which we shall denote by J will be projected into a finite point I′ in p', viz., into the point where the parallel to p through S cuts p'. Similarly one point J in p will be projected into the point I′ at infinity in p'. This point J is of course the point where

the parallel to p' through S cuts p. We thus see that every point in p is projected into a single point in p'.

A glance at fig. 5 shows that a segment AB will be projected into a segment A′B′ which is not equal to it, at least not as a rule; and also that the ratio AC : CB is not equal to the ratio A′C′ : C′B′ formed by the projections. These ratios will become equal only if p and p' are parallel, for in this case the triangle SAB is similar to the triangle SA′B′. Between three points in a line and their projections there exists therefore in general no relation. But between four points a relation does exist.

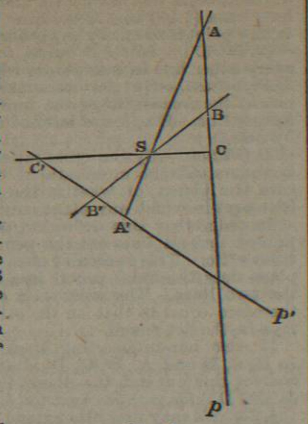


Fig. 5.

§ 13. Let A, B, C, D be four points in p, A′, B′, C′, D′ their projections in p', then the ratio of the two ratios AC : CB and AD : DB into which C and D divide the segment AB is equal to the corresponding expression between A′, B′, C′, D′. In symbols we have—

$$\frac{AC}{CB}:\frac{AD}{DB}=\frac{A'C'}{C'B'}:\frac{A'D'}{D'B'}.$$

This is easily proved by aid of similar triangles.

Through the points A and B on p draw parallels to p', which cut the projecting rays in C_2, D_2, B_2 and A_1, C_1, D_1, as indicated in fig. 6. The two triangles ACC_2 and BCC_1 will be similar, as will also be the triangles ADD_2 and BDD_1.

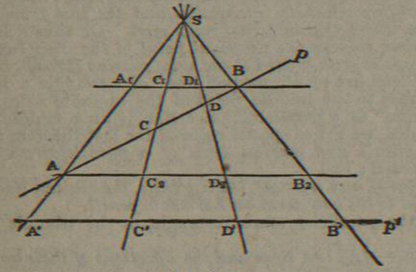


Fig. 6.

We have therefore—

$$\frac{AC}{CB}=\frac{AC_2}{C_1B}, \quad \frac{AD}{DB}=\frac{AD_2}{D_1B},$$

where account is taken of the sense.

Hence—

$$\frac{AC}{CB}:\frac{AD}{DB}=\frac{AC_2}{C_1B}:\frac{AD_2}{D_1B}=\frac{AC_2}{AD_2}:\frac{C_1B}{D_1B};$$

but

$$\frac{AC_2}{AD_2}=\frac{A'C'}{A'D'} \text{ and } \frac{C_1B}{D_1B}=\frac{C'B'}{D'B'};$$

so that the above expression becomes

$$\frac{A'C'}{A'D'}:\frac{C'B'}{D'B'},$$

which is equal to $\frac{A'C'}{C'B'}:\frac{A'D'}{D'B'}$ as required.

This result is of fundamental importance.

The expression formed has been called by Chasles the anharmonic ratio of the four points A, B, C, D. Instead of this Professor Clifford has proposed the shorter and more expressive name of "cross-ratio." We shall adopt the latter. We have then the

Fundamental Theorem.—*The cross-ratio of four points in a line is equal to the cross-ratio of their projections on any other line which lies in the same plane with it.*

§ 14. Before we draw conclusions from this result, we must investigate the meaning of a cross-ratio somewhat more fully.

If four points A, B, C, D are given, and we wish to form their cross-ratio, we have first to divide them into two groups of two, the points in each group being taken in a definite order. Thus, let A, B be the first, C, D the second pair, A and C being the first points in each pair. The cross-ratio is then the ratio AC : CB divided by AD : DB. This will be denoted by (ABCD), so that

$$(ABCD)=\frac{AC}{CB}:\frac{AD}{DB}.$$

This is easily remembered. In order to write it out, make first the two lines for the fractions, and put above and below these the letters A and B in their places, thus, $\frac{A}{\ B}:\frac{A}{\ B}$; and then fill up, crosswise, the first by C and the other by D.

§ 15. If we take the points in a different order, the value of the cross-ratio will change. We can do this in twenty-four different ways by forming all permutations of the letters. But of these twenty-four cross-ratios groups of four are equal, so that there are really only six different ones.

We have the following rules:—

I. If in a cross-ratio the two groups be interchanged, its value remains unaltered.

$$(ABCD)=(CDAB).$$

II. If in a cross-ratio the two points belonging one of the two groups be interchanged, the cross-ratio changes into its reciprocal.

$$(ABCD)=\frac{1}{(ABDC)}.$$

III. If in a cross-ratio the two middle letters be interchanged, the cross-ratio κ changes into its complement $1-\kappa$.

$$(ABCD)=1-(ACBD).$$

The first two are easily proved by writing out their values. The third is proved by aid of the formula (3), § 10,

$$BC \, . \, AD+CA \, . \, BD+AB \, . \, CD=0.$$

If we divide this by CB . AD we get

$$-1+\frac{CA \, . \, BD}{CB \, . \, AD}+\frac{AB \, . \, CD}{CB \, . \, AD}=0,$$

or,

$$\frac{AC}{CB}:\frac{AD}{DB}+\frac{AB}{BC}:\frac{AD}{DC}=1;$$

that is,

$$(ABCD)+(ACBD)=1,$$

which was to be proved.

IV. From II. it follows at once that if we interchange the elements in each pair, the cross-ratio remains unaltered; and thus we see that

$$(ABCD)=(CDAB) \qquad \text{by I.}$$
$$=(BADC)=(DCBA) \qquad \text{by II.}$$

§ 16. By aid of these rules we get the following results:—

$$(ABCD)=(BADC)=(CDAB)=(DCBA)=\kappa.$$
$$(ABDC)=(BACD)=(CDBA)=(DCAB)=\frac{1}{\kappa}.$$
$$(ACBD)=(BDAC)=(CADB)=(DBCA)=1-\kappa.$$
$$(ACDB)=(BDCA)=(CABD)=(DBAC)=\frac{1}{1-\kappa}.$$
$$(ADBC)=(BCAD)=(CBDA)=(DACB)=\frac{\kappa-1}{\kappa}.$$
$$(ADCB)=(BCDA)=(CBAD)=(DABC)=\frac{\kappa}{\kappa-1}.$$

In the theorem that the cross-ratio of four points equals that of the projections, the points have, of course, to be taken in the same order.

§ 17. If one of the points of which a cross-ratio is formed is the point at infinity in the line, the cross-ratio changes into a simple ratio. It is convenient to let the point at infinity occupy the last place in the symbolic expression for the cross-ratio. Thus if I is a point at infinity, we have

$$(ABCI)=-\frac{AC}{CB},$$

because

$$AI:IB=-1.$$

Every common ratio of three points in a line may thus be expressed as a cross-ratio, by adding the point at infinity to the group of points.

§ 18. If the points have special positions, the cross-ratios κ may have such a value that, of the six different ones, two and two become equal. If the first two shall be equal, we get $\kappa=\frac{1}{\kappa}$, or $\kappa^2=1$, $\kappa=\pm 1$.

If we take $\kappa=+1$, we have (ABCD)=1, or $\frac{AC}{CB}=\frac{AD}{DB}$; that is, the points C and D coincide, provided that A and B are different. This is too special a case to be of much interest.

If, however, we take $\kappa=-1$, so that (ABCD)=−1, we have $\frac{AC}{CB}=-\frac{AD}{DB}$. *Hence* C *and* D *divide* AB *internally and externally in the same ratio.*

The four points are in this case said to be *harmonic points*, and C *and* D *are said to be harmonic conjugates with regard to* A *and* B.