rotating round AB, ABD, ABC, and ABDC will generate a cone, a hemisphere, and a cylinder respectively.

Now draw two parallel planes EFGH and E′F′G′H′ very near each other and perpendicular to AB, and draw FF′ and GG′ parallel to AB. then, by § 80,

$$\text{volume generated by EHH'E'} = \pi \mathrm{EH}^2 \times \mathrm{EE'},$$
$$\text{,, \quad ,, \quad EGG'E'} = \pi \mathrm{EG}^2 \times \mathrm{EE'},$$
$$\text{,, \quad ,, \quad EFF'E'} = \pi \mathrm{EF}^2 \times \mathrm{EE'}.$$

Thus volume generated by EFF′E′ + volume generated by EGG′E′

$$= \pi(\mathrm{EF}^2 + \mathrm{EG}^2) \times \mathrm{EE'} = \pi(\mathrm{EA}^2 + \mathrm{EG}^2) \times \mathrm{EE'}$$
$$= \pi(\mathrm{AG}^2) \times \mathrm{EE'} = \pi \mathrm{EH}^2 \times \mathrm{EE'}$$
$$= \text{volume generated by EHH'E'}.$$

Therefore in the limit, when the number of slices is indefinitely increased, and their thickness indefinitely diminished, we have volume of cone generated by AF + volume of spherical zone generated by CG = volume of cylinder generated by CH.

Let r = radius of sphere, h = AE = height of zone ACGE, then

$$\text{volume of cone} = \tfrac{1}{3}\pi h^2 \times h = \tfrac{1}{3}\pi h^3, \text{ and}$$
$$\text{volume of cylinder} = \pi r^2 \times h,$$

therefore
$$\text{volume of spherical zone} = \pi r^2 h - \tfrac{1}{3}\pi h^3 = \tfrac{1}{3}\pi h(3r^2 - h^2).$$

The height of a hemisphere is r,

therefore $\quad$ volume of hemisphere $= \tfrac{1}{3}\pi r(3r^2 - r^2) = \tfrac{2}{3}\pi r^3$,

and $\quad$ volume of whole sphere $= \tfrac{4}{3}\pi r^3$,

a result readily obtainable by the infinitesimal calculus, or by inscribing within the sphere a series of triangular pyramids whose vertices all meet at the centre of the sphere, and the angles of whose bases all rest on the surface. In the limit the altitude of each pyramid becomes the radius of the sphere, and the sum of the bases of the pyramids the surface of the sphere; hence

$$\text{volume} = \tfrac{1}{3}\mathrm{S} \times r = \tfrac{1}{3} \times 4\pi r^2 \times r = \tfrac{4}{3}\pi r^3.$$

The volume of the circumscribing cylinder $= \pi r^2 \times 2r = 2\pi r^3$,

therefore volume of sphere = $\tfrac{2}{3}$ volume of circumscribing cylinder.

§ 92. Let S denote the surface of a sphere and V its volume, then from §§ 87 and 91 we have

$$(\alpha)\ r = \frac{\sqrt{\mathrm{S}}}{2\sqrt{\pi}} = \sqrt[3]{\frac{3}{4\pi}} \times \sqrt[3]{\mathrm{V}}:$$
$$(\beta)\ \mathrm{S} = \sqrt[3]{\pi(6\mathrm{V})^2};$$
$$(\gamma)\ \mathrm{V} = \frac{1}{6\sqrt{\pi}}\sqrt{(\mathrm{S})^3};$$

formulæ which give the radius in terms of the surface or volume, the surface in terms of the volume, and the volume in terms of the surface.

§ 93. *Volume of a Spherical Shell.*—Let r and r_1 denote the radii of the two spheres, then

$$\text{volume of shell} = \mathrm{V} = \tfrac{4}{3}\pi r_1^3 - \tfrac{4}{3}\pi r^3$$
$$= \tfrac{4}{3}\pi(r_1^3 - r^3)$$
$$= \tfrac{4}{3}\pi(r_1 - r)(r_1^2 + r_1 r + r^2).$$

Now let $r_1 - r = h$, then

$$\mathrm{V} = \tfrac{4}{3}\pi r_1^2 h\left(1 + \frac{r}{r_1} + \frac{r^2}{r_1^2}\right).$$

If h be small compared with r_1, then r/r_1 is very nearly equal to 1, and we have approximately

$$\mathrm{V} = \tfrac{4}{3}\pi r_1^2 h(1 + 1 + 1) = 4\pi r_1^2 h.$$

Again, if h be nearly equal to r_1, r is very small, and r/r_1 is also very small, so that we have approximately

$$\mathrm{V} = \tfrac{4}{3}\pi r_1^2 h.$$

§ 94. *Volume of a Spherical Segment.*—Let CRC′ (fig. 54) be a section of a spherical segment whose altitude RQ is p, then, if OQ = h, volume of segment CRC′ = volume of hemisphere − volume of zone AA′C′C

$$= \tfrac{2}{3}\pi r^3 - \tfrac{1}{3}\pi h\{3r^2 - h^2\}, \text{ § 91.}$$
$$= \tfrac{2}{3}\pi r^3 - \tfrac{1}{3}\pi(r - p)\{3r^2 - (r - p)^2\}$$
$$= \tfrac{1}{3}\pi p^2(3r - p).$$

If we put $p = 2r$, we obtain as before volume of sphere $= \tfrac{4}{3}\pi r^3$.

Again if CQ $= a_1$ we have

$$\mathrm{CQ}^2 = a_1^2 = \mathrm{RQ}.\mathrm{R'Q} = p(2r - p),$$

whence $r = \dfrac{a_1^2 + p^2}{2p}$,

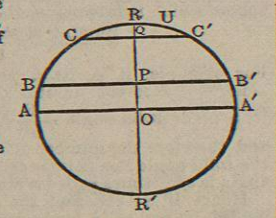

Fig. 54.

therefore $\quad$ volume of segment $= \tfrac{1}{6}\pi p(3a_1^2 + p^2)$.

§ 95. *Volume of a Spherical Frustum.*—When one of the terminating planes passes through the centre we have already found that the volume

$$= \tfrac{1}{3}\pi h(r^2 - h^2),$$

where h is its altitude.

Now suppose that neither of the terminating planes passes through the centre; for example, to find the volume of the frustum BB′C′C.

Let RQ $= p$ and RP $= q$, then

$$\mathrm{BB'C'C} = \text{segment RBB'} - \text{segment RCC'}$$
$$= \tfrac{1}{6}\pi q(3a_2^2 + q^2) - \tfrac{1}{6}\pi p(3a_1^2 + p^2),$$

where a_1 and a_2 are the radii of the ends CC′ and BB′.

Let $q - p = h$ = height of frustum, and, since from the geometry of the figure,

$$\frac{a_1^2 + p^2}{p} = \frac{a_2^2 + q^2}{q} = 2r,$$

we have $\quad$ volume $= \tfrac{1}{6}\pi h\{3(a_1^2 + a_2^2) + h^2\}$,

a result which may also be obtained by considering BB′C′C as the difference of the two zones AA′C′C and AA′B′B.

D. *Spheroid.*

§ 96. *Surface of a Prolate Spheroid.*—The prolate spheroid is the solid generated by the revolution of an ellipse about its major axis. If S be the surface generated by an arc of the curve, then

$$\mathrm{S} = 2\pi\int y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}dx, \text{ taken between proper limits.}$$

In the case before us

$$\mathrm{S} = 2\pi b^2 + 2\pi\frac{ab}{e}\sin^{-1}e,$$

where e is the eccentricity (INFINITESIMAL CALCULUS, art. 179).

§ 97. *Surface of an Oblate Spheroid.*—The oblate spheroid is the solid generated by the revolution of an ellipse about its minor axis (fig. 55).

Here surface $= 2\pi a^2 + \pi\dfrac{b^2}{e}\log_e\dfrac{1+e}{1-e}$ (INFINITESIMAL CALCULUS, art. 179).

§ 98. *Volume of a Spheroid.*—We have volume of prolate spheroid

$$= \pi\int_{-a}^{+a} b^2\left(1 - \frac{x^2}{a^2}\right)dx = 2\pi b^2\int_0^a\left(1 - \frac{x^2}{a^2}\right)dx = \tfrac{4}{3}\pi ab^2.$$

Similarly volume of oblate spheroid $= \tfrac{4}{3}\pi a^2 b$.

Thus,

$$(\alpha)\quad \frac{\text{volume of prolate spheroid}}{\text{volume of oblate spheroid}} = \frac{\tfrac{4}{3}\pi ab^2}{\tfrac{4}{3}\pi a^2 b} = \frac{b}{a};$$
$$(\beta)\quad \frac{\text{sphere described on major axis}}{\text{prolate spheroid}} = \frac{\tfrac{4}{3}\pi a^3}{\tfrac{4}{3}\pi ab^2} = \frac{a^2}{b^2};$$
$$(\gamma)\quad \frac{\text{sphere described on minor axis}}{\text{oblate spheroid}} = \frac{\tfrac{4}{3}\pi b^3}{\tfrac{4}{3}\pi a^2 b} = \frac{b^2}{a^2}.$$

§ 99. *Volume of a Segment of a Spheroid.*

(α) *The prolate spheroid.*—This segment is generated by the revolution of AMP (fig. 23, p. 20) about AM, and hence

$$\text{its volume} = \pi\int_0^h y^2dx = \pi\frac{b^2}{a^2}\int_0^h(2ax - x^2)dx = \frac{\pi}{3} \times \frac{b^2h^2}{a^2}(3a - h),$$

where A is the origin and AM $= h$.

(β) *The oblate spheroid.*—The segment in this case is generated by the revolution of BMP (fig. 55) about BC, and hence

$$\text{its volume} = \pi\int_0^h y^2dx = \pi\frac{a^2}{b^2}\int_0^h(2bx - x^2)dx = \frac{\pi}{3} \times \frac{a^2h^2}{b^2}(3b - h),$$

where B is the origin and BM $= h$.

§ 100. *Volume of the Frustum of a Spheroid when one of the Terminating Planes passes through the Centre.*

(α) *The prolate spheroid.*—The frustum in this case is generated by the revolution of BCMP about CM (fig. 23).

Now volume generated by BCMP
= volume generated by BCA
− volume generated by PMA

$$= \tfrac{2}{3}\pi ab^2 - \frac{\pi}{3} \times \frac{b^2h^2}{a^2}(3a - h)$$
$$= \frac{\pi}{3} \times \frac{b^2k}{a^2}(3a^2 - k^2), \text{ where}$$

k = CM = height of frustum $= a - h$.

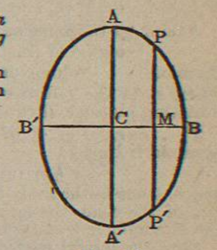

Fig. 55.

(β) *The oblate spheroid.*—We can show in a similar manner that the volume generated in this case

$$= \frac{\pi}{3} \times \frac{a^2k}{b^2}(3b^2 - k^2).$$

The above formulæ may be put into another form. Thus, in the case of the prolate spheroid, since the point P lies on the ellipse $b^2x^2 + a^2y^2 = a^2b^2$, we have

$$b^2k^2 + a^2b_1^2 = a^2b^2, \text{ where } b_1 = \mathrm{PM}, \text{ which gives}$$
$$k^2 = a^2\frac{(b^2 - b_1^2)}{b^2};$$

whence, by substitution, the volume of prolate frustum

$$= \tfrac{1}{3}\pi k(2b^2 + b_1^2).$$

Similarly we can show that the volume of the oblate frustum

$$= \tfrac{1}{3}\pi k(2a^2 + a_1^2),$$
$$\text{where } a_1 = \mathrm{PM}.$$

These formulæ play an important part in the gauging of casks.

E. *Paraboloid.*

§ 101. *Surface of a Paraboloid.*—Let the equation to the parabola be $y^2 = 4ax$, and let the coordinates of P (fig. 21, p. 19) be x_1, y_1, then the surface of the paraboloid generated by the revolution of AM about AP

$$= 2\pi\int_0^{x_1} y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}dx = 4\pi\sqrt{a}\int_0^{x_1}\sqrt{x}\sqrt{\frac{x+a}{x}}dx$$
$$= \tfrac{8}{3}\pi\sqrt{a}\left\{(x_1 + a)^{\frac{3}{2}} - a^{\frac{3}{2}}\right\}.$$

§ 102. *Volume of a Paraboloid.*—With the same notation we have

$$\text{volume} = \pi\int_0^{x_1} y^2dx = 4\pi a\int_0^{x_1} xdx = \tfrac{1}{2}\pi \times 4ax_1 \times x_1 = \tfrac{1}{2}\pi y_1^2 \times x_1;$$

or the volume of a paraboloid generated by the revolution of a part of a parabola between the vertex and any point is equal to half the volume of the circumscribing cylinder.

§ 103. If the coordinates of Q be x_2, y_2, then the volume of the frustum PP′Q′Q

$$= \tfrac{1}{2}\pi\{y_2^2x_2 - y_1^2x_1\} = 2\pi a(x_2^2 - x_1^2) = \tfrac{1}{2}\pi(y_2^2 + y_1^2)h,$$

where h = MN; hence the volume of the frustum of a paraboloid is equal to half the sum of the areas of its ends multiplied by its height.

F. *Ellipsoid.*

§ 104. *Volume of an Ellipsoid.*—The equation to the ellipsoid being

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

the equation to the elliptic section at the distance z from the origin is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 - \frac{z^2}{c^2}.$$

Now if we draw an indefinite number of parallel planes perpendicular to the axis of z, each slice will be an infinitely thin cylindrical plate, and accordingly the whole volume of the ellipsoid

$$= \int \mathrm{A}dz, \text{ where A is the area of the elliptic section.}$$

But $\mathrm{A} = \pi ab\left(1 - \dfrac{z^2}{c^2}\right)$, § 51,

therefore $\quad$ volume $= \pi ab\displaystyle\int_{-c}^{c}\left(1 - \frac{z^2}{c^2}\right)dz = \tfrac{4}{3}\pi abc$.

The sphere being an ellipsoid whose axes are all equal, we obtain as before

$$\text{volume of sphere} = \tfrac{4}{3}\pi a^3 = \tfrac{4}{3}\pi r^3.$$

G. *Hyperboloid.*

§ 105. *Volume of an Hyperboloid.*—The hyperboloid is generated by the revolution of the hyperbolic segment ANP about AN (fig. 24, p. 20). If the coordinates of P be x_1, y_1, then

$$\text{volume of hyperboloid} = \pi\int_a^{x_1} y^2dx = \pi\frac{b^2}{a^2}\int_a^{x_1}(x^2 - a^2)dx$$
$$= \pi\frac{b^2}{a^2}\left\{\frac{x_1^3}{3} - a^2x_1 + \frac{2a^3}{3}\right\} = \frac{\pi b^2h^2}{3a^2}(3a + h),$$
$$\text{where } h = \mathrm{AN} = x_1 - a.$$

Again, since x_1, y_1 is on the curve, we have

$$a^2y_1^2 - b^2(a + h)^2 = -a^2b^2, \text{ which gives } \frac{b^2}{a^2} = \frac{y_1^2}{2ah + h^2}; \text{ whence}$$
$$\text{volume of hyperboloid} = \frac{\pi hy_1^2}{3} \times \frac{3a + h}{2a + h}.$$

H. *Solids to which the "Prismoidal Formula" applies.*

§ 106. It was shown in § 72 that the volume of any polyhedron bounded by two parallel planes and by plane rectilinear figures

$$= \tfrac{1}{6}h(\mathrm{A}_1 + 4\mathrm{A}_2 + \mathrm{A}_3),$$

where A_1, A_3, and A_2 denote respectively the areas of the two ends and of the middle section.

We now proceed to show that the same formula determines the volumes of all solids bounded by two parallel planes, provided the area of any section parallel to these planes can be expressed as a rational integral algebraic function of the third degree in x, where x is the distance of the section from either plane.

Let $\phi(x) = \mathrm{A} + \mathrm{B}x + \mathrm{C}x^2 + \mathrm{D}x^3 + \ldots + \mathrm{K}x^n$ denote the area of the section in question.

Now the solid between the sections $\phi(0)$ and $\phi(4)$ is equal to the solid between the sections $\phi(0)$ and $\phi(2)$ plus the solid between the sections $\phi(2)$ and $\phi(4)$. Hence if the prismoidal formula is to hold in this case, we have

$$\tfrac{1}{6}h\{\phi(0) + 4\phi(2) + \phi(4)\}$$
$$= \tfrac{1}{12}h\{\phi(0) + 4\phi(1) + \phi(2)\} + \tfrac{1}{12}h\{\phi(2) + 4\phi(3) + \phi(4)\},$$

where h is the distance between the sections $\phi(0)$ and $\phi(4)$.

Hence we have

$$\phi(0) - 4\phi(1) + 6\phi(2) - 4\phi(3) + \phi(4) = 0.$$

Now
$$\begin{aligned}
\phi(0) &= \mathrm{A} \\
-4\phi(1) &= -4\mathrm{A} - 4\mathrm{B} - 4\mathrm{C} - 4\mathrm{D} - 4\mathrm{E} - \ldots - 4\mathrm{K} \\
+6\phi(2) &= 6\mathrm{A} + 12\mathrm{B} + 24\mathrm{C} + 48\mathrm{D} + 96\mathrm{E} + \ldots + 6\cdot2^n\mathrm{K} \\
-4\phi(3) &= -4\mathrm{A} - 12\mathrm{B} - 36\mathrm{C} - 108\mathrm{D} - 324\mathrm{E} - \ldots - 4\cdot3^n\mathrm{K} \\
+\phi(4) &= \mathrm{A} + 4\mathrm{B} + 16\mathrm{C} + 64\mathrm{D} + 256\mathrm{E} + \ldots + 4^n\mathrm{K}.
\end{aligned}$$

Therefore $\quad 0 = 0 + 0 + 0 + 0 + 24\mathrm{E} + \mathrm{PF} + \ldots + \Gamma\mathrm{K}$.

Hence E = F = ... K = 0, and therefore $\phi(x)$ must be a function of the third degree in order that the prismoidal formula may apply.

§ 107. If we take $\phi(x) = \mathrm{A} + \mathrm{B}x + \mathrm{C}x^2 + \mathrm{D}x^3$, there will be as many possible varieties as there are combinations of four things, one, two, three, and four together, *i.e.*, $2^4 - 1 = 15$ varieties. Corresponding to each of these there will be *at least one solid* the area of a section of which at a distance x from one of the parallel planes is $\phi(x) = \mathrm{A} + \mathrm{B}x + \mathrm{C}x^2 + \mathrm{D}x^3$, and *at least one solid of revolution* generated by the curve whose equation is of the form

$$\pi y^2 = \phi(x) = \mathrm{A} + \mathrm{B}x + \mathrm{C}x^2 + \mathrm{D}x^3.$$

As space prevents us discussing all the cases that may arise, we content ourselves by giving three examples as illustrations.

(α) *Volume of an ellipsoid.*—Here $\phi(x) = \mathrm{B}x + \mathrm{C}x^2$.

Let $2a$, $2b$, and $2c$ be the axes of which $2a$ is the greatest, then $h = 2a$, $\mathrm{A}_1 = 0$, $\mathrm{A}_3 = 0$, and $\mathrm{A}_2 = \pi bc$;

therefore $\quad$ volume $= \tfrac{1}{6}h(\mathrm{A}_1 + 4\mathrm{A}_2 + \mathrm{A}_3) = \tfrac{2}{6}a(4\pi bc) = \tfrac{4}{3}\pi abc$,

which agrees with the result in § 104.

(β) *Volume of a sphere.*—Here $\pi y^2 = \phi(x) = \mathrm{B}x + \mathrm{C}x^2$.

Let r be the radius of the sphere, then $h = 2r$, $\mathrm{A}_1 = 0$, $\mathrm{A}_3 = 0$, and $\mathrm{A}_2 = \pi r^2$, hence, as before (§ 91),

$$\text{volume of sphere} = \tfrac{1}{6}h(\mathrm{A}_1 + 4\mathrm{A}_2 + \mathrm{A}_3) = \frac{2r}{6}(4\pi r^2) = \tfrac{4}{3}\pi r^3.$$

(γ) *Volume of a right circular cone.*—Here $\pi y^2 = \phi(x) = \mathrm{C}x^2$.

Let r = radius of base and h the altitude, then $\mathrm{A}_1 = 0$, $\mathrm{A}_3 = \pi r^2$, and $\mathrm{A}_2 = \pi(\tfrac{1}{2}r)^2$; hence

$$\text{volume of cone} = \tfrac{1}{6}h\{\mathrm{A}_1 + 4\mathrm{A}_2 + \mathrm{A}_3\} = \tfrac{1}{6}h\{\pi r^2 + \pi r^2\} = \tfrac{1}{3}h\pi r^2$$

In a similar manner we can determine the volumes of a cylinder, a prolate spheroid, an oblate spheroid, &c.

§ 108. In general, if in any solid we have

$$\phi(x) = \mathrm{A} + \mathrm{B}x + \mathrm{C}x^2 + \mathrm{D}x^3,$$

where A, B, C, and D are known constants, then, if h be the length of the solid,

$$\mathrm{A}_1 = \phi(0) = \mathrm{A},$$
$$\mathrm{A}_2 = \phi(\tfrac{1}{2}h) = \mathrm{A} + \mathrm{B}(\tfrac{1}{2}h) + \mathrm{C}(\tfrac{1}{2}h)^2 + \mathrm{D}(\tfrac{1}{2}h)^3,$$
$$\mathrm{A}_3 = \phi(h) = \mathrm{A} + \mathrm{B}h + \mathrm{C}h^2 + \mathrm{D}h^3,$$

and therefore

$$\text{volume of solid} = \tfrac{1}{6}h(\mathrm{A}_1 + 4\mathrm{A}_2 + \mathrm{A}_3)$$
$$= \mathrm{A}h + \tfrac{1}{2}\mathrm{B}h^2 + \tfrac{1}{3}\mathrm{C}h^3 + \tfrac{1}{4}\mathrm{D}h^4.$$

I. *Solids of Revolution in General.*

§ 109. *Volume of any Solid of Revolution.*—Let $\mathrm{P}_1\mathrm{P}_2 \ldots \mathrm{P}_n$ (fig. 34) be the generating curve, and $\mathrm{A}_1 \ldots \mathrm{A}_n$ the axis of revolution. Divide the curve into portions in the points P_2, P_3, &c., and draw the chords and tangents of the small arcs $\mathrm{P}_1\mathrm{P}_2$, $\mathrm{P}_2\mathrm{P}_3$, &c., then it is evident that the solid generated by the curve is greater than the sum of the conical frusta traced out by the chords and less than the sum of the conical frusta traced out by the tangents. Hence, by increasing the number of chords, namely, by increasing the points of division of the curve, we can make the difference between these sums as small as we please, and therefore by this method we can approximate as closely as we please to the volume of the solid generated.

Assuming that the points P_1, P_2, P_3 are so near each other that the solid generated differs little from the frustum of a cone, and using the same notation as in § 63, we have volume generated by

$$\mathrm{P}_1\mathrm{P}_2\mathrm{P}_3 = \tfrac{1}{6}\pi\mathrm{A}_1\mathrm{A}_3(s_1^2 + 4s_2^2 + s_3^2) = \tfrac{1}{6}\pi 2\mathrm{A}_1\mathrm{A}_2(s_1^2 + 4s_2^2 + s_3^2)$$
$$= \tfrac{1}{3}\pi a(s_1^2 + 4s_2^2 + s_3^2);$$

similarly the volume generated by

$$\mathrm{P}_3\mathrm{P}_4\mathrm{P}_5 = \tfrac{1}{3}\pi a(s_3^2 + 4s_4^2 + s_5^2);$$

whence the volume generated by the whole curve $\mathrm{P}_1\mathrm{P}_2 \ldots \mathrm{P}_n$

$$= \tfrac{1}{3}\pi a\{s_1^2 + s_n^2 + 2(s_3^2 + s_5^2 + \ldots + s_{n-2}^2) + 4(s_2^2 + s_4^2 + \ldots + s_{n-1}^2)\};$$

or (since $\pi s_1^2=\frac{c_1^2}{4\pi}$, $\pi s_2^2=\frac{c_2^2}{4\pi}$, &c.)

$$=\frac{1}{\pi}\frac{a}{12}\{c_1^2+c_n^2+2(c_3^2+c_5^2+\ldots+c_{n-2}^2)+4(c_2^2+c_4^2+\ldots+c_{n-1}^2)\},$$

a formula more convenient in practice, as it is sometimes more easy to measure equidistant circumferences than equidistant radii.

J. *Theorems of Pappus.*

§ 110. The following general propositions concerning surfaces and solids of revolution, usually called Guldin's theorems, are worth the reader's attention.

If any plane curve revolve about any external axis situated in its plane, then

(α) the surface of the solid which is thereby generated is equal to the product of the perimeter of the revolving curve and the length of the path described by the centre of gravity of that perimeter;

(β) the volume of the solid is equal to the product of the area of the revolving curve and the length of the path described by the centre of gravity of the revolving area.

We content ourselves with an example or two of the application of these theorems, referring to the article INFINITESIMAL CALCULUS for the proofs.

Example 1.—To find the surface and volume of a circular ring.—Let a be the distance of the centre of the generating curve, in this case a circle, from the axis of rotation, and r the radius of the circle, then

$$\text{perimeter of generating curve}=2\pi r,$$
$$\text{area of generating curve}=\pi r^2, \text{ and}$$

path described by the centre of gravity either of the perimeter or area $=2\pi a$; hence

$$\text{surface of ring}=2\pi r\times 2\pi a=4\pi^2 ra, \text{ and}$$
$$\text{volume of ring}=\pi r^2\times 2\pi a=2\pi^2 r^2 a.$$

Example 2.—To find the volume swept out by an ellipse whose axes are $2a$ and $2b$, revolving about an axis in its own plane whose distance from the centre of the ellipse is c.

Here area of generating curve $=\pi ab$,
and path described by centre of gravity of area $=2\pi c$; hence

$$\text{volume generated}=\pi ab\times 2\pi c=2\pi^2 abc.$$

Example 3.—A circle of r inches radius, with an inscribed regular hexagon, revolves about an axis a inches distant from its centre, and parallel to a side of the hexagon; to find the difference in area of the generated surfaces and volumes.

$$\text{Here perimeter of circle}=2\pi r,$$
$$\text{and perimeter of hexagon}=12\times r\sin 30^\circ \text{ (§ 17)}$$
$$=6r;$$
$$\text{also area of circle}=\pi r^2,$$
$$\text{and area of hexagon}=3r^2\sin 60^\circ \text{ (§ 18, }\beta)$$
$$=\tfrac{3}{2}\sqrt{3}r^2;$$

hence difference of surfaces generated

$$=4\pi^2 ra-12\pi ar=4\pi ar(\pi-3);$$

and difference of volumes generated

$$=2\pi^2 r^2 a-3\pi r^2\sqrt{3}a$$
$$=\pi r^2 a(2\pi-3\sqrt{3}).$$

PART III. GAUGING.

§ 111. By gauging is meant the art of measuring the volume of a cask, or any portion of it. The subject is one of great interest and practical importance, but space will only permit us to discuss it very briefly. If the cask whose capacity we wish to determine be a solid of revolution, then its volume can at once be computed, either exactly or approximately, by the methods already described.

It is usual to divide casks into the following four classes according to the nature of the revolving curve:—

(α) the middle frustum of a spheroid;
(β) the middle frustum of a parabolic spindle;
(γ) two equal frusta of a paraboloid, united at their bases;
(δ) two equal frusta of a cone, united at their bases.

Casks of the second, third, and fourth variety are rarely met with in practice, and we shall accordingly confine our attention to the first kind, which is considered the true or model form of cask.

Let ABCD (fig. 56) be a section of the cask, and assume it to be the middle frustum of a prolate spheroid, then

$$\text{its volume}=\tfrac{1}{3}\pi(2b^2+b_1^2)k,$$

where $b=\text{OY}$, $b_1=\text{AX}$, and $k=\text{XX}'$ (§ 99).

YY′ is called the bung diameter, and AB or CD the head diameter.

An imperial gallon contains 277·274 cubic inches, and therefore the number of gallons in the above cask

$$=\frac{\pi(2b^2+b_1^2)k}{3\times 277\cdot 274}=\frac{\pi}{831\cdot 822}(2b^2+b_1^2)k$$
$$=\left(\frac{2d^2+d_1^2}{1059\cdot 1}\right)k, \text{ where } d=2b,\ d_1=2b_1;$$

whence we have the rule:—to the square of the head diameter add twice the square of the bung diameter, multiply the sum by the length and divide the result by 1059·1, and the answer is the content in imperial gallons.

Casks as ordinarily met with are not true spheroidal frusta, but it is better to consider them as such, calculate their capacity on this assumption, and then make allowance for the departure from the spheroidal form. The determination of the proper allowance to be made in each case is a matter depending on the skill and experience of the gauger, and proficiency in the art can only be attained by considerable practice.

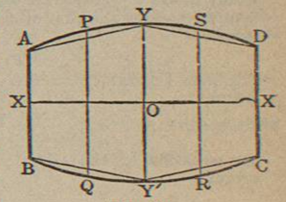

Fig. 56.

§ 112. If the cask be very little curved, we obtain an approximation to its capacity by considering it as made up of two equal frusta of a cone, united at their bases. Hence from § 83 we have

$$\text{volume of cask}=\tfrac{1}{3}\pi h(r_1^2+r_1r_3+r_3^2) \text{ nearly.}$$

Here we neglect the small volumes generated by APY, YSD, BQY′, and Y′RC; and therefore the volume is too small.

If we put $r_1r_3=r_1^2$ we obtain

$$\text{volume}=\tfrac{1}{3}\pi h(2r_1^2+r_3^2),$$

which is a little too large, and therefore the true volume lies between these two limits, and a very close approximation to it is said to be given by the formula

$$\tfrac{1}{3}\pi h\{2r_1^2+r_3^2-\tfrac{1}{3}(r_1^2-r_3^2)\}.$$

§ 113. *Ullage of a Cask.*—The quantity of liquor contained in a cask partially filled and the capacity of the portion which is empty are termed respectively the wet and dry ullage.

(α) *Ullage of a standing cask.*—By means of the method applied in § 105, the following rule is deduced:—

Add the square of the diameter at the surface, the square of the diameter at the nearest end, and the square of double the diameter half-way between; multiply the sum by the length between the surface and the nearest end, and by ·000472.

The product will be the wet or dry ullage according as the lesser portion of the cask is filled or empty.

(β) *Ullage of a lying cask.*—The ullage in this case is found approximately on the assumption that it is proportional to the segment of the bung circle cut off by the surface of the liquor. The rule adopted in practice is

$$\text{ullage}=\tfrac{5}{4}\text{ content}\times\text{segmental area.}$$

(W. T.*)

MENTAL DISEASES. See INSANITY.

MENTON (Ital., *Mentone*), a cantonal capital in the department of Alpes-Maritimes, France, situated 15 miles north-east of Nice, on the shores of the Mediterranean. The town, which has a population of about 8000, rises like an amphitheatre on a promontory by which its semicircular bay (5 miles wide at its entrance, and bounded on the W. by Cape Martin and on the E. by the cliffs of La Murtola) is divided. It is composed of two very distinct portions: below, along the sea-shore, is the town of hotels and of foreigners, which alone is accessible to wheeled vehicles; above is that of the native Mentonese, with steep, narrow, and dark streets, spread over and clinging to the mountain, around the strong castle which was once its protection against the attacks of pirates. Facing the south-east, and sheltered on the north and west by high mountains, the Bay of Menton enjoys a delicious climate, and is on this account much frequented by invalids requiring a mild and equable temperature. The mean for the year is 61° Fahr., exceeding that of Rome or of

Pisa, and equalling that of Naples. Frost occurs on the average only once in ten years; in one particular year the thermometer did not fall below 46° Fahr. In summer the heat is never very great, the temperature rarely exceeding 86° Fahr. Winter and summer are the most agreeable seasons; in autumn the rain storms are accompanied by sudden changes of temperature, and in spring the sea breezes are apt to be violent. Besides the charms of its climate, Menton offers those of an almost tropical vegetation. Lemon-trees, olive-trees, and pines, rising above each other in successive stages, adorn the surrounding slopes. The district produces forty millions of lemons yearly, and this is the principal source of its natural wealth. The olive-trees are remarkable for the great size they have attained in the course of the centuries during which they have continued to bear. Of their wood a multitude of fancy objects are made for sale to strangers.

The origin of Menton is unknown. During the Middle Ages it was successively occupied by the Saracens, the Genoese, and the princes of Anjou. In the middle of the 14th century it was purchased as a single domain by the Grimaldis, lords of Monaco. During the times of the republic and the first empire it belonged to France; but in 1815 it again became the property of the princes of Monaco, who subjected it to such exactions that in 1848 its inhabitants, weary of finding their reasonable demands put off with empty promises, proclaimed their town free and independent, under the protection of Sardinia. Menton, with the neighbouring commune of Roquebrune, was united to France in 1860, at the same time as Nice and Savoy.

MENTZ. See MAINZ.

MENZEL, WOLFGANG (1798–1873), poet, critic, and historian, was born June 21, 1798, at Waldenburg in Silesia, studied at Breslau, Jena, and Bonn, and after living for some time in Aarau and Heidelberg finally settled in Stuttgart, where, from 1830 to 1838, he had a seat in the Würtemberg "landtag." His first work, a clever and original volume of poems, entitled *Streckverse* (Heidelberg, 1823), was followed in 1824–25 by a popular *Geschichte der Deutschen* in three volumes, and in 1829 and 1830 by *Rübezahl* and *Narcissus*, the ballads upon which his reputation as a poet chiefly rests. In 1851 he published the romance of *Furore*, a lively picture of the period of the Thirty Years' War; his other very numerous writings include *Geschichte Europa's*, 1789–1815 (1853), and histories of the German war of 1866 and of the Franco-German war of 1870–71. From 1825 to 1848 Menzel edited a "Literaturblatt" in connexion with the *Morgenblatt*; in the latter year he transferred his allegiance from the Liberal to the Conservative party, and in 1852 his "Literaturblatt" was again revived in that interest. In 1866 his political sympathies again changed, and all his energies were employed to oppose the "particularism" of the Prussian "junkers" and the antiunionism of South Germany. He died on April 23, 1873. His large private library of 18,000 volumes was afterwards acquired for the university of Strasburg.

MEPHISTOPHELES, the name of one of the personifications of the principle of evil. In old popular books and puppet-plays the word appears in various forms,—as Mephistopheles, Mephistophiles, Mephistophilis, and Mephostophilis. In the *Tragical History of Doctor Faustus*, Marlowe writes "Mephistophilis"; in the *Merry Wives of Windsor* we find "Mephistophilus." The etymology of the word is uncertain. According to one theory, it may be taken to represent μηφωστοφιλης; in which case the meaning would be "one who loves not light." Another theory is that the word is a combination of the Latin "mephitis" and the Greek φίλος, signifying "one who loves noxious exhalations." Probably it is of Hebrew origin,—from מֵפֵר, a destroyer, and כַּפֵּל, taken to mean a liar. This view is supported by the fact that almost all the names of devils in the magic-books of the 16th century spring from the Hebrew. In the old Faust legends the character of Mephistopheles is simply that of a powerful and wicked being who fulfils Faust's commands in order to obtain possession of his soul. Marlowe attributes to him a certain dignity and sadness, and there can be little doubt that the Mephistophilis of the *Tragical History* suggested some important traits of Milton's Satan. The name has been made famous chiefly by Goethe, whose conception of the character varied at different periods of his career. In the fragment of *Faust* published in 1790, but written many years before, Mephistopheles has a clearly marked individuality; he is cynical and materialistic, but has a man's delight in activity and adventure, and his magical feats alone remind us that he is preternatural. In revising and extending this fragment, which forms the chief portion of the first part of *Faust*, Goethe treated Mephistopheles as the representative of the evil tendencies of nature, especially of the tendency to denial for its own sake, rather than as a living person. This character Mephistopheles maintains in the second part, where, indeed, the name often stands for a pure abstraction.

See Julius Mosen, *Faust*; Düntzer, *Erläuterungen zu Goethe's Werken: Faust*; Vischer, *Goethe's Faust*.

MEQUINEZ (the Spanish form of the Arabic *Miknása*), a town of Morocco, the ordinary residence of the emperor, is situated in a fine hilly country about 70 miles from the west coast and 35 west-south-west of Fez on the road to Sallee, in 34° N. lat. and 5° 35′ W. long. The town-wall, with its four-cornered towers, is kept in good condition; and a lower wall of wider circuit protects the luxuriant gardens with which the outskirts are embellished. In the general regularity of its streets, and in the fairly substantial character of its houses, Mequinez ranks higher than any other town in Morocco; but it possesses few buildings of any note, except the palace, and the mosque of Mulei Ismael, which serves as the royal burying-place. At one time the palace (founded in 1634) was an imposing structure, but the finest part has been allowed to go to ruin. In 1721 Windhus described it as "about 4 miles in circumference, the whole building exceeding massy, and the walls in every part very thick; the outward one about a mile long and 25 feet thick." The best part consisted of oblongs enclosing large open courts or gardens. Mortar or concrete was the principal material used for the walls, but the pillars were in many cases marble blocks of great beauty and costliness (*A Journey to Mequinez*, London, 1725). Most of the inhabitants of Mequinez are connected more or less directly with the court. Their number has been very variously estimated by different travellers. Gråberg de Hemsö gives 56,000 in 1834, Rohlfs in 1861 from 40,000 to 50,000, and Conring in 1880 about 30,000. The town was formerly called Tákarart. Edrisi refers the present name to a Berber chief Meknás.

MERAN, a favourite health resort, and the capital of a district in South Tyrol, Austria, is picturesquely situated at the foot of the vine-clad Küchelberg, on the right bank of the Passer, about half a mile above its junction with the Adige, and 45 miles to the south of Innsbruck. Meran proper consists mainly of one long narrow street, called the Laubengasse, flanked by covered arcades. In a wider sense, the name is often used to include the adjacent villages of Untermais, Obermais, and Gratsch. The most noteworthy buildings are the Gothic church of St Nicholas, with its lofty tower, dating from the 14th and 15th centuries; the Spitalkirche, built in the 15th century, and restored in 1880; and the quaint old Fürstenhaus, or residence of the counts of Tyrol. The town contains a gymnasium, a nunnery and school for girls, an institution for sick priests, and several other charitable establishments.