28. In plane geometry, reckoning the line as a curve of the first order, we have only the point and the curve. In solid geometry, reckoning a line as a curve of the first order, and the plane as a surface of the first order, we

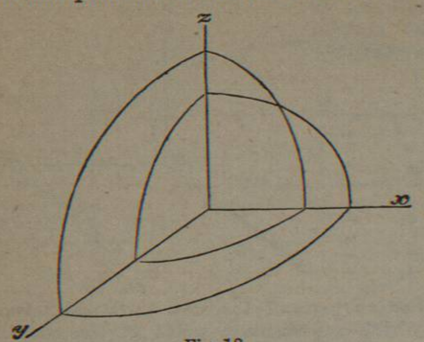

Fig. 18.

have the point, the curve, and the surface; but the increase of complexity is far greater that would hence at first sight appear. In plane geometry a curve is considered in connexion with lines (its tangents); but in solid geometry the curve is considered in connexion with lines and planes (its tangents and osculating planes), and the surface also in connexion with lines and planes (its tangent lines and tangent planes); there are surfaces arising out of the line—cones, skew surfaces, developables, doubly and triply infinite systems of lines, and whole classes of theories which have nothing analogous to them in plane geometry: it is thus a very small part indeed of the subject which can be even referred to in the present article.

In the case of a surface we have between the coordinates (x, y, z) a single, or say a onefold relation, which can be represented by a single relation $f(x, y, z)=0$; or we may consider the coordinates expressed each of them as a given function of two variable parameters p, q; the form $z=f(x, y)$ is a particular case of each of these modes of representation; in other words, we have in the first mode $f(x, y, z)=z-f(x, y)$, and in the second mode $x=p, y=q$ for the expression of two of the coordinates in terms of the parameters.

In the case of a curve we have between the coordinates (x, y, z) a twofold relation: two equations $f(x, y, z)=0$, $\phi(x, y, z)=0$ give such a relation; *i.e.*, the curve is here considered as the intersection of two surfaces (but the curve is not always the complete intersection of two surfaces, and there are hence difficulties); or, again, the coordinates may be given each of them as a function of a single variable parameter. The form $y=\phi x$, $z=\psi x$, where two of the coordinates are given in terms of the third, is a particular case of each of these modes of representation.

29. The remarks under plane geometry as to descriptive and metrical propositions, and as to the non-metrical character of the method of coordinates when used for the proof of a descriptive proposition, apply also to solid geometry; and they might be illustrated in like manner by the instance of the theorem of the radical centre of four spheres. The proof is obtained from the consideration that S and S′ being each of them a function of the form $x^2+y^2+z^2+ax+by+cz+d$, the difference $S-S'$ is a mere linear function of the coordinates, and consequently that $S-S'=0$ is the equation of the plane containing the circle of intersection of the two spheres $S=0$ and $S'=0$.

Metrical Theory.

30. The foundation in solid geometry of the metrical theory is in fact the before-mentioned theorem that if a finite right line PQ be projected upon any other line OO′ by lines perpendicular to OO′, then the length of the projection P′Q′ is equal to the length of PQ into the cosine of its inclination to P′Q′—or (in the form in which it is now convenient to state the theorem) the perpendicular distance P′Q′ of two parallel planes is equal to the inclined distance PQ into the cosine of the inclination. Hence also the algebraical sum of the projections of the sides of a closed polygon upon any line is $=0$; or, reversing the signs of certain sides and considering the polygon as made up of two broken lines each extending from the same initial to the same terminal point, the sum of the projections of the one set of lines upon any line is equal to the sum of the projections of the other set of lines upon the same line. When any of the lines are at right angles to the given line (or, what is the same thing, in a plane at right angles to the given line) the projections of these lines severally vanish.

31. Consider the skew quadrilateral QMNP, the sides QM, MN, NP being respectively parallel to the three rectangular axes Ox, Oy, Oz; let the lengths of these sides be ξ, η, ζ, and that of the side QP be $=\rho$; and let the cosines of the inclinations (or say the cosine-inclinations) of ρ to the three axes be a, β, γ; then projecting successively on the three sides and on QP we have

$$\xi, \eta, \zeta=\rho a, \rho\beta, \rho\gamma,$$

and

$$\rho=a\xi+\beta\eta+\gamma\zeta,$$

whence $\rho^2=\xi^2+\eta^2+\zeta^2$, which is the relation between a distance ρ and its projections ξ, η, ζ upon three rectangular axes. And from the same equations we obtain $a^2+\beta^2+\gamma^2=1$, which is a relation connecting the cosine-inclinations of a line to three rectangular axes.

Suppose we have through Q any other line QT, and let the cosine-inclinations of this to the axes be a', β', γ', and δ be its cosine-inclination to QP; also let p be the length of the projection of QP upon QT; then projecting on QT we have

$$p=a'\xi+\beta'\eta+\gamma'\zeta, =\rho\delta.$$

And in the last equation substituting for ξ, η, ζ their values $\rho a, \rho\beta, \rho\gamma$ we find

$$\delta=aa'+\beta\beta'+\gamma\gamma',$$

which is an expression for the mutual cosine-inclination of two lines, the cosine-inclinations of which to the axes are a, β, γ and a', β', γ' respectively. We have of course $a^2+\beta^2+\gamma^2=1$, and $a'^2+\beta'^2+\gamma'^2=1$; and hence also

$$1-\delta^2=a^2+\beta^2+\gamma^2)(a'^2+\beta'^2+\gamma'^2)-(aa'+\beta\beta'+\gamma\gamma')^2,$$
$$=(\beta\gamma'-\beta'\gamma)^2+(\gamma a'-\gamma' a)^2+(a\beta'-a'\beta)^2;$$

so that the sine of the inclination can only be expressed as a square root. These formulæ are the foundation of spherical trigonometry.

The Line, Plane, and Sphere.

32. The foregoing formulæ give at once the equations of these loci.

For first, taking Q to be a fixed point, coordinates (a, b, c) and the cosine-inclinations (a, β, γ) to be constant, then P will be a point in the line through Q in the direction thus determined; or, taking (x, y, z) for its coordinates, these will be the current coordinates of a point in the line. The values of ξ, η, ζ then are $x-a, y-b, z-c$, and we thus have

$$\frac{x-a}{a}=\frac{y-b}{\beta}=\frac{z-c}{\gamma}(=\rho),$$

which (omitting the last equation, $=\rho$) are the equations of the line through the point (a, b, c), the cosine-inclinations to the axes being a, β, γ, and these quantities being connected by the relation $a^2+\beta^2+\gamma^2=1$. This equation may be omitted, and then a, β, γ, instead of being equal, will only be proportional to the cosine-inclinations.

Using the last equation, and writing

$$x, y, z=a+a\rho, b+\beta\rho, c+\gamma\rho,$$

these are expressions for the current coordinates in terms of a parameter ρ, which is in fact the distance from the fixed point (a, b, c).

It is easy to see that, if the coordinates (x, y, z) are connected by any two linear equations, these equations can always be brought

into the foregoing form, and hence that the two linear equations represent a line.

Secondly, taking for greater simplicity the point Q to be coincident with the origin, and a', β', γ', p to be constant, then p is the perpendicular distance of a plane from the origin, and a', β', γ' are the cosine-inclinations of this distance to the axes $(a'^2+\beta'^2+\gamma'^2=1)$. P is any point in this plane, and taking its coordinates to be (x, y, z) then (ξ, η, ζ) are $=(x, y, z)$, and the foregoing equation $p=a'\xi+\beta'\eta+\gamma'\zeta$ becomes

$$a'x+\beta'y+\gamma'z=p,$$

which is the equation of the plane in question.

If, more generally, Q is not coincident with the origin, then, taking its coordinates to be (a, b, c), and writing p_1 instead of p, the equation is

$$a'(x-a)+\beta'(y-b)+\gamma'(z-c)=p_1;$$

and we thence have $p_1=p-(aa'+b\beta'+c\gamma')$, which is an expression for the perpendicular distance of the point (a, b, c) from the plane in question.

It is obvious that any linear equation $Ax+By+Cz+D=0$ between the coordinates can always be brought into the foregoing form, and hence that such equation represents a plane.

Thirdly, supposing Q to be a fixed point, coordinates (a, b, c) and the distance QP, $=\rho$, to be constant, say this is $=d$, then, as before, the values of ξ, η, ζ are $x-a, y-b, z-c$, and the equation $\xi^2+\eta^2+\zeta^2=\rho^2$ becomes

$$(x-a)^2+(y-b)^2+(z-c)^2=d^2,$$

which is the equation of the sphere, coordinates of the centre $=(a, b, c)$ and radius $=d$.

A quadric equation wherein the terms of the second order are $x^2+y^2+z^2$, viz., an equation

$$x^2+y^2+z^2+Ax+By+Cz+D=0,$$

can always, it is clear, be brought into the foregoing form; and it thus appears that this is the equation of a sphere, coordinates of the centre $-\frac{1}{2}A, -\frac{1}{2}B, -\frac{1}{2}C$, and squared radius $=\frac{1}{4}(A^2+B^2+C^2)-D$.

Cylinders, Cones, Ruled Surfaces.

33. A singly infinite system of lines or system of lines depending upon one variable parameter forms a surface; and the equation of the surface is obtained by eliminating the parameter between the two equations of the line.

If the lines all pass through a given point, then the surface is a cone; and, in particular, if the lines are all parallel to a given line, then the surface is a cylinder.

Beginning with this last case, suppose the lines are parallel to the line $x=mz$, $y=nz$, the equations of a line of the system are $x=mz+a$, $y=nz+b$,—where a, b are supposed to be functions of the variable parameter, or, what is the same thing, there is between them a relation $f(a, b)=0$: we have $a=x-mz$, $b=y-nz$, and the result of the elimination of the parameter therefore is $f(x-mz, y-nz)=0$, which is thus the general equation of the cylinder the generating lines whereof are parallel to the line $x=mz$, $y=nz$. The equation of the section by the plane $z=0$ is $f(x, y)=0$, and conversely if the cylinder be determined by means of its curve of intersection with the plane $z=0$, then, taking the equation of this curve to be $f(x, y)=0$, the equation of the cylinder is $f(x-mz, y-nz)=0$. Thus, if the curve of intersection be the circle $(x-a)^2+(y-\beta)^2=\gamma^2$, we have $(x-mz-a)^2+(y-nz-\beta)^2=\gamma^2$ as the equation of an oblique cylinder on this base, and thus also $(x-a)^2+(y-\beta)^2=\gamma^2$ as the equation of the right cylinder.

If the lines all pass through a given point (a, b, c), then the equations of a line are $x-a=a(z-c)$, $y-b=\beta(z-c)$, where a, β are functions of the variable parameter, or, what is the same thing, there exists between them an equation $f(a, \beta)=0$; the elimination of the parameter gives, therefore, $f\left(\frac{x-a}{z-c}, \frac{y-b}{z-c}\right)=0$; and this equation, or, what is the same thing, any homogeneous equation $f(x-a, y-b, z-c)=0$, or, taking f to be a rational and integral function of the order n, say $(*)(x-a, y-b, z-c)^n=0$, is the general equation of the cone having the point (a, b, c) for its vertex. Taking the vertex to be at the origin, the equation is $(*)(x, y, z)^n=0$; and, in particular, $(*)(x, y, z)^2=0$ is the equation of a cone of the second order, or quadricone, having the origin for its vertex.

34. In the general case of a singly infinite system of lines, the locus is a ruled surface (or *regulus*). If the system be such that a line does not intersect the consecutive line, then the surface is a skew surface, or scroll; but if it be such that each line intersects the consecutive line, then it is a developable, or torse.

Suppose, for instance, that the equations of a line (depending on the variable parameter θ) are $\frac{x}{a}+\frac{y}{c}=\theta\left(1+\frac{y}{b}\right)$, $\frac{x}{a}-\frac{z}{c}=\frac{1}{\theta}\left(1-\frac{y}{b}\right)$, then, eliminating θ, we have $\frac{x^2}{a^2}-\frac{z^2}{c^2}=1-\frac{y^2}{b^2}$, or say $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$, the equation of a quadric surface, afterwards called the hyperboloid of one sheet; this surface is consequently a scroll. It is to be remarked that we have upon the surface a second singly infinite series of lines; the equations of a line of this second system (depending on the variable parameter ϕ) are

$$\frac{x}{a}+\frac{z}{c}=\phi\left(1-\frac{y}{b}\right), \frac{x}{a}-\frac{z}{c}=\frac{1}{\phi}\left(1+\frac{y}{b}\right).$$

It is easily shown that any line of the one system intersects every line of the other system.

Considering any curve (of double curvature) whatever, the tangent lines of the curve form a singly infinite system of lines, each line intersecting the consecutive line of the system,—that is, they form a developable, or torse; the curve and torse are thus inseparably connected together, forming a single geometrical figure. A plane through three consecutive points of the curve (or osculating plane of the curve) contains two consecutive tangents, that is, two consecutive lines of the torse, and is thus a tangent plane of the torse along a generating line.

Transformation of Coordinates.

35. There is no difficulty in changing the origin, and it is for brevity assumed that the origin remains unaltered. We have, then, two sets of rectangular axes, Ox, Oy, Oz, and Ox_1, Oy_1, Oz_1, the mutual cosine-inclinations being shown by the diagram—

	x	y	z
x_1	a	β	γ
y_1	a'	β'	γ'
z_1	a''	β''	γ''

that is, a, β, γ are the cosine-inclinations of Ox_1 to Ox, Oy, Oz; a', β', γ' those of Oy_1, &c.

And this diagram gives also the linear expressions of the coordinates (x_1, y_1, z_1) or (x, y, z) of either set in terms of those of the other set; we thus have

$$\begin{aligned} x_1&=a x+\beta y+\gamma z, & x&=ax_1+a'y_1+a''z_1,\\ y_1&=a'x+\beta'y+\gamma'z, & y&=\beta x_1+\beta'y_1+\beta''z_1,\\ z_1&=a''x+\beta''y+\gamma''z, & z&=\gamma x_1+\gamma'y_1+\gamma''z_1, \end{aligned}$$

which are obtained by projection, as above explained. Each of these equations is, in fact, nothing else than the before-mentioned equation $p=a'\xi+\beta'\eta+\gamma'\zeta$, adapted to the problem in hand.

But we have to consider the relations between the nine coefficients. By what precedes, or by the consideration that we must have identically $x^2+y^2+z^2=x_1^2+y_1^2+z_1^2$, it appears that these satisfy the relations—

$$\begin{aligned} a^2+\beta^2+\gamma^2&=1, & a^2+a'^2+a''^2&=1,\\ a'^2+\beta'^2+\gamma'^2&=1, & \beta^2+\beta'^2+\beta''^2&=1,\\ a''^2+\beta''^2+\gamma''^2&=1, & \gamma^2+\gamma'^2+\gamma''^2&=1,\\ a'a''+\beta'\beta''+\gamma'\gamma''&=0, & \beta\gamma+\beta'\gamma'+\beta''\gamma''&=0,\\ a''a+\beta''\beta+\gamma''\gamma&=0, & \gamma a+\gamma'a'+\gamma''a''&=0,\\ aa'+\beta\beta'+\gamma\gamma'&=0, & a\beta+a'\beta'+a''\beta''&=0, \end{aligned}$$

either set of six equations being implied in the other set.

It follows that the square of the determinant

$$\begin{vmatrix} a, & \beta, & \gamma \\ a', & \beta', & \gamma' \\ a'', & \beta'', & \gamma'' \end{vmatrix}$$

is $=1$; and hence that the determinant itself is $=\pm1$. The distinction of the two cases is an important one: if the determinant is $=+1$, then the axes Ox_1, Oy_1, Oz_1 are such that they can by a rotation about O be brought to coincide with Ox, Oy, Oz respectively; if it is $=-1$, then they cannot. But in the latter case, by measuring x_1, y_1, z_1 in the opposite directions we change the signs of all the coefficients and so make the determinant to be $=+1$; hence this case need alone be considered, and it is accordingly assumed that the determinant is $=+1$. This being so, it is found that we have a further set of nine equations, $a=\beta'\gamma''-\beta''\gamma'$, &c.; that is, the coeficients arranged as in the diagram have the values

$\beta'\gamma''-\beta''\gamma'$	$\gamma'\alpha''-\gamma''\alpha'$	$\alpha'\beta''-\alpha''\beta'$
$\beta''\gamma-\beta\gamma''$	$\gamma''\alpha-\gamma\alpha''$	$\alpha''\beta-\alpha\beta''$
$\beta\gamma'-\beta'\gamma$	$\gamma\alpha'-\gamma'\alpha$	$\alpha\beta'-\alpha'\beta$

36. It is important to express the nine coefficients in terms of three independent quantities. A solution which, although unsymmetrical, is very convenient in Astronomy and Dynamics is to use for the purpose the three angles θ, ϕ, τ of fig. 19; say θ = longitude of the node; ϕ = inclination and τ = longitude of x_1 from node.

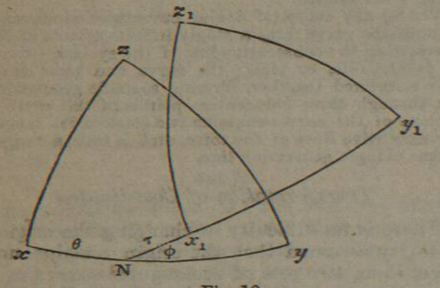

Fig. 19.

The diagram of transformation then is

	x	y	z
x_1	$\cos\tau\cos\theta-\sin\tau\sin\theta\cos\phi$	$\cos\tau\sin\theta+\sin\tau\cos\theta\cos\phi$	$\sin\tau\sin\phi$
y_1	$-\sin\tau\cos\theta-\cos\tau\sin\theta\cos\phi$	$-\sin\tau\sin\theta+\cos\tau\cos\theta\cos\phi$	$\cos\tau\sin\phi$
z_1	$\sin\theta\sin\phi$	$-\cos\theta\sin\phi$	$\cos\phi$

But a more elegant solution (due to Rodrigues) is that contained in the diagram

	x	y	z
x_1	$1+\lambda^2-\mu^2-\nu^2$	$2(\lambda\mu-\nu)$	$2(\lambda\nu+\mu)$
y_1	$2(\lambda\mu+\nu)$	$1-\lambda^2+\mu^2-\nu^2$	$2(\mu\nu-\lambda)$
z_1	$2(\nu\lambda-\mu)$	$2(\mu\nu+\lambda)$	$1-\lambda^2-\mu^2+\nu^2$

$$\div(1+\lambda^2-\mu^2+\nu^2).$$

The nine coefficients of transformation are the nine functions of the diagram, each divided by $1+\lambda^2+\mu^2+\nu^2$; the expressions contain as they should do the three arbitrary quantities λ, μ, ν; and the identity $x_1^2+y_1^2+z_1^2=x^2+y^2+z^2$ can be at once verified. It may be added that the transformation can be expressed in the quaternion form

$$ix_1+jy_1+kz_1=(1+\Lambda)(ix+jy+kz)(1+\Lambda)^{-1}$$

where Λ denotes the vector $i\lambda+j\mu+k\nu$.

Quadric Surfaces (Paraboloids, Ellipsoid, Hyperboloids).

37. It appears by a discussion of the general equation of the second order $(a, \ldots \between x, y, z, 1)^2=0$ that the proper quadric surfaces[1] represented by such an equation are the following five surfaces (a and b positive):—

(1.) $z=\frac{x^2}{2a}+\frac{y^2}{2b}$, elliptic paraboloid.

(2.) $z=\frac{x^2}{2a}-\frac{y^2}{2b}$, hyperbolic paraboloid.

(3.) $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$, ellipsoid.

(4.) $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$, hyperboloid of one sheet.

(5.) $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=-1$, hyperboloid of two sheets.

[1] The improper quadric surfaces represented by the general equation of the second order are (1) the pair of planes or plane-pair, including as a special case the twice repeated plane, and (2) the cone, including as a special case the cylinder. There is but one form of cone; but the cylinder may be parabolic, elliptic, or hyperbolic.

It is at once seen that these are distinct surfaces; and the equations also show very readily the general form and mode of generation of the several surfaces.

In the elliptic paraboloid (fig. 20), the sections by the planes of zx and zy are the parabolas

$$z=\frac{x^2}{2a},\ z=\frac{y^2}{2b},$$

having the common axis Oz; and the section by any plane $z=\gamma$ parallel to that of xy is the ellipse

$$\gamma=\frac{x^2}{2a}+\frac{y^2}{2b};$$

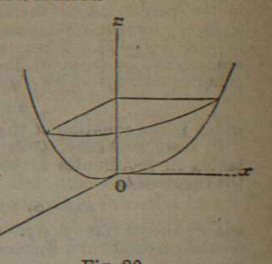

Fig. 20.

so that the surface is generated by a variable ellipse moving parallel to itself along the parabolas as directrices.

In the hyperbolic paraboloid (fig. 21) the sections by the planes of zx, zy are the parabolas $z=\frac{x^2}{2a}$, $z=-\frac{x^2}{2b}$, having the opposite axes Oz, Oz', and the section by a plane $z=\gamma$ parallel to that of xy is the hyperbola $\gamma=\frac{x^2}{2a}-\frac{y^2}{2b}$, which has its transverse axis parallel to Ox or Oy according as γ is positive or negative. The surface is thus generated by a variable hyperbola moving parallel to itself along the parabolas as directrices.

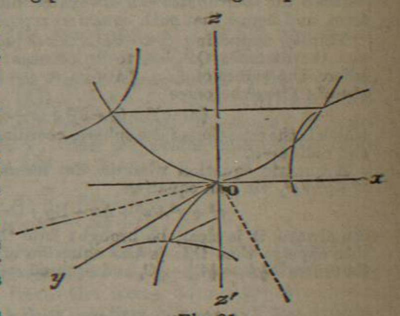

Fig. 21.

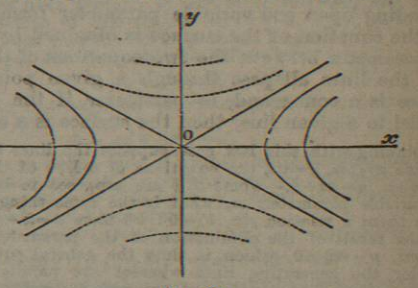

Fig. 22.

The form is best seen from fig. 22, which represents the sections by planes parallel to the plane of xy, or say the contour lines; the continuous lines are the sections above the plane of xy, and the dotted lines the sections below this plane. The form is, in fact, that of a saddle.

In the ellipsoid (fig. 23) the sections by the planes of zx, zy, and xy are each of them an ellipse, and the section by any parallel plane is also an ellipse. The surface may be considered as generated by an ellipse moving parallel to itself along two ellipses as directrices.

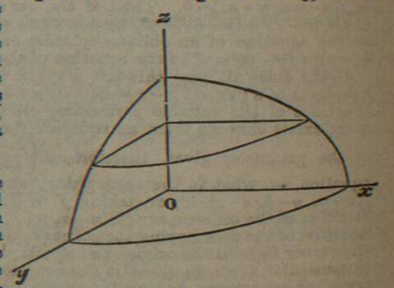

Fig. 23.

In the hyperboloid of one sheet (fig. 24), the sections by the planes of zx, zy are the hyperbolas

$$\frac{x^2}{a}-\frac{z^2}{c^2}=1,\ \frac{y^2}{b^2}-\frac{z^2}{c^2}=1,$$

having a common conjugate axis zOz'; the section by the plane of

xy, and that by any parallel plane, is an ellipse; and the surface may be considered as generated by a variable ellipse moving parallel to itself along the two hyperbolas as directrices.

In the hyperboloid of two sheets (fig. 25), the sections by the planes of zx and zy are the hyperbolas

$$\frac{z^2}{c^2}-\frac{x^2}{a^2}=1,\ \frac{z^2}{c^2}-\frac{y^2}{b^2}=1,$$

having the common transverse axis zOz'; the section by any plane $z=\pm\gamma$ parallel to that of xy, γ being in absolute magnitude $>c$, is the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{y^2}{c^2}-1$$

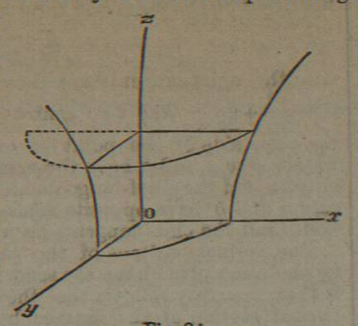

Fig. 24.

and the surface, consisting of two distinct portions or sheets, may be considered as generated by a variable ellipse moving parallel to itself along the hyperbolas as directrices.

The hyperbolic paraboloid is such (and it is easy from the figure to understand how this may be the case) that there exist upon it two singly infinite series of right lines. The same is the case with the hyperboloid of one sheet (ruled or skew hyperboloid, as with reference to this property it is termed). If we imagine two equal and parallel circular disks, their points connected by strings of equal length, so that these are the generating lines of a right circular cylinder, then by turning one of the disks about its centre through the same angle in one or the other direction, the strings will in each case generate one and the same hyperboloid, and will in regard to it be the two systems of lines on the surface, or say the two systems of generating lines; and the general configuration is the same when instead of circles we have ellipses. It has been already shown analytically that the equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ is satisfied by each of two pairs of linear relations between the coordinates.

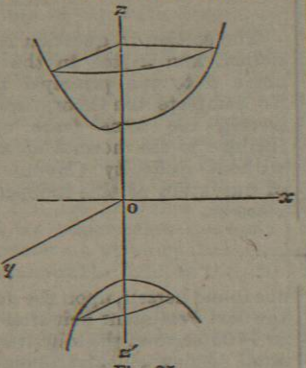

Fig. 25.

Curves; Tangent, Osculating Plane, Curvature, &c.

38. It will be convenient to consider the coordinates (x, y, z) of the point on the curve as given in terms of a parameter θ, so that dx, dy, dz, d^2x, &c., will be proportional to $\frac{dx}{d\theta}, \frac{dy}{d\theta}, \frac{dz}{d\theta}, \frac{d^2x}{d\theta^2}$, &c. But only a part of the analytical formulæ will be given. ξ, η, ζ are used as current coordinates.

The tangent is the line through the point (x, y, z) and the consecutive point $(x+dx, y+dy, z+dz)$; its equations therefore are

$$\frac{\xi-x}{dx}=\frac{\eta-y}{dy}=\frac{\zeta-z}{dz}.$$

The osculating plane is the plane through the point and two consecutive points, and contains therefore the tangent; its equation is

$$\begin{vmatrix} \xi-x, & \eta-y, & \zeta-z \\ dx, & dy, & dz \\ d^2x, & d^2y, & d^2z \end{vmatrix}=0$$

or, what is the same thing,

$$(\xi-x)(dy\,d^2z-dz\,d^2y)+(\eta-y)(dz\,d^2x-dx\,d^2z)+(\zeta-z)(dx\,d^2y-dy\,d^2x)=0.$$

The normal plane is the plane through the point at right angles to the tangent. It meets the osculating plane in a line called the principal normal; and drawing through the point a line at right angles to the osculating plane, this is called the binormal. We have thus at the point a set of three rectangular axes—the tangent, the principal normal, and the binormal.

We have through the point and three consecutive points a sphere of spherical curvature,—the centre and radius thereof being the centre, and radius, of spherical curvature. The sphere is met by the osculating plane in the circle of absolute curvature,—the centre and radius thereof being the centre, and radius, of absolute curvature. The centre of absolute curvature is also the intersection of the principal normal by the normal plane at the consecutive point.

Surfaces; Tangent Lines and Plane, Curvature, &c.

39. It will be convenient to consider the surface as given by an equation $f(x, y, z)=0$ between the coordinates; taking (x, y, z) for the coordinates of a given point, and $(x+dx, y+dy, z+dz)$ for those of a consecutive point, the increments dx, dy, dz satisfy the condition

$$\frac{df}{dx}dx+\frac{df}{dy}dy+\frac{df}{dz}dz=0,$$

but the ratio of two of the increments, suppose $dx:dy$, may be regarded as arbitrary. Only a part of the analytical formulæ will be given. ξ, η, ζ are used as current coordinates.

We have through the point a singly infinite series of right lines, each meeting the surface in a consecutive point, or say having each of them two-point intersection with the surface. These lines lie all of them in a plane which is the tangent plane; its equation is

$$\frac{df}{dx}(\xi-x)+\frac{df}{dy}(\eta-y)+\frac{df}{dz}(\zeta-z)=0,$$

as is at once verified by observing that this equation is satisfied (irrespectively of the value of $dx:dy$) on writing therein $\xi, \eta, \zeta=x+dx,\ y+dy,\ z+dz$.

The line through the point at right angles to the tangent plane is called the normal; its equations are

$$\frac{\xi-x}{\frac{df}{dx}}=\frac{\eta-y}{\frac{df}{dy}}=\frac{\zeta-z}{\frac{df}{dz}}.$$

In the series of tangent lines there are in general two (real or imaginary) lines, each of which meets the surface in a second consecutive point, or say it has three-point intersection with the surface; these are called the chief-tangents (Haupt-tangenten). The tangent-plane cuts the surface in a curve, having at the point of contact a node (double point), the tangents to the two branches being the chief-tangents.

In the case of a quadric surface the curve of intersection, *qua* curve of the second order, can only have a node by breaking up into a pair of lines; that is, every tangent-plane meets the surface in a pair of lines, or we have on the surface two singly infinite systems of lines; these are real for the hyperbolic paraboloid and the hyperboloid of one sheet, imaginary in other cases.

At each point of a surface the chief-tangents determine two directions; and passing along one of them to a consecutive point, and thence (without abrupt change of direction) along the new chief-tangent to a consecutive point, and so on, we have on the surface a chief-tangent curve; and there are, it is clear, two singly infinite series of such curves. In the case of a quadric surface, the curves are the right lines on the surface.

40. If at the point we draw in the tangent-plane two lines bisecting the angles between the chief-tangents, these lines (which are at right angles to each other) are called the principal tangents.[1] We have thus at each point of

[1] The point on the surface may be such that the directions of the principal tangents become arbitrary; the point is then an umbilicus. It is in the text assumed that the point on the surface is not an umbilicus.

the surface a set of rectangular axes, the normal and the two principal tangents.

Proceeding from the point along a principal tangent to a consecutive point on the surface, and thence (without abrupt change of direction) along the new principal tangent to a consecutive point, and so on, we have on the surface a curve of curvature; there are, it is clear, two singly infinite series of such curves, cutting each other at right angles at each point of the surface.

Passing from the given point in an arbitrary direction to a consecutive point on the surface, the normal at the given point is not intersected by the normal at the consecutive point; but passing to the consecutive point along a curve of curvature (or, what is the same thing, along a principal tangent) the normal at the given point is intersected by the normal at the consecutive point; we have thus on the normal two centres of curvature, and the distances of these from the point on the surface are the two principal radii of curvature of the surface at that point; these are also the radii of curvature of the sections of the surface by planes through the normal and the two principal tangents respectively; or say they are the radii of curvature of the normal sections through the two principal tangents respectively. Take at the point the axis of z in the direction of the normal, and those of x and y in the directions of the principal tangents respectively, then, if if the radii of curvature be a, b (the signs being such that the coordinates of the two centres of curvature are $z=a$ and $z=b$ respectively), the surface has in the neighbourhood of the point the form of the paraboloid

$$z=\frac{x^2}{2a}+\frac{y^2}{2b}$$

and the chief-tangents are determined by the equation $0=\frac{x^2}{2a}+\frac{y^2}{2b}$. The two centres of curvature may be on the same side of the point or on opposite sides; in the former case a and b have the same sign, the paraboloid is elliptic, and the chief-tangents are imaginary; in the latter case a and b have opposite signs, the paraboloid is hyperbolic, and the chief-tangents are real.

The normal sections of the surface and the paraboloid by the same plane have the same radius of curvature; and it thence readily follows that the radius of curvature of a normal section of the surface by a plane inclined at an angle θ to that of zx is given by the equation

$$\frac{1}{\rho}=\frac{\cos^2\theta}{a}+\frac{\sin^2\theta}{b}.$$

The section in question is that by a plane through the normal and a line in the tangent plane inclined at an angle θ to the principal tangent along the axis of x. To complete the theory, consider the section by a plane having the same trace upon the tangent plane, but inclined to the normal at an angle ϕ; then it is shown without difficulty (Meunier's theorem) that the radius of curvature of this inclined section of the surface is $=\rho\cos\phi$. (A. CA.)

GEORGE I., king of Great Britain and Ireland (*George Louis*, 1660-1727), born in 1660, was heir through his father Ernest Augustus to the hereditary lay bishopric of Osnabrück, and to the duchy of Calenburg, which formed one portion of the Hanoverian possessions of the house of Brunswick, whilst he secured the reversion of the other portion, the duchy of Celle or Zell, by his marriage (1682) with the heiress, his cousin Sophia Dorothea. The marriage was not a happy one. The morals of German courts in the end of the 17th century took their tone from the splendid profligacy of Versailles. It became the fashion for a prince to amuse himself with a mistress or more frequently with many mistresses simultaneously, and he was often content that the mistresses whom he favoured should be neither beautiful nor witty. George Louis followed the usual course. Count Königsmark—a handsome adventurer—seized the opportunity of paying court to the deserted wife. Conjugal infidelity was held at Hanover to be a privilege of the male sex. Count Königsmark was assassinated. Sophia Dorothea was divorced in 1694, and remained in seclusion till her death in 1726. When her descendant in the fourth generation attempted in England to call his wife to account for sins of which he was himself notoriously guilty, free-spoken public opinion reprobated the offence in no measured terms. In the Germany of the 17th century all free-spoken public opinion had been crushed out by the misery of the Thirty Years' War, and it was understood that princes were to arrange their domestic life according to their own pleasure.

The prince's father did much to raise the dignity of his family. By sending help to the emperor when he was struggling against the French and the Turks, he obtained the grant of a ninth electorate in 1692. His marriage with Sophia, the youngest daughter of Elizabeth the daughter of James I. of England, was not one which at first seemed likely to confer any prospect of advancement to his family. But though there were many persons whose birth gave them better claims than she had to the English crown, she found herself, upon the death of the duke of Gloucester, the next Protestant heir after Anne. The Act of Settlement in 1701 secured the inheritance to herself and her descendants. Being old and unambitious she rather permitted herself to be burthened with the honour than thrust herself forward to meet it. Her son George took a deeper interest in the matter. In his youth he had fought with determined courage in the wars of William III. Succeeding to the electorate on his father's death in 1698, he had sent a welcome reinforcement of Hanoverians to fight under Marlborough at Blenheim. With prudent persistence he attached himself closely to the Whigs and to Marlborough, refusing Tory offers of an independent command, and receiving in return for his fidelity a guarantee by the Dutch of his succession to England in the Barrier treaty of 1709. In 1714 when Anne was growing old, and Bolingbroke and the more reckless Tories were coquetting with the son of James II., the Whigs invited George's eldest son, who was duke of Cambridge, to visit England in order to be on the spot in case of need. Neither the elector nor his mother approved of a step which was likely to alienate the queen, and which was specially distasteful to himself, as he was on very bad terms with his son. Yet they did not set themselves against the strong wish of the party to which they looked for support, and it is possible that troubles would have arisen from any attempt to carry out the plan, if the deaths, first of the electress (May 28) and then of the queen (August 1, 1714), had not laid open George's way to the succession without further effort of his own.

In some respects the position of the new king was not unlike that of William III. a quarter of a century before. Both sovereigns were foreigners, with little knowledge of English politics and little interest in English legislation. Both sovereigns arrived at a time when party spirit had been running high, and when the task before the ruler was to still the waves of contention. In spite of the difference between an intellectually great man and an intellectually

small one, in spite too of the difference between the king who began by choosing his ministers from both parties, and the king who persisted in choosing his ministers from only one, the work of pacification was accomplished by George even more thoroughly than by William.

George I. was fortunate in arriving in England when a great military struggle had come to an end. He had therefore no reason to call upon the nation to make great sacrifices. All that he wanted was to secure for himself and his family a high position which he hardly knew how to occupy, to fill the pockets of his German attendants and his German mistresses, to get away as often as possible from the uncongenial islanders whose language he was unable to speak, and to use the strength of England to obtain petty advantages for his German principality. In order to do this he attached himself entirely to the Whig party, though he refused to place himself at the disposal of its leaders. He gave his confidence, not to Somers and Wharton and Marlborough, but to Stanhope and Townshend, the statesmen of the second rank. At first he seemed to be playing a dangerous game. The Tories, whom he rejected, were numerically superior to their adversaries, and were strong in the support of the country gentlemen and the country clergy. The strength of the Whigs lay in the towns and in the higher aristocracy. Below both parties lay the mass of the nation, which cared nothing for politics except in special seasons of excitement, and which asked only to be let alone. In 1715 a Jacobite insurrection in the north, supported by the appearance of the Pretender, the son of James II., in Scotland, was suppressed, and its suppression not only gave to the Government a character of stability, but displayed its adversaries in an unfavourable light as the disturbers of the peace.

Even this advantage, however, would have been thrown away, if the Whigs in power had continued to be animated by violent party spirit. What really happened was that the Tory leaders were excluded from office, but that the principles and prejudices of the Tories were admitted to their full weight in the policy of the Government. The natural result followed. The leaders to whom no regard was paid continued in opposition. The rank and file who would personally have gained nothing by a party victory were conciliated into quiescence.

This mingling of two policies was conspicuous both in the foreign and the domestic actions of the reign. In the days of Queen Anne, the Whig party had advocated the continuance of war with a view to the complete humiliation of the king of France, whom they feared as the protector of the Pretender, and in whose family connexion with the king of Spain they saw a danger for England. The Tory party on the other hand had been the authors of the peace of Utrecht, and held that France was sufficiently depressed. A fortunate concurrence of circumstances enabled George's ministers, by an alliance with the regent of France, the duke of Orleans, to pursue at the same time the Whig policy of separating France from Spain and from the cause of the Pretender, and the Tory policy of the maintenance of a good understanding with their neighbour across the Channel. The same eclecticism was discernible in the proceedings of the home Government. The Whigs were conciliated by the repeal of the Schism Act and the Occasional Conformity Act, whilst the Tories were conciliated by the maintenance of the Test Act in all its vigour. The satisfaction of the masses was increased by the general well-being of the nation.

Very little of all that was thus accomplished was directly owing to George I. The policy of the reign is the policy of his ministers. Stanhope and Townshend from 1714 to 1717 were mainly occupied with the defence of the Hanoverian settlement. After the dismissal of the latter in 1717, Stanhope in conjunction with Sunderland took up a more decided Whig policy. The Occasional Conformity Act and the Schism Act were repealed in 1719. But the wish of the liberal Whigs to modify if not to repeal the Test Act remained unsatisfied. In the following year the bursting of the South Sea bubble, and the subsequent deaths of Stanhope in 1721 and of Sunderland in 1722, cleared the way for the accession to power of Sir Robert Walpole, to whom and not to the king was due the conciliatory policy which quieted Tory opposition by abstaining from pushing Whig principles to their legitimate consequences.

Nevertheless something of the honour due to Walpole must be reckoned to the king's credit. It is evident that at his accession his decisions were by no means unimportant. The royal authority was still able within certain limits to make its own terms. This support was so necessary to the Whigs that they made no resistance when he threw aside their leaders on his arrival in England. When by his personal intervention he dismissed Townshend and appointed Sunderland, he had no such social and parliamentary combination to fear as that which almost mastered his great-grandson in his struggle for power. If such a combination arose before the end of his reign it was owing more to his omitting to fulfil the duties of his station than from the necessity of the case. As he could talk no English, and his ministers could talk no German, he absented himself from the meetings of the cabinet, and his frequent absences from England and his want of interest in English politics strengthened the cabinet in its tendency to assert an independent position. Walpole at last by his skill in the management of parliament rose as a subject into the almost royal position denoted by the name of prime minister. In connexion with Walpole the force of wealth and station established the Whig aristocracy in a point of vantage from which it was afterwards difficult to dislodge them. Yet, though George had allowed the power which had been exercised by William and Anne to slip through his hands, it was understood to the last that if he chose to exert himself he might cease to be a mere cipher in the conduct of affairs. As late as in 1727 Bolingbroke gained over one of the king's mistresses, the duchess of Kendal; and though her support of the fallen Jacobite took no effect, Walpole was not without fear that her reiterated entreaties would lead to his dismissal. The king's death in a carriage on his way to Hanover, in the night between 10th and 11th June in the same year, put an end to these apprehensions.

His only children were his successor George II., and Sophia Dorothea (1687-1757), who married in 1706 Frederick William, crown prince (afterwards king) of Prussia. She was the mother of Frederick the Great. (S. R. G.)

GEORGE II. (*George Augustus*, 1683-1760), the only son of George I., was born in 1683. In 1705 he married Wilhelmina Caroline of Anspach. In 1706 he was created earl of Cambridge. In 1708 he fought bravely at Oudenarde. At his father's accession to the English throne he was thirty-one years of age. He was already on bad terms with his father. The position of an heir-apparent is in no case an easy one to fill with dignity, and the ill treatment of the prince's mother by his father was not likely to strengthen in him a reverence for paternal authority. It was most unwillingly that, on his first journey to Hanover in 1716, George I. appointed the prince of Wales guardian of the realm during his absence. In 1717 the existing ill feeling ripened into an open breach. At the baptism of one of his children, the prince selected one godfather whilst the king persisted in selecting another. The young man spoke angrily, was ordered into arrest, and was subsequently commanded to leave St James's, and to be excluded from all court ceremonies. The prince took