make CD turn about P and move R along p, whilst QD and RD describe projective pencils about A and B. Hence Q and R describe projective rows, and hence PR, which is the polar of Q, describes a pencil projective to either.

§ 67. Two points, of which one, and therefore each, lies on the polar of the other, are said to be *conjugate with regard to the conic*; and two lines, of which one, and therefore each, passes through the pole of the other, are said to be *conjugate with regard to the conic*. Hence all points conjugate to a point P lie on the polar of P; all lines conjugate to a line p pass through the pole of p.

If the line joining two conjugate poles cuts the conic, then the poles are harmonic conjugates with regard to the points of intersection; hence one lies within the other without the conic, and all points conjugate to a point within a conic lie without it.

Of a polar-triangle any two vertices are conjugate poles, any two sides conjugate lines. If, therefore, one side cuts a conic, then one of the two vertices which lie on this side is within and the other without the conic. The vertex opposite this side lies also without, for it is the pole of a line which cuts the curve. In this case therefore one vertex lies within, the other two without. If, on the other hand, we begin with a side which does not cut the conic, then its pole lies within and the other vertices without. Hence—

Theorem.—Every polar triangle has one and only one vertex within the conic.

We add, without a proof, the theorem—

Theorem.—The four points in which a conic is cut by two conjugate polars are four harmonic points in the conic.

§ 68. If two conics intersect in four points (they cannot have more points in common, § 52), there exists one and only one four-point which is inscribed in both, and therefore one polar triangle common to both.

Theorem.—Two conics which intersect in four points have always one and only one common polar-triangle; and reciprocally,

Two conics which have four common tangents have always one and only one common polar-triangle.

The proof that these polar triangles are identical in case of a conic which have four points and also four tangents in common is left to the reader.

Diameters and Axes of Conics.

§ 69. *Diameters.*—The theorems about the harmonic properties of poles and polars contain, as special cases, a number of important metrical properties of conics. These are obtained if either the pole or the polar is moved to infinity,—it being remembered that the harmonic conjugate to a point at infinity, with regard to two points A, B, is the middle point of the segment AB. The most important properties are stated in the following theorems:—

The middle points of parallel chords of a conic lie in a line—viz., on the polar to the point at infinity on the parallel chords.

This line is called a *diameter*.

The polar of every point at infinity is a diameter.

The tangents at the end points of a diameter are parallel, and are parallel to the chords bisected by the diameter.

All diameters pass through a common point, the pole of the line at infinity.

All diameters of a parabola are parallel, the pole to the line at infinity being the point where the curve touches the line at infinity.

In case of the ellipse and hyperbola, the pole to the line at infinity is a finite point called the *centre* of the curve.

A centre of a conic bisects every chord through it.

The centre of an ellipse is within the curve, for the line at infinity does not cut the ellipse.

The centre of an hyperbola is without the curve, because the line at infinity cuts the curve. Hence also

From the centre of an hyperbola two tangents can be drawn to the curve which have their point of contact at infinity. These are called *Asymptotes* (§ 59).

To construct a diameter of a conic, draw two parallel chords and join their middle points.

To find the centre of a conic, draw two diameters; their intersection will be the centre.

§ 70. *Conjugate Diameters.*—A polar-triangle with one vertex at the centre will have the opposite side at infinity. The other two sides pass through the centre, and are called *conjugate diameters*, each being the polar of the point at infinity on the other.

Of two conjugate diameters each bisects the chords parallel to the other, and if one cuts the curve, the tangents at its ends are parallel to the other diameter.

Further—

Every parallelogram inscribed in a conic has its sides parallel to two conjugate diameters; and

Every parallelogram circumscribed about a conic has as diagonals two conjugate diameters.

This will be seen by considering the parallelogram in the first case as an inscribed four-point, in the other as a circumscribed four-side, and determining in each case the corresponding polar-triangle. The first may also be enunciated thus—

The lines which join any point on an ellipse or an hyperbola to the ends of a diameter are parallel to two conjugate diameters.

§ 71. The Circle.—*If every diameter is perpendicular to its conjugate the conic is a circle.*

For the line which joins the ends of a diameter to any point on the curve include a right angle.

A conic which has more than one pair of conjugate diameters at right angles to each other is a circle.

Let AA′ and BB′ (fig. 23) be one pair of conjugate diameters at right angles to each other, CC′ and DD′ a second pair. If we draw through the end point A of one diameter a chord AP parallel to DD′, and join P to A′, then PA and PA′ are, according to § 70, parallel to two conjugate diameters. But PA is parallel to DD′, hence PA′ is parallel to CC′, and therefore PA and PA′ are perpendicular. If we further draw the tangents to the conic at A and A′, these will be perpendicular to AA′, they being parallel to the conjugate diameter BB′. We know thus five points on the conic, viz., the points A and A′ with their tangents, and the point P. Through these a circle may be drawn having AA′ as diameter; and as through five points one conic only can be drawn, this circle must coincide with the given conic.

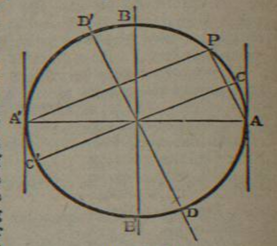

Fig. 23.

§ 72. *Axes.*—Conjugate diameters perpendicular to each other are called *axes*, and the points where they cut the curve *vertices* of the conic.

In a circle every diameter is an axis, every point on it is a vertex; and any two lines at right angles to each other may be taken as a pair of axes of any circle which has its centre at their intersection.

If we describe on a diameter AB of an ellipse or hyperbola a circle concentric to the conic, it will cut the latter in A and B (fig. 24). Each of the semicircles in which it is divided by AB will be partly within, partly without the curve, and must cut the latter therefore again in a point. The circle and the conic have thus four points A, B, C, D, and therefore one polar-triangle, in common (§ 68). Of this the centre is one vertex, for the line at infinity is the polar to this point, both with regard to the circle and the other conic. The other two sides are conjugate diameters of both, hence perpendicular to each other. This gives—

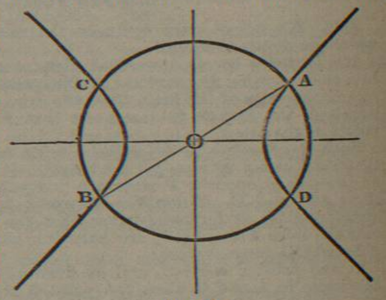

Fig. 24.

Theorem.—An ellipse as well as an hyperbola has one pair of axes.

This reasoning shows at the same time *how to construct the axis of an ellipse or of an hyperbola.*

A parabola has one axis, if we define an axis as a diameter perpendicular to the chords which it bisects. It is easily constructed. The line which bisects any two parallel chords is a diameter. Chords perpendicular to it will be bisected by a parallel diameter, and this is the axis.

§ 73. The first part of the right hand theorem in § 64 may be stated thus: any two conjugate lines through a point P without a conic are harmonic conjugates with regard to the two tangents that may be drawn from P to the conic.

If we take instead of P the centre C of an hyperbola, then the conjugate lines become conjugate diameters, and the tangents asymptotes. Hence—

Theorem.—Any two conjugate diameters of an hyperbola are harmonic conjugates with regard to the asymptotes.

As the axes are conjugate diameters at right angles to one another, it follows (§ 23)—

Theorem.—The axes of an hyperbola bisect the angles between the asymptotes.

Let O be the centre of the hyperbola (fig. 25), t any secant which cuts the hyperbola in C,D and the asymptotes in E,F, then the line OH which bisects the chord CD is a diameter conjugate to the diameter OK which is parallel to the secant t, so that OK and OM are harmonic with regard to the asymptotes. The point M therefore bisects EF. But by construction M bisects CD. It follows that DF = EC, and ED = CF; or

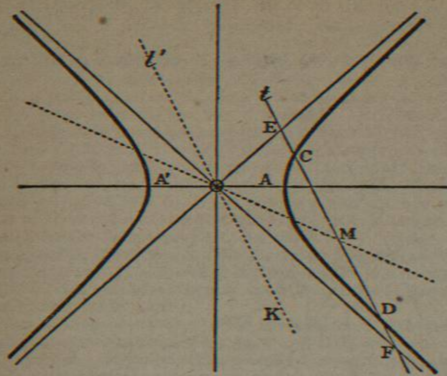

Fig. 25.

Theorem.—*On any secant of an hyperbola the segments between the curve and the asymptotes are equal.*

If the chord is changed into a tangent, this gives

The segment between the asymptotes on any tangent to an hyperbola is bisected by the point of contact.

The first part allows a simple solution of the problem to find any number of points on an hyperbola, of which the asymptotes and one point are given. This is equivalent to three points and the tangents at two of them. This construction requires measurement.

§ 74. For the parabola, too, follow some metrical properties. A diameter PM (fig. 26) bisects every chord conjugate to it, and the pole P of such a chord BC lies on the diameter. But a diameter cuts the parabola once at infinity. Hence—

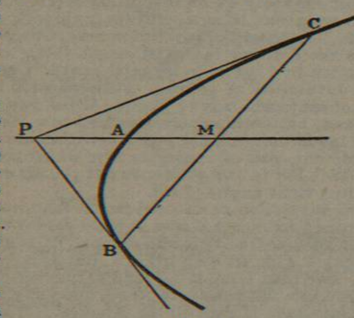

Fig 26.

Theorem.—The segment PM which joins the middle point M of a chord of a parabola to the pole P of the chord is bisected by the parabola at A.

§ 75. Two asymptotes and any two tangents to an hyperbola may be considered as a quadrilateral circumscribed about the hyperbola. But in such a quadrilateral the intersections of the diagonals and the points of contact of opposite sides lie in a line (§ 54). If therefore DEFG (fig. 27) is such a quadrilateral, then the diagonals DF and GE will meet on the line which joins the points of contact of the asymptotes, that is, on the line at infinity; hence they are parallel. From this the following theorem is a simple deduction:—

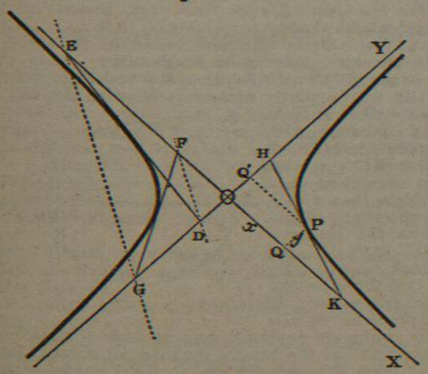

Fig. 27.

Theorem.—All triangles formed by a tangent and the asymptotes of an hyperbola are equal in area.

If we draw at a point P (fig. 27) on an hyperbola a tangent, the part HK between the asymptotes is bisected at P. The parallelogram PQOQ′ formed by the asymptotes and lines parallel to them through P will be half the triangle OHK, and will therefore be constant. If we now take the asymptotes OX and OY as oblique axes of coordinates, the lines OQ and QP will be the coordinates of P, and will satisfy the equation $xy = \text{const.} = a^2$.

Theorem.—For the asymptotes as axes of coordinates the equation of the hyperbola is

$$xy = \text{const.}$$

It is not difficult to get the equations to the ellipse and hyperbola referred to their axes as axes of coordinates. We are satisfied to have shown in one case that the curves of the second order as generated by projective pencils are the same as those which are in coordinate geometry defined by equations of the second degree.

Involution.

§ 76. If we have two projective rows, ABC on u and A′B′C′ on u', and place their bases on the same line, then each point in this line counts twice, once as a point in the row u and once as a point in the row u'. In fig. 28 we denote the points as points in the one row by letters above the line A, B, C . . ., and as points in the second row by A′, B′, C′ . . . . below the line. Let now A and B′ be the same point, then to A will correspond a point A′ in the second, and to B′ a point B in the first row. In general these points A′ and B will be different. It may, however, happen that they coincide. Then the correspondence is a peculiar one, as the following theorem shows:—

Fig. 28.

Theorem.—*If two projective rows lie on the same base, and if it happens that to one point in the base the same point corresponds, whether we consider the point as belonging to the first or to the second row, then the same will happen for every point in the base—that is to say, to every point in the line corresponds the same point in the first as in the second row.*

Proof. In order to determine the correspondence, we may assume three pairs of corresponding points in two projective rows. Let then A′, B′, C′, in fig. 29, correspond to A, B, C, so that A and B′, and also B and A′, denote the same point. Let us further denote the point C′ when considered as a point in the first row by D; then it is to be proved that the point D′, which corresponds to D, is the same point as C. We know that the cross-ratio of four points is equal to that of the corresponding row. Hence

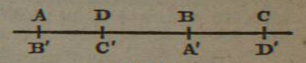

Fig. 29.

$$(ABCD) = (A'B'C'D')$$

but replacing the dashed letters by those undashed ones which denote the same points, the second cross-ratio equals (BADD′), which, according to § 15, iv., equals (ABD′D); so that the equation becomes

$$(ABCD) = (ABD'D).$$

This requires that C and D′ coincide.

§ 77. Two projective rows on the same base, which have the above property, that to every point, whether it be considered as a point in the one or in the other row, corresponds the same point, are said to be in *involution*, or to form an *involution* of points on the line.

We mention, but without proving it, that any two projective rows may be placed so as to form an involution.

An involution may be said to consist of a row of pairs of points, to every point A corresponding a point A′, and to A′ again the point A. These points are said to be conjugate.

From the definition, according to which an involution may be considered as made up of two projective rows, follow at once the following important properties:—

(1.) The cross-ratio of four points equals that of the four conjugate points.

(2.) If we call a point which coincides with its conjugate point a "focus" of the involution, we may say: An involution has either two foci, or one, or none, and is called respectively a hyperbolic, parabolic, or elliptic involution (§ 34).

(3.) In a hyperbolic involution any two conjugate points are harmonic conjugates with regard to the two foci.

For if A, A′ be two conjugate points, F_1, F_2 the two foci, then to the points F_1, F_2, A, A′ in the one row correspond the points F_1, F_2, A′, A in the other, each focus corresponding to itself. Hence $(F_1F_2AA') = (F_1F_2A'A)$—that is, we may interchange the two points AA′ without altering the value of the cross-ratio, which is the characteristic property of harmonic conjugates (§ 18).

(4.) The point conjugate to the point at infinity is called the "centre" of the involution. Every involution has a centre, unless the point at infinity be a focus, in which case we may say that the centre is at infinity.

In a hyperbolic involution the centre is the middle point between the foci.

(5.) The product of the distances of two conjugate points A, A' from the centre O is constant:—

$$OA \, . \, OA' = c.$$

Proof.—Let A, A', and B, B' be two pairs of conjugate points, O the centre, I the point at infinity, then

$$(ABOI) = (A'B'IO),$$

or

$$OA \, . \, OA' = OB \, . \, OB'.$$

In order to determine the distances of the foci from the centre, we write F for A and A' and get

$$\overline{OF}^2 = c; \quad OF = \pm\sqrt{c}.$$

Hence if c is positive OF is real, and has two values equal and opposite. The involution is hyperbolic.

If $c = 0$, $OF = 0$, and the two foci both coincide with the centre.

If c is negative, $\sqrt{c}$ becomes imaginary, and there are no foci.

Hence we may write—

In a hyperbolic involution, $OA \, . \, OA' = k^2$,
In a parabolic involution, $OA \, . \, OA' = 0$,
In an elliptic involution, $OA \, . \, OA' = -k^2$.

From these expressions it follows that conjugate points A, A' in a hyperbolic involution lie on the same side of the centre, and in an elliptic involution on opposite sides of the centre, and that in a parabolic involution one coincides with the centre.

In the first case, for instance, $OA \, . \, OA'$ is positive; hence OA and OA' have the same sign.

It also follows that two segments, AA' and BB', between pairs of conjugate points have the following positions:—in a hyperbolic involution they lie either one altogether within or altogether without each other; in a parabolic involution they have one point in common; and in an elliptic involution they overlap, each being partly within and partly without the other.

Proof.—We have $OA \, . \, OA' = OB \, . \, OB' = k^2$ in case of a hyperbolic involution. Let A and B be the points in each pair which are nearer to the centre O. If now A, A' and B, B' lie on the same side of O, and if B is nearer to O than A, so that $OB < OA$, then $OB' > OA'$; hence B' lies further away from O than A', or the segment AA' lies within BB'. And so on for the other cases.

(6.) An involution is determined—

(α) By two pairs of conjugate points. Hence also
(β) By one pair of conjugate points and the centre;
(γ) By the two foci;
(δ) By one focus and one pair of conjugate points;
(ε) By one focus and the centre.

(7.) The condition that A, B, C and A', B', C' may form an involution may be written in one of the forms—

$$(ABCC') = (A'B'C'C),$$
or
$$(ABCA') = (A'B'C'A),$$
or
$$(ABC'A') = (A'B'CA),$$

for each expresses that in the two projective rows in which A, B, C and A', B', C' are conjugate points two conjugate elements may be interchanged.

(8.) Any three pairs, A, A', B, B', C, C', of conjugate points are connected by the relation—

$$\frac{BA'}{A'C} \cdot \frac{CB'}{B'A} \cdot \frac{AC'}{C'B} = -1.$$

Proof.—We have by (7) $(ABC'A') = (A'B'CA)$, which, when worked out, gives the above relation.

The latter is easily remembered by aid of the following rule of writing the first side. First write

$$\frac{B}{C} \quad \frac{C}{A} \quad \frac{A}{B},$$

and then fill up the gaps in numerator and denominator by A', B', C' respectively.

§ 78. THEOREM.—*The sides of any four-point are cut by any line in six points in involution, opposite sides being cut in conjugate points.*

Let $A_1B_1C_1D_1$ (fig. 30) be the four-point. If its sides be cut by the line p in the points A, A', B, B', C, C', if further, C_1D_1 cuts the line A_1B_1 in C_2, and if we project the row $A.B.C_2C$ to p once from D_1 and once from C_1, we get

$$(A'B'C'C) = (BAC'C).$$

Interchanging in the last cross-ratio the letters in each pair we get

$$(A'B'C'C) = (ABCC').$$

Hence by § 77 (7) the points are in involution.

The theorem may also be stated thus:—

Theorem.—The three points in which any line cuts the sides of a triangle and the projections, from any point in the plane, of the vertices of the triangle on to the same line are six points in involution.

Or again—

The projections from any point on to any line of the six vertices

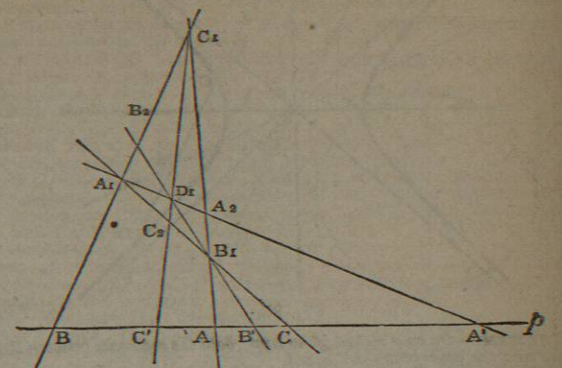

Fig 30.

of a four-side are six points in involution, the projections of opposite vertices being conjugate points.

This property gives a simple means to construct, by aid of the straight edge only, in an involution of which two pairs of conjugate points are given, to any point its conjugate.

§ 79. The theory of involution may at once be extended from the row to the flat and the axial pencil—viz., we say that there is an involution in a flat or in an axial pencil if any line cuts the pencil in an involution of points. An involution in a pencil consists of pairs of conjugate rays or planes; it has two, one, or no *focal rays* or *planes*, but nothing corresponding to a centre.

An involution in a flat pencil contains always one, and in general only one, pair of conjugate rays which are perpendicular to one another. For in two projective flat pencils exist always two corresponding right angles (§ 40).

Each involution in an axial pencil contains in the same manner one pair of conjugate planes at right angles to one another.

As a rule, there exists but one pair of conjugate lines or planes at right angles to each other. But it is possible that there are more, and then there is an infinite number of such pairs. An involution in a flat pencil, in which every ray is perpendicular to its conjugate ray, is said to be *circular*. That such involution is possible is easily seen thus:—if in two concentric flat pencils each ray on one is made to correspond to that ray on the other which is perpendicular to it, then the two pencils are projective, for if we turn the one pencil through a right angle each ray in one coincides with its corresponding ray in the other. But these two projective pencils are in involution.

A circular involution has no focal rays, because no ray in a pencil coincides with the ray perpendicular to it.

§ 80. THEOREM.—*Every elliptical involution in a row may be considered as a section of a circular involution.*

Proof.—In an elliptical involution any two segments AA' and BB' lie partly within partly without each other (fig. 31). Hence two circles described on AA' and BB' as diameters will intersect in two points E and E'. The line EE' cuts the base of the involution at a point O, which, from a well known proposition (Eucl. III. 35), has the property that $OA \, . \, OA' = OB \, . \, OB'$, for each is equal to $OE \, . \, OE'$. The point O is therefore the centre of the involution. If we wish to construct to any point C the conjugate point C', we may draw the circle through CEE'. This will cut the base in the required point C' for $OC \, . \, OC' = OA \, . \, OA'$. But EC and EC' are at right angles. Hence the involution which is obtained by joining E or E' to the points in the given involution is circular. This may also be expressed thus:—

Every elliptical involution has the property that there are two definite points in the plane, from which any two conjugate points are seen under a right angle.

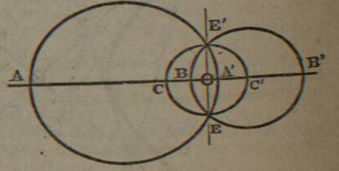

Fig. 31.

At the same time the following problem has been solved:—

Problem.—To determine the centre and also the point corresponding to any given point in an elliptical involution of which two pairs of conjugate points are given.

§ 81. By the aid of § 53, the points on a conic may be made to correspond to those on a line, so that the row of points on the conic is projective to a row of points on a line. We may also have two projective rows on the same conic, and these will be in involution as soon as one point on the conic has the same point corresponding to it all the same to whatever row it belongs. An involution of points on a conic will have the property (as follows from its definition, and from § 53) that the lines which join conjugate points of the involution to any point on the conic are conjugate lines of an involution in a pencil, and that a fixed tangent is cut by the tangents at conjugate points on the conic in points which are again conjugate points of an involution on the fixed tangent. For such involution on a conic the following theorem holds:—

THEOREM.—*The lines which join corresponding points in an involution on a conic all pass through a fixed point*; and reciprocally, *the points of intersection of conjugate lines in an involution among tangents to a conic lie on a line.*

We prove the first part only. The involution is determined by two pairs of conjugate points, say by A, A' and B, B' (fig. 32). Let AA' and BB' meet in P. If we join the points in involution to any point on the conic, and the conjugate points to another point on the conic,

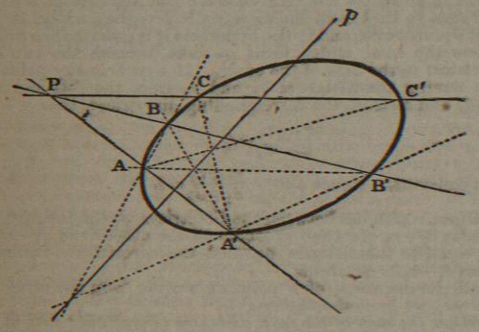

Fig. 32.

we obtain two projective pencils. We take A and A' as centres of these pencils, so that the pencils A(A'BB') and A'(AB'B) are projective, and in perspective position, because AA' corresponds to A'A. Hence corresponding rays meet in a line, of which two points are found by joining AB' to A'B and AB to A'B'. It follows that the *axis* of perspective is the polar of the point P, where AA' and BB' meet. If we now wish to construct to any other point C on the conic the corresponding point C', we join C to A' and the point where this line cuts p to A. The latter line cuts the conic again in C'. But we know from the theory of pole and polar that the line CC' passes through P.

INVOLUTION DETERMINED BY A CONIC ON A LINE.—FOCI.

§ 82. The polars, with regard to a conic, of points in a row p form a pencil P projective to the row (§ 66). This pencil cuts the base of the row p in a projective row.

If A is a point in the given row, A' the point where the polar of A cuts p, then A and A' will be corresponding points. If we take A' a point in the first row, then the polar of A' will pass through A, so that A corresponds to A'—in other words, the rows are in involution. The conjugate points in this involution are conjugate points with regard to the conic. Conjugate points coincide only if the polar of a point A passes through A—that is, if A lies on the conic. Hence—

THEOREM.—*A conic determines on every line in its plane an involution, in which those points are conjugate which are also conjugate with regard to the conic.*

If the line cuts the conic the involution is hyperbolic, the points of intersection being the foci.

If the line touches the conic the involution is parabolic, the two foci coinciding at the point of contact.

If the line does not cut the conic the involution is elliptic, having no foci.

If, on the other hand, we take a point P in the plain of a conic, we get to each line a through P one conjugate line which joins P to the pole of a. These pairs of conjugate lines through P form an involution in the pencil at P. The focal rays of this involution are the tangents drawn from P to the conic. This gives the theorem reciprocal to the last, viz.:—

THEOREM.—*A conic determines in every pencil in its plane an involution, corresponding lines being conjugate lines with regard to the conic.*

If the point is without the conic the involution is hyperbolic, the tangents from the points being the focal rays.

If the point lies on the conic the involution is parabolic, the tangent at the point counting for coincident focal rays.

If the point is within the conic the involution is elliptic, having no focal rays.

It will further be seen that the involution determined by a conic on any line p is a section of the involution, which is determined by the conic at the pole P of p.

§ 83. *Definition.*—The centre of a pencil in which the conic determines a circular involution is called a "focus" of the conic.

In other words—

A focus is such a point that every line through it is perpendicular to its conjugate line.

The polar to a focus is called a *directrix* of the conic.

From the definition it follows that:—

Every focus lies on an axis, for the line joining a focus to the centre of the conic is a diameter to which the conjugate lines are perpendicular; and

Every line joining two foci is an axis, for the perpendiculars to this line through the foci are conjugate to it. These conjugate lines pass through the pole of the line, the pole lies therefore at infinity, and the line is a diameter, hence by the last property an axis.

It follows that all *foci lie on one axis*, for no line joining a point in one axis to a point in the other can be an axis.

As the conic determines in the pencil which has its centre at a focus a circular involution, no tangents can be drawn from the focus to the conic. Hence *each focus lies within a conic*; and *a directrix does not cut the conic.*

Further properties are found by the following considerations:—

§ 84. Through a point P one line p can be drawn, which is with regard to a given conic conjugate to a given line q, viz., that line which joins the point P to the pole of the line q. If the line q is made to describe a pencil about a point Q, then the line p will describe a pencil about P. These two pencils will be projective, for the line p passes through the pole of q, and whilst q describes the pencil Q, its pole describes a projective row, and this row is perspective to the pencil P.

We now take the point P on an axis of the conic, draw any line p through it, and from the pole of p draw a perpendicular q to p. Let q cut the axis in Q. Then, in the pencils of conjugate lines, which have their centres at P and Q, the lines p and q are conjugate lines at right angles to one another. Besides, to the axis as a ray in either pencil will correspond in the other the perpendicular to the axis (§ 72). The conic generated by the intersection of corresponding lines in the two pencils is therefore the circle on PQ as diameter, *so that every line in P is perpendicular to its corresponding line in Q.*

To every point P on an axis of a conic corresponds thus a point Q, such that conjugate lines through P and Q are perpendicular.

We shall show that these *point-pairs P, Q form an involution.* To do this let us move P along the axis, and with it the line p, keeping the latter parallel to itself. Then P describes a row, p a perspective pencil (of parallels), and the pole of p a projective row. At the same time the line q describes a pencil of parallels perpendicular to p, and perspective to the row formed by the pole of p. The point Q, therefore, where q cuts the axis, describes a row projective to the row of points P. The two points P and Q describe thus two projective rows on the axis; and not only does P as a point in the first now correspond to Q, but also Q as a point in the first corresponds to P. The two rows therefore form an involution. *The centre of this involution, it is easily seen, is the centre of the conic.*

A focus of this involution has the property that any two conjugate lines through it are perpendicular; hence, it is a focus to the conic.

Such involution exists on each axis. But only one of these can have foci, because all foci lie on the same axis. The involution on one of the axis is elliptic, and appears (§ 80) therefore as the section of two circular involutions in two pencils whose centres lie in the other axis. These centres are foci, hence the one axis contains two foci, the other axis none; *or every central conic has two foci which lie on one axis equidistant from the centre.*

The axis which contains the foci is called the *principal axis*; in case of an hyperbola it is the axis which cuts the curve, because the foci lie within the conic.

In case of the parabola there is but one axis. The involution on this axis has its centre at infinity. One focus is therefore at infinity, the one focus only is finite. *A parabola has only one focus.*

§ 85. If through any point P (fig. 33) on a conic the tangent PT and the normal PN (*i.e.*, the perpendicular to the tangent through the point of contact) be drawn, these will be conjugate lines with regard to the conic, and at right angles to each other.

They will therefore cut the principal axis in two points, which are conjugate in the involution considered in § 84; hence they are harmonic conjugates with regard to the foci. If therefore the two foci F_1 and F_2 be joined to P, these lines will be harmonic with

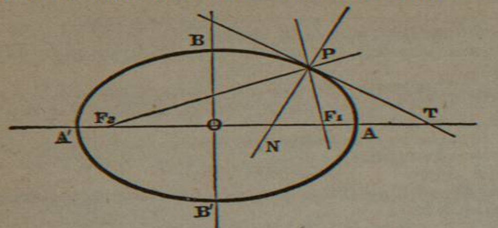

Fig. 33.

regard to the tangent and normal. As the latter are perpendicular, they will bisect the angles between the other pair. Hence—

The line joining any point on a conic to the two foci are equally inclined to the tangent and normal at that point.

In case of the parabola this becomes—

The line joining any point on a parabola to the focus and the diameter through the point, are equally inclined to the tangent and normal at that point.

From the definition of a focus it follows that—

The segment of a tangent between the directrix and the point of contact is seen from the focus belonging to the directrix under a right angle, because the lines joining the focus to the ends of this segment are conjugate with regard to the conic, and therefore perpendicular.

With equal ease the following theorem is proved :—

The two lines which join the points of contact of two tangents each to one focus, but not both to the same, are seen from the intersection of the tangents under equal angles.

§. 86. Other focal properties of a conic are obtained by the following considerations :—

Let F (fig. 34) be a focus to a conic, f the corresponding directrix, A and B the points of contact of two tangents meeting at T, and P the point where the line AB cuts the directrix. Then TF will be the polar of P (because polars of F and T meet at P). Hence TF

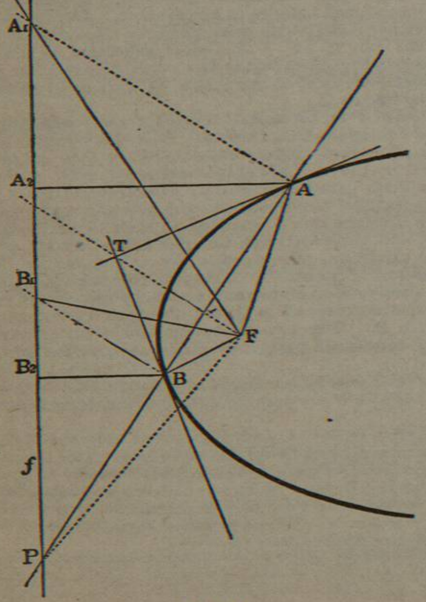

Fig. 34.

and PF are conjugate lines through a focus, and therefore perpendicular. They are further harmonic conjugates with regard to FA and FB (§§ 64 and 13), so that they bisect the angles formed by these lines. This by the way proves—

The segments between the point of intersection of two tangents to a conic and their points of contact are seen from a focus under equal angles.

If we next draw through A and B lines parallel to TF, then the points A_1, B_1 where these cut the directrix will be harmonic conjugates with regard to P and the point where FT cuts the directrix. The lines FT and FP bisect therefore also the angles between FA_1 and FB_1. From this it follows easily that the triangles FAA_1 and FBB_1 are equiangular, and therefore similar, so that

$$FA : AA_1 = FB : BB_1.$$

The triangles AA_1A_2 and BB_1B_2 formed by drawing perpendiculars from A and B to the directrix are also similar, so that $AA_1 : AA_2 = BB_1 : BB_2$.

This, combined with the above proportion, gives

$$FA : AA_2 = FB : BB_2.$$

Hence the theorem :—

The ratio of the distances of any point on a conic from a focus and the corresponding directrix is constant.

To determine this ratio we consider its value for a vertex on the principal axis. In an ellipse the focus lies between the two vertices on this axis, hence the focus is nearer to a vertex than to the corresponding directrix. Similarly in an hyperbola a vertex is nearer to the directrix than to the focus. In a parabola the vertex lies halfway between directrix and focus.

It follows in an ellipse the ratio between the distance of a point from the focus to that from the directrix is less than unity, in the parabola it equals unity, and in the hyperbola it is greater than unity.

It is here the same which focus we take, because the two foci lie symmetrical to the axis of the conic. If now P is any point on the conic having the distances r_1 and r_2 from the foci and the distances d_1 and d_2 from the corresponding directrices, then

$$\frac{r_1}{d_1} = \frac{r_2}{d_2} = e,$$

where e is constant. Hence also $\dfrac{r_1 \pm r_2}{d_1 \pm d_2} = e$.

In the ellipse, which lies between the directrices, $d_1 + d_2$ is constant, therefore also $r_1 + r_2$. In the hyperbola on the other hand $d_1 - d_2$ is constant, equal to the distance between the directrices, therefore in this case $r_1 - r_2$ is constant.

If we call the distances of a point on a conic from the focus its focal distances we have the theorem :—

In an ellipse the sum of the focal distances is constant; and in an hyperbola the difference of the focal distances is constant.

This constant sum or difference equals in both cases the length of the principal axis.

Pencil of Conics.

§ 87. Through four points A, B, C, D in a plane, of which no three lie in a line, an infinite number of conics may be drawn, viz., through these four points and any fifth one single conic. *This system of conics is called a pencil of conics.* Similarly all conics touching four fixed lines form a system such that any fifth tangent determines one and only one conic. We have here the theorems :—

Theorem.—The pairs of points in which any line is cut by a system of conics through four fixed points are in involution.

Theorem.—The pairs of tangents which can be drawn from a point to a system of conics touching four fixed lines are in involution.

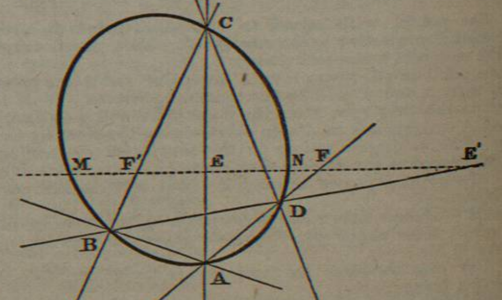

Fig. 35.

We prove the first theorem only. Let ABCD (fig. 35) be the four-point, then any line t will cut two opposite sides AC, BD in the points E, E', the pair AD, BC in points F, F'. and any conic of the system in M, N, and we have

$$A(CDMN) = B(CDMN).$$

If we cut these pencils by t we get

$$(EFMN) = (F'E'MN)$$

or

$$(EFMN) = (E'F'NM).$$

But this is, according to § 77 (7), the condition that M, N are corresponding points in the involution determined by the point pairs E, E', F, F' in which the line t cuts pairs of opposite sides of the four-point ABCD. This involution is independent of the particular conic chosen.

§ 88. There follow several important consequences :—

Theorem.—*Through four points two, one, or no conic may be drawn which touch any given line, according as the involution determined by the given four-point on the line has real, coincident, or imaginary foci.*

Theorem.—*Two, one, or no conics may be drawn which touch four given lines and pass through a given point, according as the involution determined by the given four-side at the point has real, coincident, or imaginary focal rays.*

For the conic through four points which touches a given line has its point of contact at a focus of the involution determined by the four-point on the line.

As a special case we get, by taking the line at infinity :—

Theorem.—*Through four points of which none is at infinity either two or no parabolas may be drawn.*

The problem of drawing a conic through four points and touching a given line is solved by determining the points of contact on the line, that is, by determining the foci of the involution in which the line cuts the sides of the four-point. The corresponding remark holds for the problem of drawing the conics which touch four lines and pass through a given point.

Ruled Quadric Surfaces.

§ 89. Formerly we have considered projective rows which lie in the same plane. In that case, lines joining corresponding points envelope a conic. We shall now consider projective rows whose bases do not meet. In this case, corresponding points will be joined by lines which do not lie in a plane, but on some surface, which like every surface generated by lines is called a *ruled* surface. This surface clearly contains the bases of the two rows.

If the points in either row be joined to the base of the other, we obtain two axial pencils which are also projective, those planes being corresponding which pass through corresponding points in the given rows. If A, A' be two corresponding points, a, a' the planes in the axial pencils passing through them, then AA' will be the line of intersection of the corresponding planes a, a', and also the line joining corresponding points in the rows.

If we cut the whole figure by a plane this will cut the axial pencils in two projective flat pencils, and the curve of the second order generated by these will be the curve in which the plane cuts the surface. Hence

Theorem.—*The locus of lines joining corresponding points in two projective rows which do not lie in the same plane is a surface which contains the bases of the rows, and which can also be generated by the lines of intersection of corresponding planes in two projective axial pencils. This surface is cut by every plane in a curve of the second order, hence either in a conic or in a line-pair. No line which does not lie altogether on the surface can have more than two points in common with the surface, which is therefore said to be of the second order, or is called a ruled quadric surface.*

That no line which does not lie on the surface can cut the surface in more than two points is seen at once if a plane be drawn through the line, for this will cut the surface in a conic. It follows also that

A line which contains more than two points of the surface lies altogether on the surface.

§ 90. Through any point in space one line can always be drawn cutting two given lines which do not themselves meet.

If therefore three lines in space be given of which no two meet, then through every point in either one line may be drawn cutting the other two.

Theorem.—*If a line moves so that it always cuts three given lines of which no two meet, then it generates a ruled quadric surface.*

Proof.—Let a, b, c be the given lines, and $p, q, r \ldots$ lines cutting them in the points A, A', A'' . . . ; B, B', B'' . . . ; C, C', C'' . . . respectively; then the planes through a containing p, q, r, and the planes through b containing the same lines, may be taken as corresponding planes in two axial pencils which are projective, because both pencils cut the line c in the same row C, C', C'' . . . ; the surface can therefore be generated by projective axial pencils.

Of the lines $p, q, r \ldots$ no two can meet, for otherwise the lines a, b, c which cut them would also lie in their plane. There is a single infinite number of them, for one passes through each point of a. These lines are said to form a *set* of lines on the surface.

If now three of the lines p, q, r be taken, then every line d cutting them will have three points in common with the surface, and will therefore lie altogether on it. This gives rise to a second set of lines on the surface. From what has been said the theorem follows:—

Theorem.—*A ruled quadric surface contains two sets of straight lines. Every line of one set cuts every line of the other, but no two lines of the same set meet.*

Any two lines of the same set may be taken as bases of two projective rows, or of two projective pencils which generate the surface. They are cut by the lines of the other set in two projective rows.

The plane at infinity like every other plane cuts the surface either in a conic proper or in a line-pair. In the first case the surface is called an *Hyperboloid of one sheet*, in the second an *Hyperbolic Paraboloid.*

The latter may be generated by a line cutting three lines of which one lies at infinity that is, cutting two lines and remaining parallel to a given plane.

Quadric Surfaces.

§ 91. The conics, the cones of the second order, and the ruled quadric surfaces complete the figures which can be generated by projective rows or flat and axial pencils, that is, by those aggregates of elements which are of one dimension (§§ 5, 6). We shall now consider the simpler figures which are generated by aggregates of two dimensions. The space at our disposal will not, however, allow us to do more than indicate a few of the results.

§ 92. We establish a correspondence between the lines and planes in pencils in space, or reciprocally between the points and lines in two or more planes, but consider principally pencils.

In two pencils we may either make planes correspond to planes and lines to lines, or else planes to lines and lines to planes. If hereby the condition be satisfied that to a flat, or axial, pencil corresponds in the first case a projective flat, or axial, pencil, and in the second a projective axial, or flat, pencil, the pencils are said to be *projective* in the first case and *reciprocal* in the second.

For instance, two pencils which join two points S_1 and S_2 to the different points and lines in a given plane π are projective (and in perspective position), if those lines and planes be taken as corresponding which meet the plane π in the same point or in the same line. In this case every plane through both centres S_1 and S_2 of the two pencils will correspond to itself. If these pencils are brought into any other position they will be projective (but not perspective).

The correspondence between two projective pencils is uniquely determined, if to four rays (or planes) in the one the corresponding rays (or planes) in the other are given, provided that no three rays of either set lie in a plane.

Proof.—Let a, b, c, d be four rays in the one, a', b', c', d' the corresponding rays in the other pencil. We shall show that we can find for every ray e in the first a single corresponding ray e' in the second. To the axial pencil $a\,(b, c, d \ldots)$ formed by the planes which join a to $b, c, d \ldots$, respectively corresponds the axial pencil $a'\,(b', c', d' \ldots)$, and this correspondence is determined. Hence, the plane $a'e'$ which corresponds to the plane ae is determined. Similarly the plane $b'e'$ may be found and both together determine the ray e'.

Similarly the correspondence between two reciprocal pencils is determined if for four rays in the one the corresponding planes in the other are given.

§ 93. We may now combine—

1. Two reciprocal pencils.
   Each ray cuts its corresponding plane in a point, the locus of these points is a quadric surface.
2. Two projective pencils.
   Each plane cuts its corresponding plane in a line, but a ray as a rule does not cut its corresponding ray. The locus of points where a ray cuts its corresponding ray is a twisted cubic. The lines where a plane cuts its corresponding plane are secants.
3. Three projective pencils.
   The locus of intersection of corresponding planes is a cubic surface.

Of these we consider only the first two cases.

§ 94. If two pencils are reciprocal, then to a plane in either corresponds a line in the other, to a flat pencil an axial pencil, and so on. Every line cuts its corresponding plane in a point. If S_1 and S_2 be the centres of the two pencils, and P be a point where a line a_1 in the first cuts its corresponding plane a_2, *then the line b_2 in the pencil S_2 which passes through P will meet its corresponding plane β_1 in P.* For b_2 is a line in the plane a_2. The corresponding plane β_1 must therefore pass through the line a_1, hence through P.

The points in which the lines in S_1 cut the planes corresponding to them in S_2 are therefore the same as the points in which the lines in S_2 cut the planes corresponding to them in S_1.

The locus of these points is a surface which is cut by a plane in a conic or in a line-pair and by a line in not more than two points unless it lies altogether on the surface. The surface itself is therefore called a quadric surface, or a surface of the second order.

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Every line which has three points in common with a quadric surface lies on the surface.

Every conic which has five points in common with a quadric surface lies on the surface.

Through two conics which lie in different planes, but have two points in common, and through one external point always one quadric surface may be drawn.

§ 97. *Every plane which cuts a quadric surface in a line-pair is a tangent plane.* For every line in this plane through the centre of the line-pair (the point of intersection of the two lines) cuts the surface in two coincident points and is therefore a tangent to the surface, *the centre of the line-pair being the point of contact.*

If a quadric surface contains a line, then every plane through this line cuts the surface in a line-pair (or in two coincident lines). For this plane cannot cut the surface in a conic. Hence

If a quadric surface contains one line p *then it contains an infinite number of lines, and through every point* Q *on the surface, one line* q *can be drawn which cuts* p. For the plane through the point Q and the line p cuts the surface in a line-pair which must pass through Q and of which p is one line.

No two such lines q *on the surface can meet.* For as both meet p their plane would contain p and therefore cut the surface in a triangle.

Every line which cuts three lines q *will be on the surface*; for it has three points in common with it.

Hence the quadric surfaces which contain lines are the same as the ruled quadric surfaces considered in §§ 89–93, but with one important exception. In the last investigation we have left out of consideration the possibility of a plane having only one line (two coincident lines) in common with a quadric surface.

§ 98. To investigate this case we suppose first that there is one point A on the surface through which two different lines a, b can be drawn, which lie altogether on the surface.

If P is any other point on the surface which lies neither on a nor b, then the plane through P and a will cut the surface in a second line a' which passes through P and which cuts a. Similarly there is a line b' through P which cuts b. These two lines a' and b' *may* coincide, but then they must coincide with PA.

If this happens for one point P, it happens for every other point Q. For if two different lines could be drawn through Q, then by the same reasoning the line PQ would be altogether on the surface, hence two lines would be drawn through P against the assumption. From this follows:—

If there is one point on a quadric surface through which one, but only one, line can be drawn on the surface, then through every point one line can be drawn, and all these lines meet in a point. The surface is a cone of the second order.

If through one point on a quadric surface, two, and only two, lines can be drawn on the surface, then through every point two lines may be drawn, and the surface is a ruled quadric surface.

If through one point on a quadric surface no line on the surface can be drawn, then the surface contains no lines.

Using the definitions at the end of § 95, we may also say:—

On a quadric surface the points are all hyperbolic, or all parabolic, or all elliptic.

As an example of a quadric surface with elliptical points, we mention the sphere which may be generated by two reciprocal pencils, where to each line in one corresponds the plane perpendicular to it in the other.

§ 99. *Poles and Polar Planes.*—The theory of poles and polars with regard to a conic is easily extended to quadric surfaces.

Let P be a point in space not on the surface, which we suppose not to be a cone. On every line through P which cuts the surface in two points we determine the harmonic conjugate Q of P with regard to the points of intersection. Through one of these lines we draw two planes α and β. The locus of the points Q in α is a line a, the polar of P with regard to the conic in which α cuts the surface. Similarly the locus of points Q in β is a line b. This cuts a, because the line of intersection of α and β contains but one point Q. The locus of all points Q therefore is a plane. *This plane is called the polar plane of the point* P, *with regard to the quadric surface. If* P *lies on the surface we take the tangent plane of* P *as its polar.*

The following propositions hold:—

1. *Every point has a polar plane*, which is constructed by drawing the polars of the point with regard to the conics in which two planes through the point cut the surface.

2. *If* Q *is a point in the polar of* P, *then* P *is a point in the polar of* Q, because this is true with regard to the conic in which a plane through PQ cuts the surface.

3. *Every plane is the polar plane of one point, which is called the Pole of the plane.*

The pole to a plane is found by constructing the polar planes of three points in the plane. Their intersection will be the pole.

4. *The points in which the polar plane of* P *cuts the surface are points of contact of tangents drawn from* P *to the surface*, as is easily seen. Hence:—

5. *The tangents drawn from a point* P *to a quadric surface form a cone of the second order*, for the polar plane of P cuts it in a conic.

6. *If the pole describes a line* a, *its polar plane will turn about another line* a′, as follows from 2. *These lines* a *and* a′ *are said to be conjugate with regard to the surface.*

§ 100. The pole of the line at infinity is called the *centre* of the surface. If it lies at the infinity, the plane at infinity is a tangent plane, and the surface is called a *paraboloid.*

The polar plane to any point at infinity passes through the centre, and is called a diametrical plane.

A line through the centre is called a diameter. It is bisected at the centre. The line conjugate to it lies at infinity.

If a point moves along a diameter its polar plane turns about the conjugate line at infinity; that is, *it moves parallel to itself, its centre moving on the first line.*

The middle points of parallel chords lie in a plane, viz., in the polar plane of the point at infinity through which the chords are drawn.

The centres of parallel sections lie in a diameter which is a line conjugate to the line at infinity in which the planes meet.

Twisted Cubics.

§ 101. If two pencils with centres S_1 and S_2 are made projective, then to a ray in one corresponds a ray in the other, to a plane a plane, to a flat or axial pencil a projective flat or axial pencil, and so on.

There is a double infinite number of lines in a pencil. We shall see that a single infinite number of lines in one pencil meets its corresponding ray, and that the points of intersection form a curve in space.

Of the double infinite number of planes in the pencils each will meet its corresponding plane. This gives a system of a double infinite number of lines in space. We know (§ 5) that there is a quadruple infinite number of lines in space. From among these we may select those which satisfy one or more given conditions. The systems of lines thus obtained was first systematically investigated and classified by Plücker, in his *Geometrie des Raumes.* He uses the following names:—

A *treble infinite* number of lines, that is, all lines which satisfy one condition, are said to form a *complex of lines*; *e.g.*, all lines cutting a given line, or all lines touching a surface.

A *double infinite* number of lines, that is, all lines which satisfy two conditions, or which are common to two complexes, are said to form a *congruence of lines*; *e.g.*, all lines in a plane, or all lines cutting two curves, or all lines cutting a given curve twice.

A *single infinite* number of lines, that is, all lines which satisfy three conditions, or which belong to three complexes, form a *ruled surface*; *e.g.*, one set of lines on a ruled quadric surface, or developable surfaces which are formed by the tangents to a curve.

It follows that all lines in which corresponding planes in two projective pencils meet form a congruence. We shall see this congruence consists of all lines which cut a twisted cubic twice, or of all *secants* to a twisted cubic.

§ 102. Let l_1 be the line S_1S_2 as a line in the pencil S_1. To it corresponds a line l_2 in S_2. *At each of the centres two corresponding lines meet.* The two axial pencils with l_1 and l_2 as axes are projective, and, as their axes meet at S_2, the intersections of corresponding planes form a cone of the second order (§ 58), with S_2 as centre. If π_1 and π_2 be corresponding planes, then their intersection will be a line p_2 which passes through S_2. Corresponding to it in S_1 will be a line p_1 which lies in the plane π_1, and which therefore meets p_2 at some point P. Conversely, if p_2 be any line in S_2 which meets its corresponding line p_1 at a point P, then to the plane $l_2\,p_2$ will correspond the plane $l_1\,p_1$, that is, the plane S_1S_2P. These planes intersect in p_2, so that p_2 is a line on the quadric cone generated by the axial pencils l_1 and l_2. Hence:—

All lines in one pencil which meet their corresponding lines in the other form a cone of the second order which has it centre at the centre of the first pencil, and passes through the centre of the second.

From this follows that the points in which corresponding rays meet lie on two cones of the second order which have the ray joining their centres in common, and form therefore, together with the line S_1S_2 or l_1, the intersection of these cones. Any plane cuts each of the cones in a conic. These two conics have necessarily that point in common in which it cuts the line l_1, and therefore besides either one or three other points. It follows that the curve is of the *third order* as a plane may cut it in three, but not in more than three, points. Hence:—

The locus of points in which corresponding lines on two projective pencils meet is a curve of the third order or a "twisted cubic" k, *which passes through the centres of the pencils, and which appears as the intersection of two cones of the second order, which have one line in common.*

A line belonging to the congruence determined by the pencils is a secant of the cubic; it has two, or one, or no points in common with this cubic, and is called accordingly a secant proper, a tangent, or a

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