was, not indeed the originator, but the chief exponent at the time when the anti-pædo-baptism of the congregations in which he laboured took permanent form in opposition to ordinary Protestantism on the one hand and to the theocratic ideas of the Münster type of anabaptism on the other. The original home of the views afterwards called Mennonite was in Zürich, where, as early as 1525, Grebel and Manz founded a community having for its most distinctive mark baptism upon confession of faith. The chief doctrines of these Zürich Baptists have been already stated in the article BAPTISTS, vol. iii. p. 353. The main interest of the sect lay not in dogma but in discipline. Within the communities evangelical life was reduced to a law of separation from the world, and this separation—enforced by a stringent use of excommunication and the prohibition of marriage beyond the brotherhood—involved not only abstinence from worldly vanities but refusal of civic duties (the state being held to be un-Christian)—refusal to take an oath or use the sword. In their revolt against the corruptions of the mediæval church the Reformers neither denied the continuity of the church as an organization nor impugned the Christian character of the state. The new sect did both; and their position thus appeared so radically subversive of the foundations of society that it is not surprising, under the imperfect views of toleration then current, that they became the objects of bitter persecution from Protestants as well as from Catholics. But the Grebelians had no desire, like the fanatics of Münster, to found a new theocracy in opposition to the anti-Christian state. They sought only to withdraw from what their conscience condemned, content to live as strangers upon earth, and devoting all their energy to preserve the purity of their own communities. The mediæval conception of separation from the world as the true path of Christian perfection had leavened all middle-class society in Europe, and prepared many to accept separatist views of the church as soon as they were reached by the impulse of revolt against Roman Catholicism; the pursuit of holiness in a society protected by a strict discipline is an idea which experience has shown to have a great attraction for one class of earnest minds; hence, in spite of persecutions incomparably fiercer than any of the larger Protestant bodies ever underwent, the new doctrine and praxis rapidly spread from Switzerland to Germany, Holland, and even to France. Each community was quite independent, united to the rest only by a bond of love. There was no sort of hierarchy, but only "exhorters" chosen by the congregation, of whom the most prominent were also "elders" entrusted with the administration of the sacraments—an organization so easily kept alive or reproduced that the movement could hardly be checked by any persecution short of the total annihilation which at length was actually the fate of many of the Swiss communities. The remnants of the Swiss Mennonites broke in 1620 into two parties, the stricter of which, the Ammanites or Upland Mennonites, were distinguished from the Lowland Mennonites by holding that excommunication of one party dissolved marriage, and by their rejection of buttons and the use of the razor. Their persecution lasted till 1710; a few congregations still remain and keep themselves quite distinct from Baptist bodies of more modern origin. In Germany the Mennonites are somewhat more numerous; more important are the German Mennonite colonies in southern Russia, brought thither in 1783 by the empress Catherine, which in turn have recently sent many emigrants to America. America indeed, and especially Pennsylvania, early became a refuge for the Mennonites of Switzerland, the Palatinate, and Holland, and is now the chief home of the body (175,000 in the United States and 25,000 in Canada). The oldest congregation is that of Germantown (since 1683); the most numerous of several divisions are the Old Mennonites, corresponding to the less strict of the Swiss sections.

All these communities in Europe and America are distinguished by an antique simplicity combined with antique prejudices, by indifference to the interests of the greater world, while at the same time their industry and self-concentration have made them generally well-to-do. Their religious type has varied very little in the course of centuries, as indeed is not surprising, their theology being ascetic rather than dogmatic or speculative. The Mennonites of Holland, on the other hand, have passed through an interesting and progressive history.

It was in Holland and the adjoining parts of Low Germany that the personal influence of Menno Simons (1492–1559) was mainly felt. He was originally a priest, and was pastor at his native place Witmarsum in Friesland from 1531 to 1536, when convictions long ripening in his mind compelled him to resign his cure. At this time the anti-pædo-baptist societies in the Low Countries were much agitated. The views which had just before received their political deathblow at Münster (see ANABAPTISTS) were not extinct, and even those who did not share them were by no means at one. Menno attached himself to the Obbenites, who held that on earth true Christians had no prospect but to suffer persecution, refused to use the sword, and looked for no millennium on earth. Menno became one of their elders, and by his wanderings among the scattered and oppressed communities, and especially by the natural eloquence and religious power of his numerous writings, did much to sustain the faith of his associates, to confirm the type of their religious life, and to prevent startling aberrations in doctrine or discipline. He was not an original thinker; but the love which all felt for the man, and which was kept alive for generations by his writings, gave him the place which the name of Mennonites expresses.

It may be ascribed to the influence of Menno's writings that the Dutch Mennonites, though for a time (since 1554) they broke into fractions on questions of discipline, and especially on the effect of excommunication upon marriage, never fell so far apart as regards the type of their religious life as to preclude the possibility of reunion. The Waterlanders in North Holland, who held the least strict doctrine of excommunication, soon moved farther in the direction of liberality, and exchanged the name Mennonites for that of Doopsgezinden (Baptist persuasion). In 1579 they refused to condemn any one for opinions, even on the incarnation, which the word of Scripture did not pronounce necessary to salvation. They aided William the Silent with money, and from 1581 to 1618 even accepted civil office. Meantime the stricter party had undergone various divisions, which, however, in 1627–32 were reunited on the basis of confessions essentially embodying Menno's teachings. They too had learned moderation, at least in their views of excommunication, and their antithesis to the state was softened since the cessation of persecution in 1581, but especially since in 1672 they were recognized as citizens. On the other hand, the adoption of a confession had deepened the separation between them and the liberal Doopsgezinden; but doctrine was never the fundamental principle of the Mennonite communities, confessionalism took no firm root, and the two sections gradually approached, and through a series of partial fusions became at length finally united when the Amsterdam congregations came together in 1801. The persuasion declined much in numbers in the 18th century; since then it has increased, and has now 127 congregations with nearly 50,000 members. The objection to hold civil office disappeared in 1795; that to carry arms in the war of freedom against Napoleon. Baptism on profession of faith and the refusal of the oath, tolerance in matters of doctrine without religious indifference, are the chief marks of the body, which in point of theological culture and general enlightenment, philanthropic zeal and social importance, has long stood very high.

Authorities.—The best life of Menno Simons is Cramer's, 1837. De Hoop Scheffer's article in Herzog-Plitt, *R. E.*, is excellent; only one point of consequence in his account seems to call for modification,—the book against John of Leyden, said to have been published before Menno joined the Obbenites, is almost certainly spurious. See Sepp, *Geschiedkundige Nasporingen*, i. (1872) p. 128 *sq.* The completest edition of Menno's works is that in folio, 1686. Many of them are known only in bad Dutch versions; Menno himself wrote in the "Oostersch" or East Sea Dialect of Low German. For the literature on the Mennonites in general, see De Hoop Scheffer, on whom the foregoing sketch is mainly dependent.

MENSHIKOFF, ALEXANDER DANILOVICH (1672–1729), born at Moscow on the 17th of November (o.s.) 1672, was the son of a poor man, who employed him to sell cakes about the streets of that city. In this humble occupation he attracted the attention of Lefort, one of Peter the Great's most active co-operators, who was pleased with his sprightliness, and took him into his service. Peter, soon afterwards seeing the youth at Lefort's, was also delighted with him, and took him to be his page. Menshikoff soon became indispensable to the czar, assisting him in his workshop, and displaying signal bravery in the company of his master at the siege of Azoff. He formed one of the suite of Peter during his travels, and worked with him at Saardam and Deptford. Throughout his wars with the Swedes, Menshikoff was the companion of the czar, and greatly distinguished himself. For his gallantry at the battle of the Neva, on the 7th of May (o.s.) 1703, he received the order of St Andrew. In 1704 he was made general, and at the request of the czar created a prince of the Holy Roman Empire. His house on the Vasilii-Ostroff was magnificent; there ambassadors were received, and banquets were given gorgeous with gold and silver plate. Unfortunately there is a dark side to the picture, and the favourite was guilty of extortion to such an extent as to bring him under his master's censure. On the death of Peter the position of Menshikoff became very perilous; his successes had raised about him a host of enemies eager for his downfall. The Golitzins, Dolgoroukis, and all those who formed what may be called the Old Russian party, wished to proclaim the son of Alexis emperor. Those, however, whose aggrandizement was bound up with Peter's reforms—Menshikoff, Apraksin, Bontourlin, Goloffkin, and others—were in favour of giving the crown to Peter's widow, who accordingly ascended the throne as Catherine I. During her reign the influence of Menshikoff was unbounded, and he virtually governed the country; but the empress died in 1727, after a reign of two years. She had made a will, no doubt at the instigation of the favourite, to the effect that Peter, her grandson, was to be czar under the guardianship of Menshikoff, whose daughter Mary was to be married to the youthful sovereign. Under pretence of taking care of the young czar, Menshikoff caused him to be removed to his house and surrounded him with his creatures. He was now at the height of his power; foreign ambassadors remarked that even the great Peter himself was never feared so much. The young czar, however, showed no affection for Mary Menshikoff, and the girl was equally apathetic towards her betrothed, being in love with a member of the family of Sapieha at the time her father had forced her into the engagement. The Dolgoroukis used the aversion of the young prince to his *fiancée* as a means of creating dislike to the father. A chain of events was gradually leading to the downfall of the favourite. He was soon refused admittance to the summer palace, whither the young czar had retired. Next he was arrested, and so overpowered was he at his disgrace that he had an apoplectic stroke. In vain did he address letters both to the emperor and his sister. Shortly after, by order of the czar, the fallen magnate departed from St Petersburg, but more like a nobleman retiring to his estate than a culprit going into exile. The people regarded him with dislike, and most of them rejoiced over his fall. On his way a courier arrived with orders to take the czar's ring of betrothal from his daughter Mary and give her back her own, which had been worn by Peter II. Menshikoff was not permitted to pass through Moscow, but was conducted to Oranienburg, in the government of Riazan, and there placed under strict surveillance. Soon afterwards the whole family was banished to Siberia, and arrived at Berezoff towards the end of 1727. Menshikoff's wife died on the journey, and was buried near Kazan. On the arrival of the prisoners they were lodged in a wooden house, consisting of four rooms. But Menshikoff did not long endure the horrors of exile in this inclement region. According to Mannstein, he died (November 12, o.s., 1729) of an apoplectic stroke, because there was no one at Berezoff, as he himself remarked, who understood how to open a vein. The young czar ordered the release from exile of the two remaining children of Menshikoff,—his daughter Mary had died at Berezoff in the same year as her father,—and restored some of their property to them.

MENSHIKOFF, ALEXANDER SERGEIEVICH (1787–1869), great-grandson of Peter's favourite, born in 1787, entered the Russian service as attaché to the embassy at Vienna. He accompanied the emperor Alexander throughout his campaigns against Napoleon, and attained the rank of general, but retired from active service in 1823. He then devoted himself to naval matters, and put the Russian marine, which had fallen into decay during the reign of Alexander, on an efficient footing. On the outbreak of the Crimean War he was appointed commander-in-chief, and suffered a severe defeat at the Alma. On the death of the emperor Nicholas in 1855 he was recalled, ostensibly on account of failing health. He died in 1869.

MENSURATION

MENSURATION, or the art of measuring, involves the construction of measures, the methods of using them, and the investigation of rules by which magnitudes which it may be difficult or impossible to measure directly are calculated from the ascertained value of some associated magnitude. It is usual, however, to employ the term mensuration in the last of these senses; and we may therefore define it to be that department of mathematical science by which the various dimensions of bodies are calculated from the simplest possible measurements.

The determination of the lengths and directions of straight lines, including what are familiarly known as problems in heights and distances, generally depends on the solution of triangles, and will be discussed in the articles TRIGONOMETRY and SURVEYING. The remaining portions of the subject are the determinations of the lengths of curves, the areas of plane or other figures, and the volumes and surfaces of solids; and it is of mensuration as thus restricted that the present article will discuss some of the more important problems.

§ 1. *Units of Length, Area, and Volume.*—In measuring any magnitude we select some standard or "unit" to measure by. Thus in measuring length we take for unit an inch, a foot, or a yard. From the unit of length we derive the units of area and volume. Thus we define the unit of area to be the area of the square described upon the unit of length, and the unit of volume to be the volume of the cube whose edge is the unit of length or whose side is the unit of area. For example, if an inch be taken as the unit of length, the square whose side is 1 inch is the unit of area, and the cube whose edge is 1 inch is the unit of volume. The length of a line, the area of a surface, and the volume of a solid are then expressed by the numbers, whole or fractional, of units of length, area, and volume which they respectively contain. Hence, if l denote the linear unit, the length of a line which contains a units is al, or simply a since l is unity; similarly the area of a surface which contains b units of area is bm, or simply b, where m is the unit of area.

§ 2. *Commensurable and Incommensurable Magnitudes.*—When two magnitudes have a common measure, that is, when another magnitude can be found which is contained in each an exact number of times, they are said to be "commensurable." Thus a line $4\frac{1}{4}$ and another $3\frac{1}{2}$ inches long are commensurable; for, if $\frac{1}{9}$ inch be taken as unit of length, the former contains the unit nine times, and the latter seven times. If no common measure can be found, the two magnitudes are said to be "incommensurable." For instance, 1 and $\sqrt{2}$ have no common measure; for $\sqrt{2}=1{\cdot}4142\ldots$ an interminable decimal, and hence no unit, however small, can be found which will be contained in each an exact number of times. If, however, we take $\sqrt{2}=1{\cdot}4$, the error will be less than $\frac{1}{10}$; if $\sqrt{2}=1{\cdot}414$, the error will be less than $\frac{1}{1000}$, &c. Hence, by taking a sufficient number of figures, we can find a fraction which will differ from $\sqrt{2}$ by less than any assignable quantity, and therefore we can always find two commensurable magnitudes that will represent two incommensurable ones to any degree of accuracy we please. In what follows we need therefore only consider commensurable lines.

§ 3. *Area of a Rectangle.*—Let the side AB (fig. 1) contain a units and the side BC b units of length. If we divide AB into a equal parts, each equal to the unit of length, and similarly BC into b equal parts, and if through the points of division we draw lines parallel to the sides of the rectangle, these lines will divide the rectangle into a series of rectangles, each of which is the unit of area, since each is a square whose sides are of unit length. As we have a rows of these rectangles, and b in each row, the whole number of rectangles will be ab. Therefore

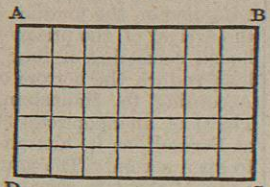

Fig. 1.

$$\text{area of ABCD} = ab \text{ units of area} = ab$$

PART I.—PLANE FIGURES.

Section I.—Plane Figures contained by Straight Lines.

A. *The Rectangle.*

§ 4. Let ABCD (fig. 2) be a rectangle, and let AB = CD = a, BC = DA = b, AC = c, and the angle BAC = α; it is required to find its area. Since a rectangle is completely determined when two independent data, one of which at least is a length, are given connecting its parts, we can determine its area in the following cases.

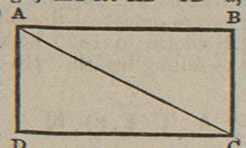

Fig. 2.

(α) *When its length* a *and its breadth* b *are given.*—It has already been proved (§ 3) that

$$\text{area of ABCD} = ab;$$

or the area of a rectangle is equal to its length multiplied by its breadth.

Example.—Let $a = 12$ feet 6 inches and $b = 9$ inches, then

$$\text{area of ABCD} = 12{\cdot}5 \times {\cdot}75 = 9{\cdot}375 \text{ square feet.}$$

If we make use of logarithms in the above calculation we have

$$\log \text{area} = \log a + \log b.$$
$$\log a = \log 12{\cdot}5 = 1{\cdot}0969100$$
$$\log b = \log {\cdot}75 = \bar{1}{\cdot}8750613$$

therefore $\log \text{area} = {\cdot}9719713;$

hence $\text{area} = 9{\cdot}375.$

(β) *When a side* a *and the diagonal* c *are given.*—By Euclid i. 47 we have

$$b^2 = c^2 - a^2, \text{ or } b = \sqrt{c^2-a^2},$$

therefore $\text{area of ABCD} = ab = a\sqrt{c^2-a^2};$

or $\log \text{area} = \log a + \frac{1}{2}\log(c+a) + \frac{1}{2}\log(c-a)$

Example.—Let $a = 238$ and $c = 456$, then

$$\log a = \log 238 = 2{\cdot}3765770$$
$$\tfrac{1}{2}\log(c+a) = \tfrac{1}{2}\log 694 = 1{\cdot}4206797$$
$$\tfrac{1}{2}\log(c-a) = \tfrac{1}{2}\log 218 = 1{\cdot}1692282$$

therefore $\log \text{area} = 4{\cdot}9664849;$

hence $\text{area} = 92573{\cdot}1.$

(γ) *When a side* a *and* α *its inclination to the diagonal are given.*—Since

$$\frac{b}{a} = \tan\alpha, \quad b = a\tan\alpha,$$

and therefore $\text{area of ABCD} = ab = a^2\tan\alpha;$

or $\log \text{area} = 2\log a + \text{L}\tan\alpha - 10.$

Example.—Let $a = 36$ and $\alpha = 32°\ 25'\ 15''$, then

$$2\log a = 2\log 36 = 3{\cdot}1126050$$
$$\text{L}\tan\alpha = \text{L}\tan 32°\ 25'\ 15'' = 9{\cdot}8028622$$

therefore $\log \text{area} = 12{\cdot}9154672 - 10 = 2{\cdot}9154672;$

hence $\text{area} = 823{\cdot}127.$

(δ) *When the diagonal* c *and* α *its inclination to either of the sides are given.*—We have

$$a = c\cos\alpha, \text{ and } b = c\sin\alpha,$$

therefore area of ABCD $= ab = c^2\sin\alpha\cos\alpha = \frac{1}{2}c^2\sin 2\alpha;$

or $2\,\text{area} = c^2\sin 2\alpha,$

and hence $\log 2\,\text{area} = 2\log c + \text{L}\sin 2\alpha - 10.$

§ 5. A square being a rectangle whose sides are equal, we can at once determine its area. When one datum, which must be a length, is given the square is completely determined and hence we have only two cases to consider.

(α) *When the side is given.*—From § 4, α, we have at once

$$\text{area of square} = ab = a \times a = a^2.$$

(β) *When the diagonal* c *is given.*—From § 4, β, we have

$$a^2 + a^2 = c^2, \text{ or } a^2 = \tfrac{1}{2}c^2;$$

hence $\text{area of square} = a^2 = \frac{1}{2}c^2$, or $2\,\text{area} = c^2;$

and therefore $\log 2\,\text{area} = 2\log c.$

B. *Right-angled Triangles.*

§ 6. The diagonal of every rectangle divides it into two equivalent right-angled triangles (Eucl. i. 34), and hence the area of the right-angled triangle ABC (fig. 2) is equal to half the area of the corresponding rectangle ABCD.

C. *Triangles Generally.*

§ 7. In every triangle there are six elements to be considered, namely, the three sides and the three angles. If any three of these six be given, provided one is a length, the triangle is completely determined, and hence its area can be found.

§ 8. *Length of Perpendiculars of a Triangle.*—In the triangle ABC (fig. 3) let BC = a, CA = b, AB = c, AD the perpendicular from A on BC = h, BD = x, and CD = y.

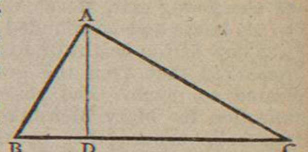

Fig. 3

Since BDA and CDA are right angles, we have

$$c^2 = x^2 + h^2 \text{ and } b^2 = y^2 + h^2,$$

and therefore

$$b^2 - c^2 = y^2 - x^2 = (y+x)(y-x) = a(y-x);$$

whence

$$y - x = \frac{b^2-c^2}{a}.$$

But $y + x = a$, and, by solving these equations, we obtain

$$y = \frac{b^2+a^2-c^2}{2a}$$

Again

$$h^2 = b^2 - y^2 = b^2 - \left(\frac{b^2+a^2-c^2}{2a}\right)^2 = \frac{(2ab)^2-(b^2+a^2-c^2)^2}{4a^2}$$
$$= \frac{(a+b+c)(b+c-a)(c+a-b)(a+b-c)}{4a^2};$$

hence

$$h = \frac{1}{2a}\sqrt{(a+b+c)(b+c-a)(c+a-b)(a+b-c)}.$$

Now let $a+b+c = 2s$, then $b+c-a = 2(s-a)$, $c+a-b = 2(s-b)$, and $a+b-c = 2(s-c)$.

Therefore, on substituting and reducing, we obtain

$$h = \frac{2}{a}\sqrt{s(s-a)(s-b)(s-c)}.$$

Similarly the perpendiculars from B and C on the opposite sides are respectively

$$\frac{2}{b}\sqrt{s(s-a)(s-b)(s-c)}, \text{ and } \frac{2}{c}\sqrt{s(s-a)(s-b)(s-c)}.$$

§ 9. We now proceed to investigate formulæ for the area of a triangle in the following important cases.

(α) *When the base* a *and the altitude* h *are given.*—Since a triangle is equal to half a rectangle of the same base and altitude, we have at once

$$\text{area ABC} = \tfrac{1}{2}ah.$$

Example.—Let $a = 40$ chains and $h = 14{\cdot}52$ chains, then

$$\text{area} = \tfrac{1}{2} \times 40 \times 14{\cdot}52 = 290{\cdot}4 \text{ square chains.}$$

(β) *When two sides* a *and* c *and the included angle* B *are given.*—From fig. 3 $\frac{h}{c} = \sin B$, and therefore $h = c\sin B$;

hence $\text{area} = \frac{1}{2}ah = \frac{1}{2}ac\sin B;$

or $\log 2\,\text{area} = \log a + \log c + \text{L}\sin B - 10.$

Example.—Let $a = 40$, $c = 30$, and $B = 30°$, then

$$\text{area} = \tfrac{1}{2}ac\sin B = \tfrac{1}{2} \times 40 \times 30 \times \tfrac{1}{2} = 300.$$

(γ) *When the three sides* a, b, c *are given.*—From § 8

$$h = \frac{2}{a}\sqrt{s(s-a)(s-b)(s-c)},$$

and therefore

$$\text{area} = \tfrac{1}{2}ah = \tfrac{1}{2}a \times \frac{2}{a}\sqrt{s(s-a)(s-b)(s-c)} = \sqrt{s(s-a)(s-b)(s-c)};$$

or $\log \text{area} = \frac{1}{2}\{\log s + \log(s-a) + \log(s-b) + \log(s-c)\}.$

Since $2s = a+b+c$, we have

$$\text{area of triangle} = \tfrac{1}{4}\sqrt{2(a^2b^2+b^2c^2+c^2a^2)-(a^4+b^4+c^4)}.$$

Example 1.—Let $a = 13$, $b = 14$, and $c = 15$, then

$$s = \tfrac{1}{2}(13+14+15) = 21, \quad s-a = 21-13 = 8,$$
$$s-b = 21-14 = 7, \text{ and } s-c = 21-15 = 6;$$

therefore $\text{area} = \sqrt{21 \times 8 \times 7 \times 6} = 84.$

Example 2.—Let $a = 255$, $b = 238$, and $c = 221$, then

$$\log s = \log 357 = 2{\cdot}5526682$$
$$\log(s-a) = \log 102 = 2{\cdot}0086002$$
$$\log(s-b) = \log 119 = 2{\cdot}0755470$$
$$\log(s-c) = \log 136 = 2{\cdot}1335389$$

therefore $\log \text{area} = \frac{1}{2}(8{\cdot}7703543) = 4{\cdot}3851771;$

hence $\text{area} = 24276.$

(δ) *When any two angles* B *and* C *and the adjacent side* a *are given.*—Since

$$\frac{c}{a} = \frac{\sin C}{\sin A}, \quad c = \frac{a\sin C}{\sin A}$$

and therefore (by β)

$$\text{area} = \tfrac{1}{2}ac\sin B = \frac{a^2\sin B\sin C}{2\sin A}, \text{ where } A = 180° - (B+C),$$

or $\log 2\,\text{area} = 2\log a + \text{L}\sin B + \text{L}\sin C + \text{L}\,\text{cosec}\,A - 30.$

Since all the angles of a triangle are given when any two are given, we can find the area of a triangle when any two angles and any one side are given. Thus, when A, B, and c are given, we know C also, and the problem reduces to a case of the preceding.

(ϵ) *When the three medians* α, β, γ *are given.*—If a, b, c be the three sides of a triangle, and α, β, γ the three medians, *i.e.*, the lines drawn from the angles to the middle points of the opposite sides, then by well-known geometrical propositions we have

$$4(\alpha^2+\beta^2+\gamma^2) = 3(a^2+b^2+c^2),$$
$$16(\alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\alpha^2) = 9(a^2b^2+b^2c^2+c^2a^2),$$

and

$$16(\alpha^4+\beta^4+\gamma^4) = 9(a^4+b^4+c^4).$$

Now (§ 9, γ)

area of triangle $= \frac{1}{4}\sqrt{2(a^2b^2+b^2c^2+c^2a^2)-(a^4+b^4+c^4)},$

therefore $= \frac{1}{3}\sqrt{2(\alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\alpha^2)-(\alpha^4+\beta^4+\gamma^4)}.$

D. *Parallelograms.*

§ 10. The opposite sides and angles of a parallelogram being equal, three independent data, one of which at least is a length, are necessary and sufficient to determine it completely.

In the parallelogram ABCD (fig. 4) let BC = DA = a, AB = CD = b, AC = c, AE = h, the angle ABC = α and AOD = β.

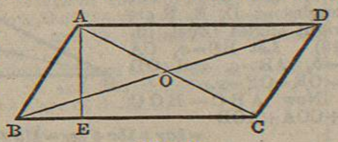

Fig. 4.

Since the diagonal AC divides the parallelogram into two equivalent triangles, we obtain

(α) area of ABCD $= 2$ area of triangle ABC $= 2 \times \frac{1}{2}a \times h$ (§ 9, α) $= ah;$

(β) area of ABCD $= 2$ area ABC $= 2 \times \frac{1}{2}ab\sin\alpha$ (§ 9, β) $= ab\sin\alpha;$

or $\log \text{area} = \log a + \log b + \text{L}\sin\alpha - 10;$

(γ) area of ABCD $= 2$ area ABC $= 2(\text{ABO} + \text{CBO})$

$= 2\{\frac{1}{2}\text{BO.AO}\sin\text{AOB} + \frac{1}{2}\text{BO.CO}\sin\text{COB}\} = 2\{\frac{1}{2}\text{BO.AC}\sin\beta\}$

$= \frac{1}{2}\text{BD.AC}\sin\beta = \frac{1}{2}cd\sin\beta,$

or $\log 2\,\text{area} = \log c + \log d + \text{L}\sin\beta - 10$

§ 11. If the parallelogram be *equiangular* (a rectangle), $c = d$, and area $= \frac{1}{2}c^2\sin\beta$. If it be *equilateral* (a rhombus), $\beta = 90°$, and area $= \frac{1}{2}cd$. If it be both *equiangular* and *equilateral* (a square), $c = d$ and $\beta = 90°$, and area $= \frac{1}{2}c^2$ as before (§ 5, β).

E. *Trapeziums.*

§ 12. To determine a trapezium completely four data are necessary and sufficient.

In the trapezium ABCD (fig. 5) let BC = a, CD = b, DA = c, AB = d; and AE perpendicular to BC = h, and draw AF parallel to CD, then

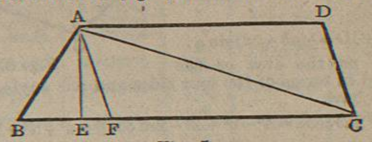

Fig. 5.

(α) area ABCD = area ABC + area ADC

$$= \tfrac{1}{2}ah + \tfrac{1}{2}ch$$
$$= \tfrac{1}{2}(a+c)h;$$

or the area is equal to half the sum of the parallel sides multiplied by the perpendicular between them.

Again, area of ABF $= \frac{1}{2}\text{BF} \times \text{AE}$ (§ 9, α) $= \frac{1}{2}(a-c)h,$

also area of ABF $= \sqrt{s(s-\text{AB})(s-\text{BF})(s-\text{FA})},$

where $2s = \text{AB} + \text{BF} + \text{FA};$

hence $h = \frac{2}{a-c}\sqrt{s(s-\text{AB})(s-\text{BF})(s-\text{FA})}$, therefore

(β) area of ABCD $= \frac{1}{2}(a+c)h = \frac{a+c}{a-c}\sqrt{s(s-\text{AB})(s-\text{BF})(s-\text{FA})}$

$$= \tfrac{1}{4}\frac{a+c}{a-c}\sqrt{(-a+b+c+d)(a+b-c-d)(a+b-c+d)(a-b-c+d)};$$

since AB $= d$, BF $= a-c$, and FA = CD $= b$.

Thus we can find the area of a trapezium in terms of its sides.

§ 13. If $c = 0$, ABCD becomes a triangle, and its area

$$= \tfrac{1}{4}\sqrt{(-a+b+d)(a+b-d)(a+b+d)(a-b+d)}.$$

Again, if $c = a$, then also $b = d$, and ABCD becomes a parallelogram, and its area takes the indeterminate form $\frac{0}{0}$, as it should do since four sides do not completely determine a parallelogram.

F. *Quadrilaterals Generally.*

§ 14. A quadrilateral is completely determined when five independent data are given. We consider the following cases.

(α) *When any diagonal and the perpendiculars on it from the opposite vertices are given.*

The quadrilateral ABCD (fig. 6) $= \text{ABD} + \text{BCD}$

$$= \tfrac{1}{2}\text{BD.AE} + \tfrac{1}{2}\text{BD.CF}$$
$$= \tfrac{1}{2}\text{BD}(\text{AE} + \text{CF});$$

or the area is equal to half the product of the diagonal and the sum of the perpendiculars.

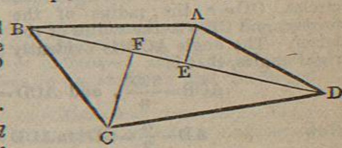

Fig. 6.

If the diagonal BD fall without the figure, as in the concave quadrilateral ABCD (fig. 7), then it is clear that area ABCD $= \frac{1}{2}\text{BD}(\text{AE} - \text{CF}).$

(β) *When the diagonals and their included angle are given.*—In the quadrilateral ABCD (fig. 8, p. 16) let BD $= h$, AC $= k$, and angle DEA $= \alpha$, then

$$\text{ABCD} = \text{ABD} + \text{BCD}$$
$$= \tfrac{1}{2}\text{BD.AE}\sin\alpha + \tfrac{1}{2}\text{BD.CE}\sin\alpha \text{ (§ 10, } \gamma)$$
$$= \tfrac{1}{2}h(\text{AE} + \text{CE})\sin\alpha$$
$$= \tfrac{1}{2}hk\sin\alpha;$$

or the area is equal to the product of the diagonals and the sine of their contained angle.

The same result holds when one of the diagonals falls without the quadrilateral, as in fig. 7, as the reader can easily verify.

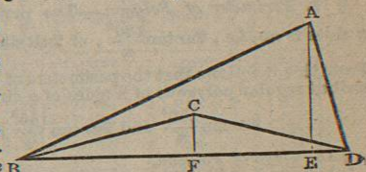

Fig. 7.

(γ) *When the four sides and the angle between the diagonals are given.*—If a, b, c, d be the sides and α the angle between the diagonals it can easily be shown that

$$\text{area of quadrilateral} = \tfrac{1}{4}(a^2 - b^2 + c^2 - d^2)\tan\alpha.$$

(3) *When the four sides are given and the opposite angles are supplementary.*—In fig. 8 let AB = a, BC = b, CD = c, DA = d, AC = k, angle ABC = α, angle CDA = β, and let $\alpha+\beta=180°$, then

area of ABCD = ABC + ADC
$= \frac{1}{2}ab\sin\alpha + \frac{1}{2}cd\sin\beta.$

But
$\sin\beta = \sin(180° - \alpha) = \sin\alpha,$

therefore

area of ABCD $= \frac{1}{2}(ab+cd)\sin\alpha.$

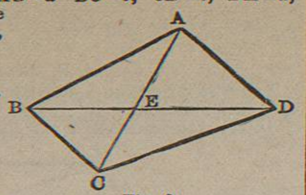

Fig. 8.

This gives us the area of the quadrilateral in terms of the four sides and one angle.

Again we have

$$a^2+b^2-2ab\cos\alpha = k^2 = c^2+d^2-2cd\cos\beta = c^2+d^2+2cd\cos\alpha,$$

therefore
$$\cos\alpha = \frac{a^2+b^2-c^2-d^2}{2(ab+cd)},\text{ and hence}$$
$$1+\cos\alpha = \frac{(a+b+c-d)(a+b-c+d)}{2(ab+cd)}\text{ and}$$
$$1-\cos\alpha = \frac{(c+d+a-b)(c+d-a+b)}{2(ab+cd)}$$

From this we obtain
$$\sin^2\alpha = (1+\cos\alpha)(1-\cos\alpha)$$
$$= \frac{(b+c+d-a)(c+d+a-b)(d+a+b-c)(a+b+c-d)}{4(ab+cd)^2}$$

Now let
$$a+b+c+d = 2s,$$
then
$$\tfrac{1}{2}(ab+cd)\sin\alpha = \sqrt{(s-a)(s-b)(s-c)(s-d)}\,;$$
therefore
$$\text{area of ABCD} = \sqrt{(s-a)(s-b)(s-c)(s-d)},$$
or
$$\log\text{area} = \tfrac{1}{2}\{\log(s-a)+\log(s-b)+\log(s-c)+\log(s-d)\}.$$

If $d=0$, the quadrilateral becomes a triangle, and its area is $\sqrt{(s-a)(s-b)(s-c)s}$ as before. In extracting the square root of $\sin\alpha$ we take the positive sign, since the angle α is less than two right angles.

G. *Regular Polygons.*

§ 15. Since a regular polygon is both equilateral and equiangular, a circle can be inscribed within it and also described about it, and thus the n straight lines drawn from the common centre of the two circles to the n vertices of the polygon divide it into n triangles equal in every respect. Therefore the area of the polygon is equal to n times the area of any one of these triangles.

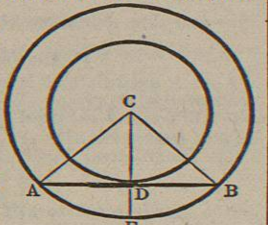

Fig. 9.

§ 16. *Radius of Inscribed and Circumscribed Circles.*—Let AB (fig. 9) = a be a side of a regular polygon of n sides; let C be the centre of the inscribed and circumscribed circles, CD = r the radius of the former, and CE = R the radius of the latter. The angle ACB is evidently equal to the nth part of four right angles, that is

$$\angle\text{ACB} = \frac{360°}{n},\text{ and ACD} = \tfrac{1}{2}\text{ACB} = \frac{180°}{n}.$$

Now
$$\text{AD} = \frac{a}{2} = \text{CD}\tan\text{ACD} = r\tan\frac{180°}{n},$$
and
$$\text{AD} = \frac{a}{2} = \text{AC}\sin\text{ACD} = \text{R}\sin\frac{180°}{n};$$
therefore
$$r = a\times\tfrac{1}{2}\cot\frac{180°}{n},$$
and
$$\text{R} = a\times\tfrac{1}{2}\operatorname{cosec}\frac{180°}{n}.$$

§ 17. *Perimeter of Polygon.*—The perimeter of the polygon of n sides is na, *i.e.*, $2nr\tan\frac{180°}{n}$, or $2n\text{R}\sin\frac{180°}{n}$.

From this it follows that the perimeters of the inscribed and circumscribed regular polygons of n sides of a circle of radius r are

$$2nr\sin\frac{180°}{n}\text{ and }2nr\tan\frac{180°}{n}\text{ respectively.}$$

§ 18. *Area of Polygon.*

(α) *In terms of r.*—The area of polygon
$$= n\text{ACB} = n\text{AD}.\text{CD} = n\times r^2\tan\frac{180°}{n}$$

(β) *In terms of* R.—The triangle ACB
$$= \tfrac{1}{2}\text{AC}.\text{CB}\sin\text{ACB} = \tfrac{1}{2}\text{R}^2\sin\frac{360°}{n}$$
and therefore area of polygon $= \frac{1}{2}n\text{R}^2\sin\frac{360°}{n}$.

(γ) *In terms of a.*—The triangle ACB
$$= \tfrac{1}{2}\text{AB}.\text{CD} = \frac{a}{2}\times r = \frac{a}{2}\times\tfrac{1}{2}a\cot\frac{180°}{n} = \frac{a^2}{4}\cot\frac{180°}{n},$$
and therefore area of polygon $= a^2\times\frac{n}{4}\cot\frac{180°}{n}$,
$$\log 4\,\text{area} = \log n + \text{L}\cot\frac{180°}{n} + 2\log a - 10.$$

From α and β it follows that the areas of the inscribed and circumscribed regular polygons of n sides of a circle of radius r are
$$\tfrac{1}{2}nr^2\sin\frac{360°}{n}\text{ and }nr^2\tan\frac{180°}{n}\text{ respectively.}$$

§ 19. In the formula (§ 18, γ) for the area of a polygon, the factor $\frac{n}{4}\cot\frac{180°}{n}$ has a definite value for every value of n, and hence, if we find its value once for all for a large number of values of n, and tabulate the results, we can find the area of a regular polygon of n sides by multiplying the square of its side by the appropriate tabular value.

Again, if $a=1$ we have
$$r = \tfrac{1}{2}\cot\frac{180°}{n}\text{ and R} = \tfrac{1}{2}\operatorname{cosec}\frac{180°}{n};$$
and thus we obtain in a similar manner the radius of the inscribed and circumscribed circles by multiplying the side by the appropriate tabular value of $\frac{1}{2}\cot\frac{180°}{n}$ and $\frac{1}{2}\operatorname{cosec}\frac{180°}{n}$ respectively.

§ 20. The following table contains the values of $\frac{n}{4}\cot\frac{180°}{n}$ and their logarithms, and the values of $\frac{1}{2}\cot\frac{180°}{n}$ and $\frac{1}{2}\operatorname{cosec}\frac{180°}{n}$ for all values of n from 3 to 12.

n	$\frac{n}{4}\cot\frac{180°}{n}$	Logarithms.	$\frac{1}{2}\cot\frac{180°}{n}$	$\frac{1}{2}\operatorname{cosec}\frac{180°}{n}$
3	0·4330127	$\bar{1}$·6365006	0·28867	·57735
4	1·0000000	0·0000000	0·50000	·70710
5	1·7204774	0·2356490	0·68819	·85065
6	2·5980762	0·4146519	0·86602	1·0000
7	3·6339124	0·5603745	1·0383	1·1523
8	4·8284271	0·6380568	1·2071	1·3065
9	6·1818242	0·7911166	1·3737	1·4619
10	7·6942088	0·8861640	1·5388	1·6180
11	9·3656407	0·9715375	1·7028	1·7747
12	11·1961524	1·0490688	1·8660	1·9318

Let A denote the area of a polygon of n sides and A′ the corresponding tabular value of $\frac{n}{4}\cot\frac{180°}{n}$, then
$$\text{A} = a^2\text{A}',$$
or
$$\log\text{A} = 2\log a + \log\text{A}'.$$

H.—*Length of the Radius of the Inscribed, Escribed, and Circumscribed Circles of a Triangle.*

§ 21. (α) *Radius of Inscribed Circle.*—Let O (fig. 10) be the centre of the circle inscribed in the triangle ABC and touching the sides in D, E, and F. Join OA, OB, and OC. The angles at D, E, F are right angles (Eucl. iii. 18). Let BC = a, CA = b, AB = c, and OD = OE = OF = r.

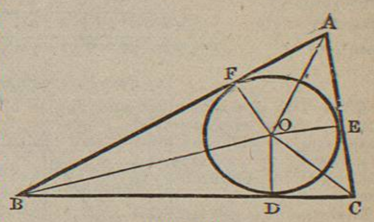

Fig. 10.

Now ABC = BOC + COA + AOB
$$= \tfrac{1}{2}ar + \tfrac{1}{2}br + \tfrac{1}{2}cr = \tfrac{1}{2}(a+b+c)r = rs\,;$$
whence
$$r = \frac{\text{area of ABC}}{s} = \frac{\sqrt{s(s-a)(s-b)(s-c)}}{s}.$$

(β) *Radius of Escribed Circles.*—Let OD = OE = OF = r_a, then

ABC = ACO + ABO − BOC
$$= \tfrac{1}{2}br_a + \tfrac{1}{2}cr_a - \tfrac{1}{2}ar_a = \tfrac{1}{2}(b+c-a)r_a = r_a(s-a),$$
and
$$r_a = \frac{\text{area of ABC}}{s-a} = \frac{\sqrt{s(s-a)(s-b)(s-c)}}{s-a} = \sqrt{\frac{s(s-b)(s-c)}{s-a}}$$
Similarly
$$r_b = \sqrt{\frac{s(s-a)(s-c)}{s-b}}\text{ and }r_c = \sqrt{\frac{s(s-a)(s-b)}{s-c}}.$$

Radius of Circumscribed Circle.—Let AD (fig. 12) = p the perpendicular from A on the side BC, and AE = 2R the diameter

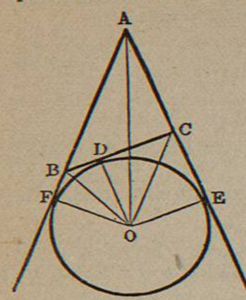

Fig. 11.

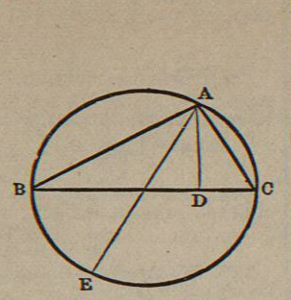

Fig. 12.

of the circle, then (Eucl. vi. C) we have
$$2\text{R}\times p = b\times c,$$
therefore
$$2\text{R}\times ap = abc.$$
Now $ap = 2\Delta$, where Δ denotes the area of ABC;

hence
$$\text{R} = \frac{abc}{4\Delta} = \frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}}.$$

Example.—Let $a=13$, $b=14$, and $c=15$; then r will be found to be 4, r_a 10½, r_b 12, r_c 14, and R 8⅛.

SECTION II.—PLANE FIGURES CONTAINED BY CURVED LINES.

A. *The Circle.*

§ 22. *Circumference of a Circle.*—If we inscribe in any circle a regular polygon of n sides, and also circumscribe a regular polygon of the same number of sides, it is clear that the perimeter of the circle is intermediate between the perimeters of the inscribed and circumscribed polygons, and that the difference between the perimeters of the inscribed and circumscribed polygons can be made as small as we please by sufficiently increasing n. A similar statement holds with reference to the areas of the circle and the inscribed and circumscribed polygons. With the above assumptions it is easily proved that the circumference of a circle bears a constant ratio to its diameter. Hence we have

Circumference = C = constant × radius = constant × r.

It is usual to denote this constant by 2π, and therefore

$\text{C} = 2\pi r = \pi d$, where d is the diameter of the circle.

§ 23 *Numerical Value of π.*—The constant π being, as can be easily proved, an interminable decimal, we can only approximate to its value, but this we can do to any degree of accuracy we please.

If s and σ denote respectively a side of the inscribed and circumscribed polygons of n sides, and s' and σ' a side of the inscribed and circumscribed polygons of $2n$ sides, it can easily be shown that

$$(\alpha)\ \sigma = \frac{rs}{\sqrt{r^2-(\frac{1}{2}s)^2}},\qquad(\beta)\ s'^2 = 2r\left\{r-\sqrt{r^2-(\tfrac{1}{2}s)^2}\right\},$$
$$(\gamma)\ \sigma' = \frac{rs'}{\sqrt{r^2-(\frac{1}{2}s')^2}},$$

where r is the radius of the circle.

If we take $r=\frac{1}{2}$ we find, by means of these formulæ, and by assuming the value of s when $n=6$, that the perimeter of inscribed polygon of 96 sides = 3·140 . . . . , and the perimeter of circumscribed polygon of 96 sides = 3·142. . . .

From this we learn that the circumference of the circle, in this case π, is greater than 3·140 . . . and less than 3·142 . . . . , and therefore as far as the second place of decimals

$$\pi = 3·14.$$

By taking greater and greater values of n we obtain closer and closer approximations to π.

The preceding method for approximating to the value of π is the simplest afforded by elementary geometry, and is also the oldest; but better and more rapid methods are furnished by the higher mathematics. The calculation of π has been carried to 707 places of decimals, the following being the first 20 figures in the result:—

3·14159265358979323846.

For all practical purposes it is sufficient to take

$$\pi = 3·14159\text{ or } = \frac{355}{113}.$$

§ 24. The following table contains the functions of π that are of most frequent occurrence in mensuration:—

	Number.	Logarithm.		Number.	Logarithm.
π	3·1415927	0·4971499	π^2	9·8696044	0·9942997
2π	6·2831853	0·7981799	$\frac{1}{6\pi^2}$	0·0168869	$\bar{2}$·2275490
4π	12·5663706	1·0992099	$\sqrt{\pi}$	1·7724539	0·2485750
$\frac{1}{2}\pi$	1·5707963	0·1961199	$\sqrt[3]{\pi}$	1·4645919	0·1657166
$\frac{1}{3}\pi$	1·0471976	0·0200286	$\frac{1}{\sqrt{\pi}}$	0·5641896	$\bar{1}$·7514251
$\frac{1}{4}\pi$	0·7853982	$\bar{1}$·8950899	$\frac{2}{\sqrt{\pi}}$	1·1283792	0·0524551
$\frac{1}{6}\pi$	0·5235988	$\bar{1}$·7189986	$\frac{1}{2\sqrt{\pi}}$	0·2820948	$\bar{1}$·4503951
$\frac{1}{8}\pi$	0·3926991	$\bar{1}$·5940599	$\sqrt[3]{\frac{6}{\pi}}$	1·2407010	0·0936671
$\frac{1}{12}\pi$	0·2617994	$\bar{1}$·4179686	$\sqrt[3]{\frac{3}{4\pi}}$	0·6203505	$\bar{1}$·7926371
$\frac{4}{3}\pi$	4·1887902	0·6220886	$\log_e\pi$	1·1447299	0·0587030
$\frac{\pi}{180}$	0·0174533	$\bar{2}$·2418774			
$\frac{1}{\pi}$	0·3183099	$\bar{1}$·5028501			
$\frac{4}{\pi}$	1·2732395	0·1049101			
$\frac{1}{4\pi}$	0·0795775	$\bar{2}$·9007901			
$\frac{180}{\pi}$	57·2957795	1·7581226			

§ 25. *Units of Angular Measurement.*—In measuring lines we select some line of constant length as the standard or unit; similarly in measuring angles we require to take some angle of constant magnitude as unit angle. The right angle is by its nature the simplest unit angle, but, for convenience, we take the $\frac{1}{90}$th part of a right angle for unit, and call it a degree, which is subdivided into sixty equal parts called minutes, and these again into sixty equal parts called seconds. For theoretical purposes we define the unit angle to be *the angle subtended at the centre of a circle by an arc equal to the radius.* This angle we call a "radian." In many treatises the radian measure of an angle is called the circular measure.

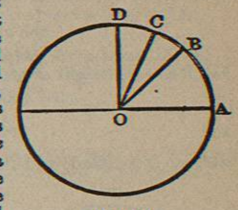

Fig. 13.

§ 26. *The radian is a constant angle.*—Let OA (fig. 13) = arc AB = r, then AOB = radian, and let AOD = 90°; then
$$\text{arc AD} = \tfrac{1}{4}\times 2\pi r = \tfrac{1}{2}\pi r\,;$$
and, since angles at the centre of a circle are proportional to the arcs on which they stand (Eucl. vi. 33),
$$\frac{\text{number of degrees in radian AOB}}{\text{number of degrees in AOD}} = \frac{\text{AB}}{\text{AD}} = \frac{r}{\frac{1}{2}\pi r} = \frac{2}{\pi}\,;$$
therefore number of degrees in radian
$$= 90°\times\frac{2}{\pi} = 57°·2957795 = \text{constant.}$$

§ 27. *Number of Radians in any Angle.*—Let AOC (fig. 13) be any angle, AOB the radian, and AC = s; then
$$\frac{\text{number of radians in AOC}}{\text{one radian}} = \frac{\text{AC}}{\text{AB}} = \frac{s}{r}\,;$$
therefore number of radians in AOC $= \frac{s}{r}$.

If AOC = 90°, then $s = \frac{1}{2}\pi r$, and number of radians $= \frac{1}{2}\pi$; there are thus π radians in two and 2π in four right angles.

When $r=1$ we have number of radians = s, and hence in some treatises for the number of radians in an angle we find the length of the arc given.

§ 28. *To transfer from degrees to radians and conversely.*—Let x denote the number of degrees in an angle, and θ the number of radians in the same, then, since $\frac{x}{180°} = \frac{\theta}{\pi}$,
$$(\alpha)\ \theta = \frac{\pi x°}{180},\quad(\beta)\ x° = \frac{180}{\pi}\theta.$$

§ 29. The following table contains the values of θ for values of x up to 180°, and also for minutes and seconds.

Degrees.	Radian.	Degrees.	Radian.	Degrees.	Radian.	Minutes.	Radian. ·0	Seconds.	Radian. ·000
1	·0174533	61	1·0646508	121	2·1118484	1	002909	1	0048
2	·0349066	62	1·0821041	122	2·1293017	2	005818	2	0097
3	·0523599	63	1·0995574	123	2·1467550	3	008727	3	0145
4	·0698132	64	1·1170107	124	2·1642083	4	011636	4	0194
5	·0872665	65	1·1344640	125	2·1816616	5	014544	5	0242
6	·1047198	66	1·1519173	126	2·1991149	6	017453	6	0291
7	·1221730	67	1·1693706	127	2·2165682	7	020362	7	0339
8	·1396263	68	1·1868239	128	2·2340214	8	023271	8	0388
9	·1570796	69	1·2042772	129	2·2514747	9	026180	9	0436
10	·1745329	70	1·2217305	130	2·2689280	10	029089	10	0485
20	·3490659	80	1·3962634	140	2·4434610	20	058178	20	0970
30	·5235988	90	1·5707963	150	2·6179939	30	087266	30	1454
40	·6981317	100	1·7453293	160	2·7925268	40	116355	40	1939
50	·8726646	110	1·9198622	170	2·9670597	50	145444	50	2424
60	1·0471976	120	2·0943951	180	3·1415927	60	174533	60	2909