As an example of the use of this table we proceed to find the value of θ when $x=68^\circ\ 45'\ 17''\cdot 8$.

$$\begin{aligned}\text{When } x&=68^\circ & \theta&=1\cdot 1868239,\\ x&=40' & \theta&=\ \ 0116355,\\ x&=\ 5' & \theta&=\cdot 0014544,\\ x&=10'' & \theta&=\cdot 0000485,\\ x&=\ 7'' & \theta&=\ \ 0000339,\\ \text{and when } x&=0''\cdot 8 & \theta&=\underline{\cdot 0000039},\\ \text{therefore when } x&=68^\circ\ 45'\ 17''\ 8 & \theta&=1\cdot 2000001.\end{aligned}$$

§ 30 Combining the results of §§ 27 and 28 we obtain

$$(\alpha)\quad \theta=\frac{s}{r},\ \text{and}\ x=\frac{180}{\pi}\cdot\frac{s}{r};$$

$$(\beta)\quad r=\frac{s}{\theta}=\frac{180}{\pi}\ \frac{s}{x};$$

$$(\gamma)\quad s=r\theta=\frac{\pi}{180}\ rx.$$

§ 31. *Length of Arcs of Circles.*—The following are the more important cases:—

(α) *In terms of the chord of the arc and the radius of the circle.*—Let AB (fig 14) $=2c$, AC $=r$, and AEB $=s$, then

AD $=\frac{1}{2}$AB $=c=r\sin\frac{1}{2}$C, whence C is known, and therefore the arc s is found (§ 30, γ).

(β) *In terms of the height of the arc and the radius of the circle.*—Let DE $=h=$ height of arc, then

$$\text{CD}=\text{CE}-\text{DE}=r-h,$$

and

$$\cos\tfrac{1}{2}\text{C}=\frac{\text{CD}}{\text{AC}}=\frac{r-h}{r},$$

whence C is found, and therefore s.

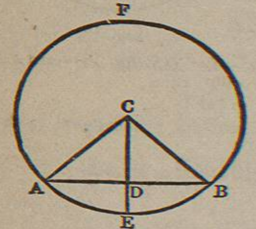

Fig. 14.

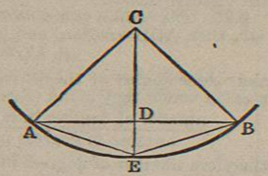

Fig. 15.

§ 32. *Huygens's Approximation to the Length of a Circular Arc.*—Let AB (fig. 15) $=p$ be the chord of the arc AEB, and AE $=$ EB $=q$ that of half the arc, then the arc AEB $=\frac{1}{3}(8q-p)$ approximately.

For, let r denote the radius, s the arc AEB, and 2θ the angle ACB, then $\theta=\frac{s}{2r}$ Again, AB $=p=2$AD $=2r\sin\theta=2r\sin\frac{s}{2r}$; and similarly $q=2r\sin\frac{s}{4r}$

Now
$$\sin\theta=\theta-\frac{\theta^3}{\lfloor 3}+\frac{\theta^5}{\lfloor 5}-\&\text{c.}:$$

therefore
$$p=2r\left\{\frac{s}{2r}-\frac{1}{\lfloor 3}\left(\frac{s}{2r}\right)^3+\frac{1}{\lfloor 5}\left(\frac{s}{2r}\right)^5-\&\text{c.}\right\}$$
$$=s-\frac{s^3}{2.3\ 4.r^2}+\frac{s^5}{2.3.4.5.16.r^4}-\&\text{c.}$$

Similarly
$$8q=16r\left\{\frac{s}{4r}-\frac{1}{\lfloor 3}\left(\frac{s}{4r}\right)^3+\frac{1}{\lfloor 5}\left(\frac{s}{4r}\right)^5-\&\text{c.}\right\}$$
$$=4s-\frac{s^3}{2.3.4.r^2}+\frac{s^5}{2.3.4.5.64.r^4}-\&\text{c.}$$

Hence, neglecting powers of $\frac{s}{r}$ beyond the fourth, we obtain

$$\frac{8q-p}{3}=s\left(1-\frac{s^4}{7680r^4}\right)=s$$

approximately.

In practice it is sometimes more convenient to use the equivalent form

$$s=2q+\tfrac{1}{3}(2q-p).$$

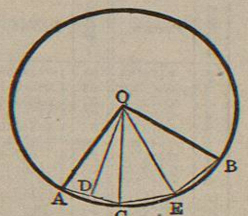

Fig. 16.

§ 33. *Area of Sector of a Circle.*—Let the sector be OAB (fig. 16). Divide the arc AB into n equal parts, and draw the chords of these. Let P denote the perimeter of the broken line AB, A the area of the polygon AOB, and p the altitude of any one of the n equal triangles of which this polygon is made up.

Now in the limit, when n is indefinitely increased, P becomes the arc AB $=s$, a result which we symbolize thus—

$$\mathrm{L}_{n=\infty}\ \mathrm{P}=\text{arc AB}=s.$$

Similarly
$$\mathrm{L}_{n=\infty}\ p=\text{radius}=r$$

Again, the area of the sector is equal to the area of the polygon when the broken line AB becomes the arc AB, that is,

$$\text{sector}=\mathrm{L}_{n=\infty}\mathrm{A}=\mathrm{L}_{n=\infty}\tfrac{1}{2}p\times\mathrm{P}=\tfrac{1}{2}\mathrm{L}_{n=\infty}p\times\mathrm{L}_{n=\infty}\mathrm{P}$$
$$=\tfrac{1}{2}rs$$

§ 34. Let S denote the area of a sector of a circle, then, by means of the above result and § 27, we have

$$(\alpha)\quad \mathrm{S}=\tfrac{1}{2}sr,$$
$$(\beta)\quad \mathrm{S}=\tfrac{1}{2}r\theta\,.\,r=\tfrac{1}{2}r^2\theta.$$

§ 35. We proceed to find the area of a sector of a circle in the following additional cases .—

(α) *When the chord of the sector and the radius of the circle are given.*—In fig. 14 let AB $=2c$, and let AC $=r$, then we have

$$\sin\frac{\text{ACB}}{2}=\frac{\text{AD}}{\text{AC}}=\frac{c}{r};$$

whence ACB and therefore θ is known, and S can be found by § 34.

ACB has two values, the smaller one giving the area of the minor, and the larger that of the major conjugate sector.

(β) *When the chord and height of the chord are given.*—Let DE (fig 14) $=h$ and AB $=2c$, then

$$\text{AC}^2=r^2=\text{AD}^2+\text{DC}^2=c^2+(r-h)^2,\ \text{whence}$$

$r=\frac{c^2+h^2}{2h}$, and therefore by previous case the area can be found.

(γ) *When the chord and angle subtended at the centre are given.*—Let AB (fig. 14) $=2c$ and ACB $=\theta$, then

$$\frac{c}{r}=\sin\frac{\text{ACB}}{2},\ \text{or}\ r=\frac{c}{\sin\frac{1}{2}\theta},$$

therefore area of sector $=\frac{1}{2}r^2\theta=\frac{1}{2}\left(\frac{c}{\sin\frac{1}{2}\theta}\right)^2\times\theta.$

§ 36. *Area of a Circle.*—The circle being a sector whose arc is the whole circumference we obtain at once

$$\text{area of circle}=\tfrac{1}{2}r\times s=\tfrac{1}{2}r\times 2\pi r=\pi r^2.$$

An independent proof of this proposition might be given by means of the inscribed and circumscribed polygons, and from the area of a circle the area of a sector can be deduced. The infinitesimal calculus affords a simple and elegant proof (see § 44).

§ 37. If A denote the area, r the radius, $d=2r$ the diameter, and C the circumference of a circle, we have

$$(\alpha)\quad \mathrm{A}=\pi r^2,$$
$$(\beta)\quad \mathrm{A}=\tfrac{1}{2}\times 2\pi r\times r=\tfrac{1}{2}\mathrm{C}r,$$
$$(\gamma)\quad \mathrm{A}=\frac{4\pi^2r^2}{4\pi}=\frac{\mathrm{C}^2}{4\pi},$$
$$(\delta)\quad \mathrm{A}=\frac{\pi 4r^2}{4}=\frac{\pi d^2}{4}$$

Whence we see that the area of a circle is obtained by multiplying

(α) the square of its radius by $\pi=3\cdot 14159$,

(β) the radius by half the circumference,

(γ) the square of the circumference by $\frac{1}{4\pi}=\cdot 07957$

(δ) the square of the diameter by $\frac{1}{4}\pi=\cdot 78539$.

§ 38. Again, from the above formulæ we deduce

$$(\alpha)\quad r=\frac{1}{\sqrt{\pi}}\mathrm{A}=\cdot 5641896\times\mathrm{A},$$
$$(\beta)\quad d=\frac{2}{\sqrt{\pi}}\mathrm{A}=1\cdot 1283792\times\mathrm{A},$$
$$(\gamma)\quad c=2\sqrt{\pi}\mathrm{A}=3\cdot 5449077\times\mathrm{A},$$

thus obtaining radius, diameter, and circumference from area.

§ 39. *Area of a Circular Ring.*—Let r and r_1 denote the radii of the outer and inner circles respectively (fig. 17), then the area of the space between them

$$=\pi r^2-\pi r_1^2=\pi(r+r_1)(r-r_1).$$

The circles need not be concentric, and the reader should note that the area of the ring is equal to the area of an ellipse whose major and minor axes are $r+r_1$ and $r-r_1$ (see § 51).

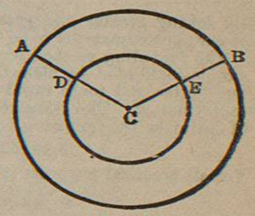

Fig. 17.

§ 40. *Area of the Sector of an Annulus.*—Let angle ACB $=\theta$ in fig. 17, then the area of ABED

$$=\text{sector ACB}-\text{sector DCE}$$
$$=\tfrac{1}{2}r^2\theta-\tfrac{1}{2}r_1^2\theta,$$
$$=\tfrac{1}{2}\theta(r+r_1)(r-r_1).$$

Again, let AB $=l$, DE $=l_1$, and CA $-$ CD $=r-r_1=h$, then

$$r=\frac{l}{\theta}\ \text{and}\ r_1=\frac{l_1}{\theta};\ \text{therefore}$$

$$r+r_1=\frac{1}{\theta}(l+l_1),\ \text{and}$$

$$\text{area of sector}=\tfrac{1}{2}\theta(r+r_1)(r-r_1)=\tfrac{1}{2}\theta\left(\frac{l+l_1}{\theta}\right)h=\tfrac{1}{2}h(l+l_1).$$

§ 41. *Area of a Segment of a Circle.*—(α) *When the radius and the angle subtended at the centre are given.*—In fig. 14, let AEB be a segment of a circle, then its area

$$=\text{sector ACB}-\text{triangle ACB}.$$
$$=\tfrac{1}{2}r^2\theta-\tfrac{1}{2}r^2\sin\theta\ (\S\S\ 9,\ 34)$$
$$=\tfrac{1}{2}r^2(\theta-\sin\theta).$$

If the segment be greater than a semicircle $\sin\theta$ is negative and the formula becomes

$$\tfrac{1}{2}r^2(\theta+\sin\theta),$$

as is also geometrically evident.

We might in a similar manner find the area of a segment of a circle

(β) when the chord and radius are given,

(γ) when the chord and its height are given,

(δ) when the radius and height of the chord are given,

(ε) when the chord and angle subtended by the chord are given.

In all these cases the method of proceeding is obvious, a segment being the difference between a sector and a triangle.

§ 42. *Area of a Lune.*—Let ADB and ACB (fig. 18) be two segments of circles, then the area of the lune ADBC

$$=\text{segment ADB}-\text{segment ACB}.$$

Hence if we so choose our data that we can determine the areas of the two segments we have only to take their difference to find the area of the lune.

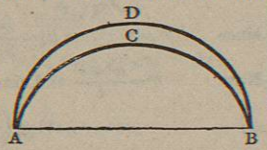

Fig. 18.

§ 43. *Area of a Circular Zone.*—Let AB and CD (fig. 19) be two parallel chords, then the area of the zone ABCD

$$=\text{circle}-\text{segment AHB}-\text{segment DFC};$$

or
$$=\text{segment AED}+\text{trapezium ABCD}+\text{segment BGC}$$
$$=2\ \text{segment AED}+\text{trapezium ABCD}.$$

§ 44. The Infinitesimal Calculus (*q.v.*) furnishes a simple and elegant proof of the formulæ for the areas of a circle and a sector. If $y=\phi(x)$ be the equation to a plane curve referred to rectangular axes, then the area between the curve, the axis of x, and two ordinates corresponding to the abscissæ a and b is represented by the integral

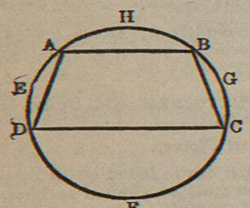

Fig. 19.

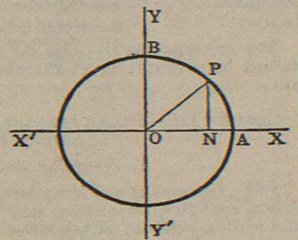

Fig. 20.

$$\int_a^b\phi(x)dx.$$

Let x and y be the coordinates of P (fig. 20), then if OP $=r$ the equation to the circle is $y^2=r^2-x^2$, and therefore

$$\text{area of quadrant AOB}=\int_0^r\sqrt{r^2-x^2}dx$$

$$=\left[\frac{r^2}{2}\sin^{-1}\frac{x}{r}+\frac{x}{2}\sqrt{r^2-x^2}\right]_0^r=\frac{r^2}{2}\times\frac{\pi}{2}=\frac{\pi r^2}{4},$$

and therefore area of whole circle $=\pi r^2$.

§ 45. If the equation to a plane curve be given in polar coordinates, the area bounded by two radii and the curve is equal to

$$\tfrac{1}{2}\int_{\theta_1}^{\theta_2}r^2d\theta,$$

where θ_1 and θ_2 are the values of θ corresponding to the limiting radii.

For example, let AOP (fig. 20) be θ, then area of circle

$$=\frac{r^2}{2}\int_0^{2\pi}d\theta=\frac{2\pi r^2}{2}=\pi r^2\ \text{as before.}$$

The area of a sector can be found in a similar manner.

B. *The Parabola.*

§ 46. *Length of an Arc of any Plane Curve.*—If a plane curve be referred to rectangular axes, then the length of any arc of the curve

$$=s=\int\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}}dx=\int\left\{1+\left(\frac{dx}{dy}\right)\right\}^{\frac{1}{2}}dy$$

taken between proper limits, *i.e.*, the extremities of the arc. See Infinitesimal Calculus.

§ 47. *Arc of a Parabola.*—Let the axes of coordinates be the axis

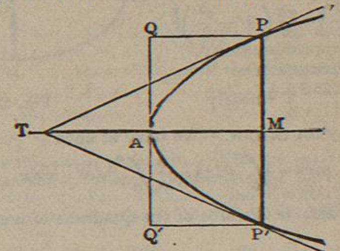

Fig. 21.

of x and the tangent at the vertex A (fig. 21), then, the equation to the parabola being $y^2=2mx$, where $m=2a=\frac{1}{2}$ latus rectum, we have

$$\frac{dx}{dy}=\frac{y}{m},\ \text{and hence}$$

$$s=\text{arc AP}=\int_0^{y_1}\left\{1+\frac{y^2}{m^2}\right\}^{\frac{1}{2}}dy$$
$$=\frac{y_1\sqrt{y_1^2+m^2}}{2m}+\frac{m}{2}\log_e\left(\frac{y_1+\sqrt{y_1^2+m^2}}{m}\right);$$

therefore whole arc PAP′ $=\frac{y_1}{m}\sqrt{y_1^2+m^2}+m\log_e\left(\frac{y_1+\sqrt{y_1^2+m^2}}{m}\right)$.

Since $y_1^2=2mx_1$, the formula may be written

$$\text{arc PAP}'=\sqrt{4x_1^2+y_1^2}+\frac{y_1^2}{2x_1}\log_e\left(\frac{2x_1+\sqrt{4x_1^2+y_1^2}}{y_1}\right).$$

§ 48. *Area of a Parabola.*—Taking the equation to the parabola in the form $y^2=4px$, we get

$$\text{area of segment PAP}'\ (\text{fig. }21)=2\int_0^{x_1}2\sqrt{px}dx$$
$$=\tfrac{8}{3}p^{\frac{1}{2}}x_1^{\frac{3}{2}}=\tfrac{4}{3}x_1y_1=\frac{y_1^3}{3p}.$$

From these formulæ we see that the area of a parabolic segment varies directly as the cube of the square root of the abscissa, and directly as the cube of the ordinate, and that it is equal to $\frac{2}{3}$ rectangle PQQ′P′, or $\frac{2}{3}$ triangle PTP′.

A similar relation holds for the segment cut off by any chord, and thus the area of any parabolic segment can be determined in terms of any data that are sufficient to determine the segment.

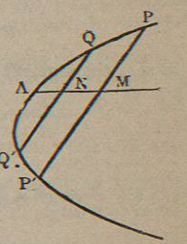

Fig. 22.

§ 49. *Area of a Parabolic Zone.*—Let PM (fig. 22) $=y_1$, QN $=y_2$, AM $=x_1$, AN $=x_2$, and let the ordinates be inclined to the axis at an angle α.

$$\text{Area of zone PQQ}'\text{P}'=\text{segment PAP}'-\text{segment QAQ}'$$
$$=\frac{y_1^3-y_2^3}{3p}\times\sin\alpha.$$

Now $y_1^2=4px_1$ and $y_2^2=4px_2$, therefore $\frac{y_1^2-y_2^2}{4(x_1-x_2)}=p$, and hence on substituting for p we have area of zone

$$=\tfrac{4}{3}(x_1-x_2)\left(\frac{y_1^3-y_2^3}{y_1^2-y_2^2}\right)\sin\alpha=\tfrac{4}{3}\frac{x_1-x_2}{y_1+y_2}(y_1^2+y_1y_2+y_2^2)\sin\alpha.$$

C. *The Ellipse.*

§ 50. *Circumference of an Ellipse.*—The equation to the ellipse being $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where a and b are the semiaxes, we have

$$\frac{dy}{dx}=-\frac{b^2x}{a^2y}\text{, and therefore (fig. 23)}$$

$$\text{arc of quadrant AB}=\int_0^a\left\{\frac{a^2-e^2x^2}{a^2-x^2}\right\}^{\frac{1}{2}}dx\text{, where }e^2=\frac{a^2-b^2}{a^2}.$$

This integral may be shown to be equal to the series

$$\frac{\pi a}{2}\left(1-\frac{e^2}{2^2}-\frac{1\,.\,3e^4}{2^2\,.\,4^2}-\frac{1\,.\,3^2\,.\,5e^6}{2^2\,.\,4^2\,.\,6^2}-\text{\&c.}\right),$$

a rapidly converging series when e is a small fraction.

By taking more and more terms of the above series we can approximate as nearly as we please to the circumference of an ellipse. For example, we have quadrant AB

$$=\frac{\pi a}{2}\left(1-\frac{e^2}{4}\right)=\frac{\pi a}{2}\left(1-\frac{e^2}{2}\right)^{\frac{1}{2}}$$

to a first approximation; hence whole circumference

$$=\pi\left\{\frac{(2a)^2+(2b)^2}{2}\right\}^{\frac{1}{2}}\text{ nearly}$$

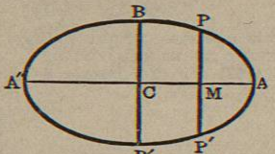

Fig. 23

§ 51. *Area of an Ellipse.*—We have at once

$$\text{area}=4\int_0^a ydx=4\frac{b}{a}\int_0^a\sqrt{a^2-x^2}dx.$$

But $\int_0^a\sqrt{a^2-x^2}dx$ is the area of the quadrant of a circle of radius a. Thus

$$\text{area of ellipse}=4\frac{b}{a}\,\frac{\pi a^2}{4}\ (\S\ 44)$$
$$=\pi ab.$$

The following proof is worth the reader's attention. By a well-known theorem in conic sections the orthogonal projection of a circle on a given plane is an ellipse. Now, if A denote the area of any plane figure, A′ the area of the projected figure, and θ the angle between the planes it can easily be shown, by dividing the two areas by planes indefinitely near to each other and perpendicular to the common section of the planes, that

$$A\cos\theta=A'.$$

In the case of the circle and ellipse $A=\pi a^2$ and $\cos\theta=\frac{b}{a}$;

hence area of ellipse $=\pi a^2\times\frac{b}{a}=\pi ab$.

§ 52. *Area of an Ellipse in terms of a Pair of Conjugate Diameters.*—Let a' and b' denote the semiconjugate diameters, and α the angle between them, then by an elementary property of the ellipse

$$ab=a'b'\sin\alpha;$$

hence area of ellipse $=\pi a'b'\sin\alpha$.

D. *The Hyperbola.*

§ 53. *Area of a Segment of an Hyperbola.*—The equation of an hyperbola being $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, we have

$$y=\frac{b}{a}\sqrt{x^2-a^2}\text{; hence (fig. 24)}$$

$$\text{area of the segment PAP}'=2\frac{b}{a}\int_a^{x_1}\sqrt{x^2-a^2}dx$$
$$=\frac{b}{a}x_1\sqrt{x_1^2-a^2}-ab\log_e\left(\frac{x_1+\sqrt{x_1^2-a^2}}{a}\right)$$
$$=x_1y_1-ab\log_e\left(\frac{x_1}{a}+\frac{y_1}{b}\right)$$

§ 54. *Area of a Sector of an Hyperbola.*—The sector PAP′C is equal to triangle PCP′ − segment PAP′

$$=x_1y_1-\left\{x_1y_1-ab\log_e\left(\frac{x_1}{a}+\frac{y_1}{b}\right)\right\}$$
$$=ab\log_e\left(\frac{x_1}{a}+\frac{y_1}{b}\right).$$

§ 55. *Area of a Zone of an Hyperbola.*—In fig. 24 the zone PP′Q′Q

$$=\text{segment QAQ}'-\text{segment PAP}'$$
$$=x_2y_2-ab\log_e\left(\frac{x_2}{a}+\frac{y_2}{b}\right)-x_1y_1+ab\log_e\left(\frac{x_1}{a}+\frac{y_1}{b}\right)$$
$$=x_2y_2-x_1y_1-ab\log_e\left(\frac{ay_2+bx_2}{ay_1+bx_1}\right)\text{, where}$$

x_1, y_1 and x_2, y_2 are the coordinates of P and Q respectively.

If the axes of coordinates be inclined at an angle α, we multiply the above results by $\sin\alpha$ to obtain the correct areas.

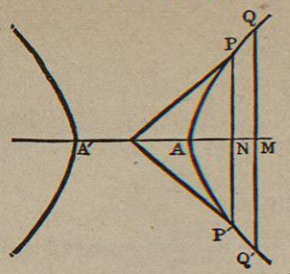

Fig. 24.

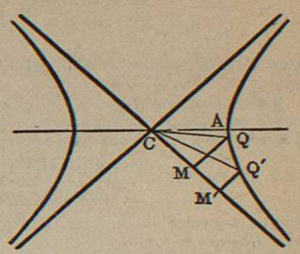

Fig. 25.

§ 56. *Area bounded by an Hyperbola and its Asymptotes.*—The equation of an hyperbola referred to its asymptotes is of the form

$$xy=c^2.$$

Let CM′ (fig. 25) $=x_1$, CM $=x_2$, Q′M′ $=y_1$, Q M $=y_2$, then, if α be the angle between the asymptotes,

$$\text{area of QMM}'\text{Q}'=\int_{x_2}^{x_1}y\sin\alpha dx$$
$$=c^2\sin\alpha\int_{x_2}^{x_1}\frac{dx}{x}=c^2\sin\alpha\log_e\left(\frac{x_1}{x_2}\right)=c^2\sin\alpha\log_e\left(\frac{y_2}{y_1}\right),$$

since $x_1=\frac{c^2}{y_1}$ and $x_2=\frac{c^2}{y_2}$.

Now $c^2=\frac{a^2+b^2}{4}$ and $\sin\alpha=\frac{2ab}{a^2+b^2}$, and therefore

(α) $$\text{area}=\tfrac{1}{2}ab\log_e\left(\frac{x_1}{x_2}\right)=\tfrac{1}{2}ab\log_e\left(\frac{y_2}{y_1}\right).$$

Again, let MM′ $=x_1-x_2=p$, then

$$c^2=x_1y_1=x_2y_2=\frac{py_1y_2}{y_2-y_1}\text{, therefore}$$

(β) $$\text{QMM}'\text{Q}'=\frac{py_1y_2}{y_2-y_1}\log_e\left(\frac{x_1}{x_2}\right)\sin\alpha=\frac{py_1y_2}{y_2-y_1}\log_e\left(\frac{y_2}{y_1}\right)\sin\alpha;$$

Again, since

$$\tfrac{1}{2}x_1y_1\sin\alpha=\tfrac{1}{2}c^2\sin\alpha=\tfrac{1}{2}x_2y_2\sin\alpha,$$

we have triangle QCM = Q′CM′, and hence the sector QCQ′ = QMM′Q′.

The corresponding results for a rectangular hyperbola are obtained by substituting in the above formulæ $\frac{1}{2}a^2$ for c^2 and 1 for $\sin\alpha$.

Section III.—Plane Irregular, Rectilineal, and Curvilineal Figures.

A. *Irregular Rectilineal Figures.*

§ 57. The area of any irregular polygon can be found by dividing it into triangles, trapeziums, &c., in the most convenient manner, and adding together all the areas. For example,

ABCDEF (fig. 26) = CKB + BKHA + AHF + FGE + EGID + DIC.

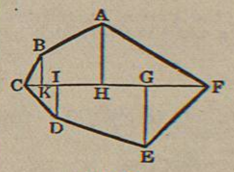

Fig. 26.

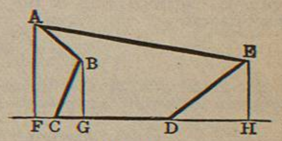

Fig. 27.

It may sometimes happen that some of the component figures have to be subtracted instead of added; for example,

ABCDE (fig. 27) = AFHE + BCG − AFGB − EDH.

§ 58. Again, the irregular rectilineal figure $P_1P_2\ .\ .\ .\ P_8P_9$ (fig. 28) can be broken up into a series of triangles and trapeziums as shown in the figure, and hence its area can be found.

§ 59. A figure made up of straight lines may be measured by cutting it up into triangles by lines drawn from some one vertex to the others. For example (fig. 29),

ABCDEF = ABC + ACD + ADE + AEF.

If the polygon be concave some of the triangles will have to be subtracted.

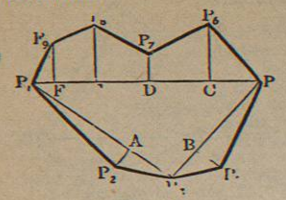

Fig. 28.

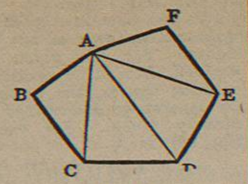

Fig. 29.

§ 60. *Area of a Polygon in terms of the Coordinates of its Angular Points.*—Let the coordinates of P, Q, R (fig. 30) be (x_1, y_1), (x_2, y_2), and (x_3, y_3) respectively, and let the axes be inclined at an angle α. Draw PL, QM, and RN parallel to OY, then

$$LM=OM-OL=x_2-x_1,$$
$$MN=ON-OM=x_3-x_2,$$

and

$$NL=ON-OL=x_3-x_1.$$

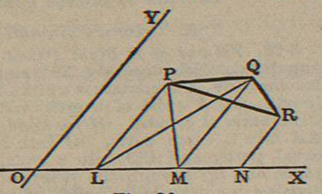

Fig. 30.

Now PQR = PLMQ + QMNR − PLNR.

But PLMQ = PLM + QMP = PLM + QLM

$$=\tfrac{1}{2}PL\cdot LM\sin\alpha+\tfrac{1}{2}QM\cdot LM\sin(180^\circ-\alpha)\ (\S\ 9,\ \beta)$$
$$=\tfrac{1}{2}(x_2-x_1)(y_1+y_2)\sin\alpha.$$

Similarly $QMNR=\tfrac{1}{2}(x_3-x_2)(y_2+y_3)\sin\alpha,$

and $PLNR=\tfrac{1}{2}(x_3-x_1)(y_3+y_1)\sin\alpha$; hence

$$\text{area of PQR}=\tfrac{1}{2}\sin\alpha\{y_1(x_2-x_3)+y_2(x_3-x_1)+y_3(x_1-x_2)\};$$

or in the notation of determinants

$$=\tfrac{1}{2}\sin\alpha\begin{vmatrix}1&1&1\\x_1&x_2&x_3\\y_1&y_2&y_3\end{vmatrix}.$$

When the axes are rectangular $\sin\alpha=\sin 90^\circ=1$, and the formula for the area becomes

$$\tfrac{1}{2}\{y_1(x_2-x_3)+y_2(x_3-x_1)+y_3(x_1-x_2)\}$$
$$=\tfrac{1}{2}\begin{vmatrix}1&1&1\\x_1&x_2&x_3\\y_1&y_2&y_3\end{vmatrix}.$$

§ 61. The area of any rectilineal figure of n sides can be found by taking any point within the figure and joining it to the n vertices of the figure, thus dividing it into n triangles the area of each of which can be obtained as in the preceding case.

We may, however, find the area of the figure directly.

For example, in fig. 31

PQRST = PP′T′T + TT′S′S + SS′R′R − RR′Q′Q − QQ′P′P,

and in fig. 32

PQRSTU = PP′U′U + RR′Q′Q + TT′S′S − PP′Q′Q − RR′S′S − TT′U′U.

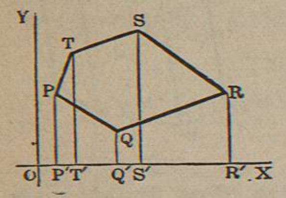

Fig. 31.

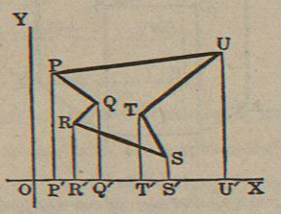

Fig. 32.

B. *Irregular Curvilineal Figures.*

§ 62. *Length of any Curve.*—If we divide the given arc into an even number of intervals and regard these as approximately circular, we can find an approximation to the length of the arc by means of Huygens's formula, § 32. For example, if we divide ABC (fig. 33) into four parts in D, B, and E, and draw the chords AD, AB, DB, BE, BC, and EC, then

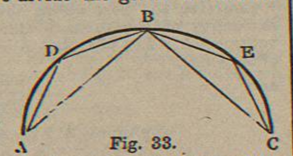

Fig. 33.

arc AC = AD + DB + BE + EC + ⅓(AD + DB + BE + EC − AB − BC) approximately

For other methods of approximation, see Rankine's *Rules and Tables.*

§ 63. *Area of an Irregular Curvilineal Figure.*—For rough approximations the following, called the trapezoidal method, may be used:—

Divide A_1A_n (fig. 34) into n equal parts, and through the points

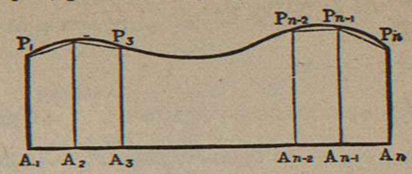

Fig. 34.

of division draw the ordinates called by surveyors offsets, A_1P_1, A_2P_2, &c.

Let $A_1P_1=s_1$, $A_2P_2=s_2$, &c., $A_nP_n=s_n$, and $A_1A_2=A_2A_3=\ .\ .\ .\ =A_{n-1}A_n=a.$

Join P_1P_2, P_2P_3, &c., then the area of the polygon $A_1A_nP_nP_2P_1$

$$=A_1A_2P_2P_1+A_2A_3P_3P_2+\ .\ .\ .\ .\ +A_{n-1}A_nP_nP_{n-1}$$
$$=\tfrac{1}{2}a(s_1+s_2)+\tfrac{1}{2}a(s_2+s_3)+\ .\ .\ .\ .\ +\tfrac{1}{2}a(s_{n-1}+s_n)\ (\S\ 13,\ a)$$
$$=a\{\tfrac{1}{2}(s_1+s_n)+s_2+s_3+\ .\ .\ .\ .+s_{n-1}\}.$$

If we take n sufficiently great the difference between the area of the polygon and the curvilineal figure can be made as small as we please, in other words, the smaller we make a the more accurately will the above formula represent the area of the curvilineal figure.

The curve may either be wholly convex or wholly concave to the line A_1A_n, or partly convex and partly concave.

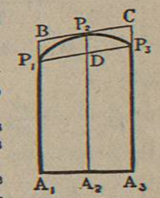

Fig. 35.

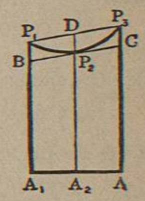

Fig. 36.

§ 64. *Simpson's Rule.*—Let A_1A_n (fig. 34) be divided into an *even* number of equal parts, and as before through the points of division draw the ordinates A_1P_1, A_2P_2, &c.

Let $A_1A_3P_3P_1$ (figs. 35, 36) be a part of the figure thus divided; join P_1P_3, and through P_2 draw BC parallel to P_1P_3 to meet A_1P_1 in B and A_3P_3 in C. Conceive a parabola to be drawn through $P_1P_2P_3$ having its axis parallel to the ordinates, then

$$A_1P_1P_2P_3A_3=\text{trapezium }A_1P_1DP_3A_3\pm\text{parabolic segment }P_1P_2P_3$$
$$=a(s_1+s_3)\pm\tfrac{4}{3}a\{s_2\sim\tfrac{1}{2}(s_1+s_3)\}=\tfrac{1}{3}a(s_1+4s_2+s_3).$$

Now when the points P_1, P_2, P_3 are near each other the parabolic curve will coincide very nearly with the given curve; hence

$$A_1P_1P_2P_3A_3=\tfrac{1}{3}a(s_1+4s_2+s_3)\text{ very nearly.}$$

Similarly $A_3P_3P_5A_5=\tfrac{1}{3}a(s_3+4s_4+s_5)$, &c.;

hence whole area of figure

$$=\tfrac{1}{3}a\{s_1+s_n+2(s_3+s_5+\ .\ .\ .\ +s_{n-2})+4(s_2+s_4+\ .\ .\ .\ .\ +s_{n-1})\};$$

whence the rule:—add together the two extreme ordinates, twice the sum of the intermediate odd ordinates, and four times the sum of the even ones, and multiply this result by one-third of the common distance between the ordinates; the result is the area,—*accurately* if the curved boundary be the arc of a parabola, in other cases approximately.

The curve may either be wholly convex or wholly concave to the line A_1A_n, or partly convex and partly concave, provided in the latter case the points of contrary flexure occur only at the odd ordinates, for otherwise the intermediate arcs could not be even approximately parabolic. When points of contrary flexure occur ordinates may be drawn at these points, and the intermediate arcs being found separately may be added to obtain the whole area.

§ 65. In the two preceding sections we investigated two formulæ for approximating to the areas of curvilinear figures. We now proceed to consider the subject more generally.

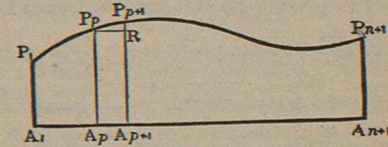

Fig. 37.

Let the equation to the curve $P_1P_pP_{n+1}$ (fig. 37) agree with the equation $y = A + Bx + Cx^2 + \ldots + Kx^n$ for $n+1$ points between P_1 and P_{n+1}, then the area of the curvilinear figure bounded by the straight lines A_1P_1, A_1A_{n+1}, and $A_{n+1}P_{n+1}$ and the curve P_1P_{n+1} will agree very nearly with the curvilinear figure bounded by the same straight lines and the curve whose equation is $y = A + Bx + Cx^2 + \ldots + Kx^n$, and the greater the number of common points the closer will be the agreement.

Let A_1A_{n+1} be divided into n equal parts, each equal to h, then $A_1A_{n+1} = nh$. Now

when $x=0$, $y = y_1 = A$;

when $x=h$, $y = y_2 = A + Bh + Ch^2 + \ldots + Kh^n$;

when $x=2h$, $y = y_3 = A + B(2h) + C(2h)^2 + \ldots + K(2h)^n$;

. . . .

when $x=ph$, $y = y_{p+1} = A + B(ph) + C(ph)^2 + \ldots + K(ph)^n$.

. . . .

when $x=nh$, $y = y_{n+1} = A + B(nh) + C(nh)^2 + \ldots + K(nh)^n$.

From these $n+1$ equations the $n+1$ quantities A, B, ... K can be determined as functions of $y_1, y_2, \ldots y_{n+1}$, and h.

Next let A_1A_{n+1} be divided into m equal parts each equal to h. Thus $mh = nh$ and hence $h = \frac{nh}{m}$

Now the area of the rectangle $A_pA_{p+1}P_pR = A_pA_{p+1} \times A_pP_p$

But $A_pP_p = y_p = A + B(ph) + C(ph)^2 + \ldots + K(ph)^n$

$$= A + B\frac{p.nh}{m} + C\left(\frac{p.nh}{m}\right)^2 + \ldots + K\left(\frac{p.nh}{m}\right)^n,$$

since $$h = \frac{nh}{m};$$

and $$A_pA_{p+1} = h = \frac{nh}{m};$$

therefore area of A_pR

$$nh\left\{\frac{A}{m} + Bnh\frac{p}{m^2} + Cn^2h^2\frac{p^2}{m^3} + \ldots + Kn^nh^n\frac{p^n}{m^{n+1}}\right\}.$$

Hence the area of the whole figure

$$= L_{m=\infty}\sum_{p=1}^{p=m} nh\left\{\frac{A}{m} + Bnh\frac{p}{m^2} + Cn^2h^2\frac{p^2}{m^3} + \ldots + Kn^nh^n\frac{p^n}{m^{n+1}}\right\}$$

$$= L_{m=\infty} nh\left\{A\frac{S_0}{m} + Bnh\frac{S_1}{m^2} + Cn^2h^2\frac{S_2}{m^3} + \ldots + Kn^nh^n\frac{S_n}{m^{n+1}}\right\},$$

where $$S_n = 1^n + 2^n + 3^n + \ldots + m^n.$$

Now if we take the limit of each of the terms

$$\frac{S_0}{m}, \frac{S_1}{m^2}, \frac{S_2}{m^3}, \ldots \frac{S_n}{m^{n+1}},$$

we obtain area of curvilinear figure

$$= nh\left\{A + \frac{B}{2}nh + \frac{C}{3}n^2h^2 + \ldots + \frac{K}{n+1}n^nh^n\right\}$$

From this general result we can deduce "Simpson's Rule" and also another rule called "Weddle's Rule."

Thus let $n=2$; that is, assume that the curve under consideration has three points in common with the curve whose equation is $y = A + Bx + Cx^2$. *i.e.*, with a parabola, then

$$y_1 = A,$$
$$y_2 = A + Bh + Ch^2,$$
$$y_3 = A + 2Bh + 4Ch^2.$$

Now the area is approximately

$$= 2h\{A + \tfrac{1}{2}B2h + \tfrac{1}{3}C2^2h^2\}$$
$$= \tfrac{1}{3}h\{6A + 6Bh + 8Ch^2\}$$
$$= \tfrac{1}{3}h\{y_1 + 4y_2 + y_3\}, \text{ Simpson's Rule.}$$

If we now put $n=6$, we have area of curvilinear figure

$$= 6h\{A + \tfrac{1}{2}B6h + \tfrac{1}{3}C6^2h^2 + \tfrac{1}{4}D6^3h^3 + \tfrac{1}{5}E6^4h^4 + \tfrac{1}{6}F6^5h^5 + \tfrac{1}{7}G6^6h^6\}.$$

Now $$y_1 = A,$$
$$y_2 = A + Bh + Ch^2 + \ldots + Gh^6,$$
. . . . . .
$$y_7 = A + B(6h) + C(6h)^2 + \ldots G(6h)^6.$$

From this system of equations we can determine A, B, C, . . . G, and substituting the values so obtained in the above expression we obtain the following remarkable formula for the approximate area:

$$\text{area} = \tfrac{3}{10}h\{(y_1 + y_3 + y_5 + y_7) + y_4 + 5(y_2 + y_4 + y_6)\}$$

This formula, called Weddle's Rule, gives the closest approximation to the curvilinear area that can be obtained by any simple rule.

We are now in a position to find the approximate area of any irregular plane figure. For the given figure can be divided into plane rectilinear and curvilinear figures, the areas of which can be separately determined by the rules already given. For example, APQRS (fig. 38)

$$= ABC + APD + BRC - DQB - ASC.$$

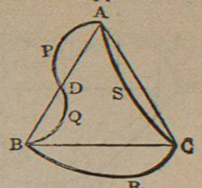

Fig. 38.

PART II. SOLIDS.

Section I. Solids contained by Planes.

A. *Prisms, Pyramids, and Prismatoids*

§ 66. *Volume of a Right Prism.*—First let the prism be a rectangular parallelepiped (fig. 39), and let the side AB contain a units of length, BC b units of length, and CD c units of length. If we divide AB into a equal parts, BC into b equal parts, and CD into c equal parts, and if, through the points of division we draw planes parallel to the sides of the parallelepiped, these planes will divide it into a series of parallelepipeds, whose edges are each equal to the unit of length. Each horizontal layer contains ab of these cubes, and since there are c layers the whole number of cubes will be abc. But each of these is the unit of volume, and therefore

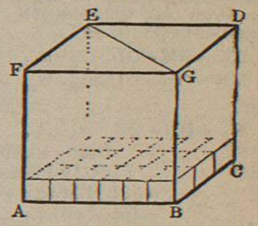

Fig. 39.

volume of ABCD $= abc = ab \times c =$ area of base ABC $\times$ altitude c.

In the above demonstration we have assumed the edges to be commensurable, but from § 2 it follows that the proof will hold also when the edges are incommensurable. If the parallelepiped be cut by a plane BGE it will be divided into two equal triangular right prisms, and hence

volume of right triangular prism $= \tfrac{1}{2}ab \times c =$ area of its base $\times$ altitude.

Since every prism can be divided into triangular prisms as in fig. 40. we have at once

volume of right prism A′ABCDE $=$ A′ABC $+$ A′ACD $+$ A′ADE
$=$ ABC $\times$ BB′ $+$ ACD $\times$ CC′ $+$ ADE $\times$ DD′
$=$ (ABC $+$ ACD $+$ ADE) $\times$ altitude
(since BB′ $=$ CC′ $=$ DD′ $=$ altitude)
$=$ area of base ABCDE $\times$ altitude.

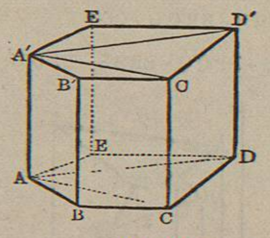

Fig. 40.

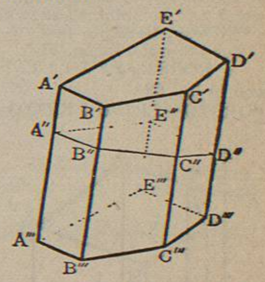

Fig. 41.

§ 67. *Volume of an Oblique Prism.*—Draw the right section A″B″C″D″E″ (fig. 41), and let A′ denote its area and A the area of the base A′B′C′D′E′. Let l denote the length of the prism, h its altitude, and α the angle between the planes A′B′C′D′E′ and A″B″C″D″E″.

Conceive the part above the right section placed at the other extremity of the prism. Then we have a right prism, whose volume $= A' \times l$ (§ 66); but $A' = A\cos\alpha$, since A′ is the projection of A (§ 51),

and $$l = \frac{h}{\cos\alpha}: \text{ hence}$$

$$\text{volume} = A' \times l = A\cos\alpha \times \frac{h}{\cos\alpha} = A \times h;$$

or the volume of any prism is equal to the area of its base multiplied by its altitude.

§ 68. *Surface of a Prism.*—Since the lines A″B″, B″C″, &c. (fig. 41), which make up the perimeter of the right section are all in one plane perpendicular to the parallel edges A′A‴, B′B‴, &c., they are perpendicular to these edges and are therefore the altitudes of the parallelograms A′B′B‴A‴, B′C′C‴B‴, &c., respectively. The lateral surface of the prism is equal to the sum of these parallelograms, and therefore

$$= A'A''' \times A''B'' + B'B''' \times B''C'' + \ldots$$
$$= A'A'''(A''B'' + B''C'' + \ldots),$$

since $$A'A''' = B'B''' = \&c.;$$

or the lateral surface of any prism is equal to the perimeter of its right section multiplied by the length of the prism.

If the prism be *right*, that is, if the faces be perpendicular to the base, then its lateral surface is equal to the perimeter of its base multiplied by its length.

The whole surface of any prism is obtained by adding to the lateral surface the areas of its bases.

§ 69. If the prism be *regular*, that is, if the bases be regular polygons, then

area of base $= a^2 \times \frac{n}{4}\cot\frac{180^\circ}{n}$ (§ 18, γ), where n is the number of sides each of length a, and therefore

$$\text{volume} = a^2 \times \frac{n}{4}\cot\frac{180^\circ}{n} \times h,$$

where h is the altitude of the prism.

Again, if the prism be right and regular, then

$$\text{its lateral surface} = nah + 2a^2 \times \frac{n}{4}\cot\frac{180^\circ}{n}.$$

§ 70. *Volume of a Pyramid.*—Let VABC (fig. 42) be for simplicity a triangular pyramid. Divide VA into n equal portions, and through the points of section draw planes parallel to the base ABC, and through BC and through the intersections of these planes with VBC draw planes parallel to VA. Let h denote the altitude of the pyramid, then the distance of the base of the r^{th} prism from the vertex V

$$= r \times \frac{h}{n},$$

and, if A denote the area of ABC, we have

$$\frac{\text{base of } r^{th} \text{ prism}}{A} = \frac{r^2h^2}{n^2} \times \frac{1}{h^2} = \frac{r^2}{n^2},$$

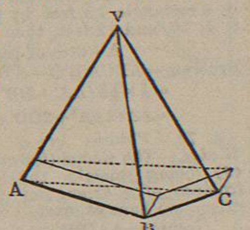

Fig. 42.

since, by a well-known theorem in solid geometry, the areas of sections of a pyramid made by planes parallel to the base are proportional to the squares of their altitudes.

Thus we have

$$\text{base of } r^{th} \text{ prism} = \frac{r^2}{n^2}A, \text{ and therefore}$$

$$\text{its volume} = \frac{r^2}{n^2}A \times \frac{h}{n} \text{ (§ 67)}$$
$$= \frac{hA}{n^3} \times r^2.$$

Therefore volume of whole pyramid

$$= hA\,L_{n=\infty}\frac{1^2 + 2^2 + \ldots + r^2 + \ldots + n^2}{n^3}$$
$$= hA\,L_{n=\infty}\frac{n(n+1)(2n+1)}{6n^3} = hA \times \tfrac{1}{3};$$

or the volume of any pyramid is equal to one-third of the area of its base multiplied by its height.

From this we see that pyramids on equal bases are to one another as their altitudes.

If the pyramid be *regular*, that is, if its base be a regular polygon the perpendicular through whose centre passes through the vertex,

$$\text{its volume} = \tfrac{1}{3} \times a^2 \times \frac{n}{4}\cot\frac{180^\circ}{n} \times h.$$

§ 71. *Surface of a Regular Pyramid.*—The lateral surface of the regular pyramid VABCDEF (fig. 43) is equal to the sum of the areas of the n congruent triangles which make up the lateral surface of the pyramid

Now area of triangle VAB $= \tfrac{1}{2}$AB $\times$ VG; hence whole lateral surface $= \tfrac{1}{2}n$AB·VG $= \tfrac{1}{2}nal$, where l is the slant height and a the length of the side of the base.

Again, if VO $= h =$ altitude of pyramid, we have

$$l = VG = \sqrt{VO^2 + OG^2} = \sqrt{h^2 + \frac{a^2}{4}\cot^2\frac{180^\circ}{n}},$$

therefore whole surface $=$ base $+$ lateral surface

$$= a^2 \times \frac{n}{4}\cot\frac{180^\circ}{n} + \tfrac{1}{2}na\sqrt{h^2 + \frac{a^2}{4}\cot^2\frac{180^\circ}{n}}$$

$$= \left(\frac{a}{2}\cot\frac{180^\circ}{n} + \sqrt{h^2 + \frac{a^2}{4}\cot^2\frac{180^\circ}{n}}\right)$$

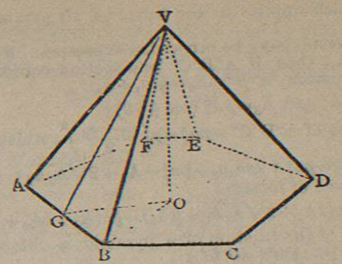

Fig. 43.

§ 72. *The Prismatoid.*—If we have a polyhedron whose bases are two polygons in parallel planes, the number of sides in each being the same or different, and if we so join the vertices of these bases that each line in order forms a triangle with the preceding line and one side of either base, the figure so formed is called a "prismatoid," and holds in stereometry a position similar to that of the trapezium in planimetry. To make the investigation of the volume of the prismatoid as simple as possible, we take the case where the lower base is a polygon of four and the upper one of three sides.

Let ABCDEFG (fig. 44) be the prismatoid, of which ABC or A_1 is the upper and DEFG or A_3 the lower base, and let HLM be the section equidistant from the bases. Take any point P in this section and join it to the corners of the prismatoid. We thus divide the polyhedron into two pyramids PABC and PDEFG, and a series of polyhedra of which CPDE may be taken as a specimen.

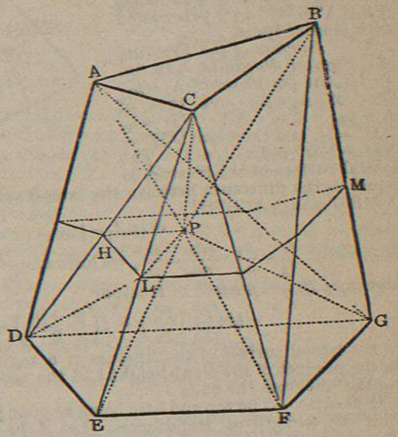

Fig. 44.

Let h be the altitude of the prismatoid, then $\tfrac{1}{2}h$ is the altitude of each of the pyramids PABC, PDEFG, and hence

$$\text{volume of PABC} = \tfrac{1}{6}hA_1, \text{ and}$$
$$\text{volume of PDEFG} = \tfrac{1}{6}hA_3.$$

Again join PH, PL, and LD, then
volume of CPDE $=$ 2 volume of CPDL,
since DE $=$ 2HL,
and volume of CPDL $=$ 2 volume of CPHL,
hence volume of CPDE $=$ 4 volume of CPHL.

Now volume of CPHL $= \tfrac{1}{6}h \times$ area of HLP, and therefore volume of CPDE $= \tfrac{4}{6}h \times$ area of HLP.

Similarly the volume of every such polyhedron is $\tfrac{4}{6}h \times$ the area of its own portion of the middle section. Hence if A_2 denote the area of the middle section we have

$$\text{volume of prismatoid} = \tfrac{1}{6}hA_1 + \tfrac{1}{6}hA_3 + \tfrac{4}{6}hA_2$$
$$= \tfrac{1}{6}h(A_1 + 4A_2 + A_3),$$

§ 73. *Volume of the Frustum of a Pyramid.*—Let A′A‴B′B‴C′C‴ (fig. 45) be a frustum of the pyramid VA′B′C′, and let A_1 and A_3 denote the areas of the ends A′B′C′, A‴B‴C‴ respectively, let VP $= x =$ altitude of pyramid VA‴B‴C‴ and let PQ $= h =$ altitude of frustum.

Now $$\left(\frac{x}{x+h}\right)^2 = \frac{A_3}{A_1}, \text{ whence } x = \frac{h\sqrt{A_3}}{\sqrt{A_1} - \sqrt{A_3}}.$$

Again frustum $= VA'B'C' - VA'''B'''C'''$

$$= \tfrac{1}{3}\left\{ A_1(x+h) - A_3 x \right\}$$
$$= \tfrac{1}{3}\left\{ A_1\left(\frac{h\sqrt{A_1}}{\sqrt{A_1}-\sqrt{A_3}}\right) - A_3\left(\frac{h\sqrt{A_3}}{\sqrt{A_1}-\sqrt{A_3}}\right)\right\}$$
$$= \tfrac{1}{3}h(A_1 + \sqrt{A_1A_3} + A_3)\ ;$$

a formula which applies to the frusta of all pyramids regular and irregular.

The above result may be otherwise expressed. For, let $A'B' = a_1$ and $A'''B''' = a_3$, then, if $A''B''C''$ be a section equidistant from the ends of the frustum, $A''B'' = a_2 = \tfrac{1}{2}(a_1 + a_3)$.

Now $A_1 = pa_1^2$ and $A_3 = pa_3^2$ (see § 70);

hence $A_2 =$ area of $A''B''C'' = pa_2^2 = p\left(\frac{a_1+a_3}{2}\right)^2$, which gives

$$4A_2 = pa_1^2 + 2pa_1a_3 + pa_3^2 = A_1 + 2\sqrt{A_1A_3} + A_3\ ;$$

therefore volume of frustum

$$= \tfrac{1}{6}h(2A_1 + 2\sqrt{A_1A_3} + 2A_3) = \tfrac{1}{6}h(A_1 + 4A_2 + A_3)\ ;$$

or the volume of the frustum of a pyramid is obtained by adding the areas of the ends to four times the area of the middle section, and multiplying the sum by one-sixth of the altitude.

The above result can be obtained at once from § 72, since $A'B'C'A'''B'''C'''$ is a prismatoid with similar bases.

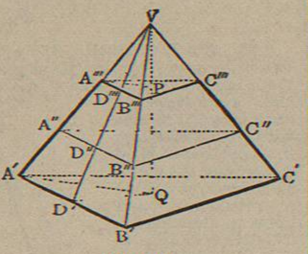

Fig. 45.

§ 74. *Surface of the Frustum of a Regular Pyramid.*—In fig. 45 let the perimeter of $A_1 = p_1$, that of $A_2 = p_2$, and that of $A_3 = p_3$, and let $VD' = l_1$, $VD''' = l_3$, and therefore $D'D''' = VD' - VD''' = l_1 - l_3 = l$. The lateral surface of the frustum is equal to the difference between the lateral surfaces of the pyramids $VA'B'C'$ and $VA'''B'''C'''$,

$$= \tfrac{1}{2}p_1l_1 - \tfrac{1}{2}p_3l_3.$$

But, since $\frac{l_1}{l_3} = \frac{a_1}{a_3} = \frac{na_1}{na_3} = \frac{p_1}{p_3}$, we have $l_1 = \frac{p_1 l}{p_1 - p_3}$ and $l_3 = \frac{p_3 l}{p_1 - p_3}$,

therefore lateral surface of frustum

$$= \tfrac{1}{2}l\left(\frac{p_1^2 - p_3^2}{p_1 - p_3}\right) = l\left(\frac{p_1+p_3}{2}\right) = lp_2\ ;$$

or the lateral surface of the frustum of a regular pyramid is equal to the product of the slant height and the perimeter of the section equidistant from the ends.

Otherwise.—The top and base being regular polygons, the inclined faces are congruent trapeziums. Let l be the height of each trapezium, and let there be n of them, then

$$\text{area of each face} = \frac{l}{2}\left(\frac{p_1}{n} + \frac{p_3}{n}\right),$$

and therefore the area of lateral surface $= \frac{l}{2}(p_1 + p_3) = lp_2$.

§ 75. If h the altitude of the frustum be given, we deduce the slant height and then proceed as before. Thus let $VP = h_3$, $VQ = h_1$, and using the same notation as in §§ 72, 73, and 74 we have

$$\frac{h_1}{h_1 - h_3} = \frac{a_1}{a_1 - a_3},\ \text{which gives}\ h_1 = \frac{a_1(h_1 - h_3)}{a_1 - a_3} = \frac{a_1 h}{a_1 - a_3}.$$

Again $\quad l_1^2 = h_1^2 + \tfrac{1}{4}a_1^2 \cot^2\frac{180^\circ}{n}$, and $l = \left(\frac{a_1 - a_3}{a_1}\right)l_1\ ;$

whence l is known since l_1 is known in terms of h.

When the pyramid is irregular the lateral planes are non-congruent trapeziums, the areas of which can be found separately by § 12, and hence the whole surface.

§ 76. *Volume of the Frustum of a Triangular Prism.*—Let A denote the area of ABC (fig. 46), and let h_1, h_2, h_3 be the altitudes of A′, B′, C′ respectively with reference to the plane ABC. Divide the frustum into three pyramids B′A′AC, B′ABC, and B′A′CC′ by the planes B′AC and B′A′C. These three pyramids are respectively equal to BA′AC, B′ABC, and ABCC′;

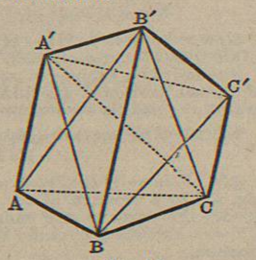

Fig. 46.

hence volume of frustum $= \tfrac{1}{3}h_1A + \tfrac{1}{3}h_2A + \tfrac{1}{3}h_3A$
$$= \tfrac{1}{3}A(h_1 + h_2 + h_3).$$

§ 77. If the prism be right or oblique, the volume of a frustum is equal to one-third of the area of its right section multiplied by the sum of the parallel edges. For divide the frustum AA′B′C′ (fig. 47) into two frusta by a plane A″B″C″ of area A at right angles to the edges, then

$$AA'B'C' = AA''B''C'' + A''A'B'C'$$
$$= \tfrac{1}{3}A(AA'' + BB'' + CC'') + \tfrac{1}{3}A(A'A'' + B'B'' + C'C'')$$
$$= \tfrac{1}{3}A(AA'' + BB'' + CC'' + A'A'' + B'B'' + C'C'')$$
$$= \tfrac{1}{3}A(AA' + BB' + CC').$$

Again, since every prism can be divided into triangular prisms, we can find by repeated applications of the above proposition the volume of the frustum of any prism whatever. For example, if

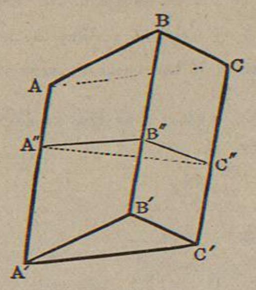

Fig. 47.

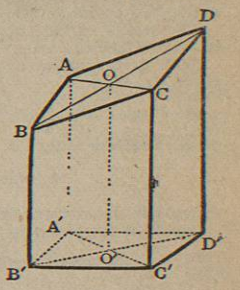

Fig. 48.

the base of the frustum of a right prism AA′B′C′D′ (fig. 48) be a rectangle 12 feet by 6 feet, and the parallel edges in order 6 4, 10, and 12 feet, then

$$A = \text{area of base} = 12 \times 6 = 72\ \text{square feet}.$$

Frustum $= ABCA'B'C' + ADCA'D'C'$
$$= \tfrac{1}{3} \times \tfrac{1}{2}A(AA' + BB' + CC') + \tfrac{1}{3} \times \tfrac{1}{2}A(AA' + CC' + DD')$$
$$= \tfrac{1}{6}A(2AA' + 2CC' + BB' + DD') = 576.$$

§ 78. *Volume of a Wedge.*—The wedge (fig. 49) being merely the frustum of a triangular prism, we have at once

$$\text{volume} = \tfrac{1}{3}A(FE + AD + BC),$$

where A is the area of its right section; otherwise, the wedge may be considered a prismatoid whose upper base is a straight line, and hence its volume $= \tfrac{1}{6}h(4A_2 + A_3)$, since $A_1 = 0$.

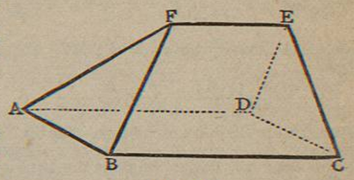

Fig. 49.

B. *Regular Polyhedra.*

§ 79. The regular polyhedra are five in number, namely, the *tetrahedron*, *cube*, *octahedron*, *dodecahedron*, and *icosahedron*, whose solid angles are formed respectively by three equilateral triangles, three squares, four equilateral triangles, three pentagons, and five equilateral triangles.

Since a regular polyhedron admits of having a sphere inscribed within it and described about it, it can easily be shown that the volume of the polyhedron

$$= a^3 \times \frac{nl}{24} \times \frac{\cos\frac{\pi}{m}\cot^2\frac{\pi}{n}}{\left\{-\cos\left(\frac{\pi}{m}+\frac{\pi}{n}\right)\cos\left(\frac{\pi}{m}-\frac{\pi}{n}\right)\right\}^{\frac{1}{2}}}\ ;$$

and from § 18, γ, it follows that the surface of the polyhedron

$$= a^2 \times \frac{ln}{4}\cot\frac{\pi}{n},$$

where $\quad l =$ the number of faces,
$m =$ the number of faces in each solid angle,
$n =$ the number of edges in each face,
and $\quad a =$ the length of each side.

The following table contains the surfaces and volumes for the five regular polyhedra whose edge is 1.

Polyhedron.	Surface.	Volume.
Tetrahedron	1·7320508	0·1178511
Cube	6·0000000	1·0000000
Octahedron	3·4641016	0·4714043
Dodecahedron	20·6457788	7·6631189
Icosahedron	8·6602540	2·1816950

The surface and volume of a regular polyhedron whose edge is a is obtained by multiplying the surface and volume of a similar polyhedron whose edge is 1 by a^2 and a^3 respectively.

Section II. Solids contained by Surfaces which are not all Planes.

A. *The Cylinder.*

§ 80. *Volume of a Cylinder* (fig. 50).—Inscribe in the cylinder a polygonal prism of which the number of sides may be increased indefinitely. Then in the limit the base of the prism becomes the base of the cylinder, and the volume of the prism the volume of the cylinder. Now by § 67 we have

volume of prism $\quad$ = area of base × altitude;
hence $\quad$ volume of cylinder = area of base × altitude.

§ 81. *Surface of a Right Cylinder.*—As above, in the limit the base of the prism becomes the base of the cylinder, and the surface of the prism the surface of the cylinder. Now the lateral surface of prism

= perimeter of right section × length
= perimeter of base × length, in the case of a right prism (§ 68);

hence lateral surface of right cylinder = circumference of base × length.

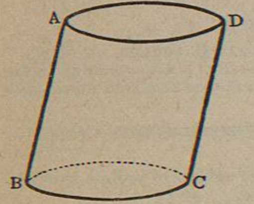

Fig. 50.

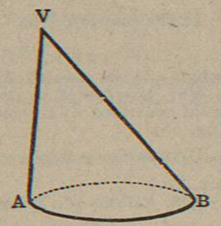

Fig. 51.

B. *The Cone.*

§ 82. *Volume of a Cone* (fig. 51).—Inscribe within the cone a pyramid of which the number of sides may be indefinitely increased, then in the limit the base of the pyramid becomes the base of the cone and the volume of the pyramid the volume of the cone.

By § 70 $\quad$ volume of pyramid $= \tfrac{1}{3}$ base × altitude,
and hence $\quad$ volume of cone $= \tfrac{1}{3}$ base × altitude.

§ 83. *Volume of the Frustum of a Cone.*—From § 73 we find that the volume of the frustum of a pyramid

$$= \tfrac{1}{3}h(A_1 + \sqrt{A_1A_3} + A_3)\ ;$$

hence, since in the limit the frustum of the pyramid becomes the frustum of the cone, we have

$$\text{volume of conical frustum} = \tfrac{1}{3}h(A_1 + \sqrt{A_1A_3} + A_3),$$

where A_1 and A_3 are the areas of the terminating planes of the frustum.

Let the terminating planes be circles of radii r_1 and r_3, then volume of frustum

$$= \tfrac{1}{3}h(\pi r_1^2 + \pi r_1r_3 + \pi r_3^2) = \tfrac{1}{3}\pi h(r_1^2 + r_1r_3 + r_3^2).$$

Again, by the same section we have

volume of frustum of pyramid $= \tfrac{1}{6}h(A_1 + 4A_2 + A_3)$,
therefore $\quad$ volume of conical frustum $= \tfrac{1}{6}\pi h(r_1^2 + 4r_2^2 + r_3^2)$,

where r_2 is the radius of the circular section parallel to the terminating planes and equidistant from them.

§ 84. *Surface of a Right Cone.*—The lateral surface of a regular pyramid is by § 71

$$= \tfrac{1}{2}\ \text{perimeter of base} \times \text{slant height};$$

hence, since in the limit the surface of the pyramid becomes the surface of the cone, the lateral surface of a right cone is equal to half the circumference of its base multiplied by the slant height

Thus the lateral surface of a right cone of slant height l and the radius of whose base is r is equal to

$$\tfrac{1}{2} \times 2\pi r \times l = \pi rl,$$

and $\quad$ whole surface = lateral surface + area of base
$$= \pi rl + \pi r^2$$
$$= \pi r(l + r).$$

Again, if h, the altitude of the cone, be given, we have

$$l = \sqrt{h^2 + r^2},$$

and therefore $\quad$ whole surface $= \pi r(\sqrt{h^2 + r^2} + r)$.

§ 85. *Surface of the Frustum of a Right Cone.*—The lateral surface of the frustum of a regular pyramid is equal to the product of the slant height and the perimeter of its middle section (§ 74); hence in the limit we find that the lateral surface of the frustum of a right cone is equal to the product of its slant height and the circumference of the section equidistant from its parallel faces.

Let r_1 and r_3 denote the radii of the ends of the frustum, and l the length of the slant height, then

$$r_2 = \tfrac{1}{2}(r_1 + r_3) = \text{radius of middle section},$$

and therefore

$$\text{lateral surface} = 2\pi r_2 \times l = 2\pi \times \tfrac{1}{2}(r_1 + r_3) \times l = \pi l(r_1 + r_3),$$

and $\quad$ whole surface $= \pi r_1^2 + \pi l(r_1 + r_3) + \pi r_3^2$.

If h, the altitude of the frustum, be given, we have

$$l = \sqrt{h^2 + (r_1 - r_3)^2}.$$

C. *The Sphere.*

§ 86. *Surface of a Spherical Zone.*—Let AB (fig. 52) be a small arc of the sphere, and let AA′, BB′ be perpendicular to the axis XX′, to find the surface of the zone generated by the arc AB. Join AB, and draw OP perpendicular to AB, BD parallel to XX′, and PP′ parallel to AA′ or BB′. The chord AB generates the frustum of a cone, whose lateral surface $= 2\pi PP' \times AB$.

But, since the triangles ABD and OPP′ are similar,

$$\frac{AB}{BD} = \frac{OP}{PP'}$$

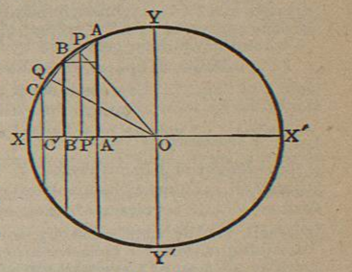

Fig. 52.

therefore $\quad$ area of conical frustum $= 2\pi.OP.BD = 2\pi.OP.A'B'$.
Similarly $\quad$ the area of the frustum generated by $BC = 2\pi.OQ.B'C'$.

But in the limit when the chords AB, BC, &c., are indefinitely diminished, the perpendiculars OP, OQ, &c., become each $= r$, and hence by summing all the areas we get in the limit

$$\text{area of zone} = 2\pi r \times (\text{projection of arc on axis of revolution}).$$

Hence the convex surface of a segment of a sphere is equal to the circumference of a great circle multiplied by the height of the segment or zone.

§ 87. *Surface of a Sphere.*—The whole sphere being a zone whose height is $2r$, we obtain at once

$$\text{surface of sphere} = 2\pi r \times 2r = 4\pi r^2\ ;$$

or the surface of a sphere is equal to four great circles.

The total surface of the cylinder circumscribing the sphere of radius r is $6\pi r^2$, hence the surface of the sphere $= \tfrac{2}{3}$ surface of circumscribing cylinder.

§ 88. *Surface of a Lune, a Spherical Triangle, and a Spherical Polygon.*—It is shown in spherical trigonometry that

(α) the area of a lune included between two great circles of a sphere of radius r, and whose inclination is θ radians, is

$$2\theta r^2\ ;$$

(β) the area of a spherical triangle whose angles are A, B, C is

$$(A + B + C - \pi)r^2\ ;$$

(γ) the area of a spherical polygon of r sides is

$$\{P - (r-2)\pi\}r^2,\ \text{where P is the sum of its angles.}$$

§ 89. *Measurement of Solid Angles.*—A convenient unit for the measurement of plane angles is the "radian." If we assume that each unit of surface of a sphere subtends the same solid angle at the centre, we can deduce a very convenient unit for the measurement of solid angles. This unit, which has received the name "steradian," we define to be the solid angle subtended at the centre of a sphere by a portion of the surface whose area is r^2.

§ 90. *Number of Steradians in an Angle.*—Let A be the angle at the centre of a sphere, and let S be the portion of the surface of the sphere which it intercepts, then

$$\frac{\text{number of steradians in A}}{1} = \frac{S}{r^2}.$$

For example, if A be a plane solid angle, S = a hemisphere $= 2\pi r^2$; hence the number of steradians in a plane solid angle $= \frac{S}{r^2} = \frac{2\pi r^2}{r^2} = 2\pi$, and therefore the number of steradians in the solid angle at a point $= 4\pi$. This solid angle is sometimes called a steregon.

Hence, if we can find the surface subtended by any solid angle, we can always find its magnitude in terms of the unit solid angle.

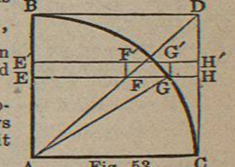

Fig. 53.

§ 91. *Volume of a Sphere.*—Let ABC (fig. 53) be the quadrant of a circle, draw DB and DC tangents to it, then, if AD be joined and the whole figure be conceived as