to break the beam (M_1) cannot be calculated from the ultimate tensile or compressive strength of the material by using the formula $M_1 = f_t I/y_1$, or $M_1 = f_c I/y_2$. When experiments are made on the ultimate strength of bars to resist bending, it is not unusual to apply a formula of this form to calculate an imaginary stress f, which receives the name of the modulus of transverse rupture. Let the section be such that $y_1 = y_2$. Then the modulus of transverse rupture is defined as $f = M_1 y_1/I$. This mode of stating the results of experiment on transverse strength is unsatisfactory, inasmuch as the modulus of rupture thus determined will vary with different forms of section. Thus a plastic material for which f_c and f_t are equal, if tested in the form of an I beam in which the flanges form practically the whole area of section, will have a modulus of rupture sensibly equal to f_t or f_c. On the other hand, if the material be tested in the form of a rectangular bar, the modulus of rupture may approach a value one and a half times as great. For in the latter case the distribution of stress may approach an ultimate condition in which half the section is in uniform tension f_t, and the other half in uniform compression of the same intensity. The moment of stress is then $\frac{1}{4} f_t bh^2$, b being the breadth and h the depth of the section; but by definition of the modulus of rupture f, $M = \frac{1}{6} fbh^2$. In tables of the modulus of transverse rupture the values are generally to be understood as referring to bars of rectangular section. Values of this modulus for some of the principal materials of engineering are given in the article BRIDGES, vol. iv. p. 292.

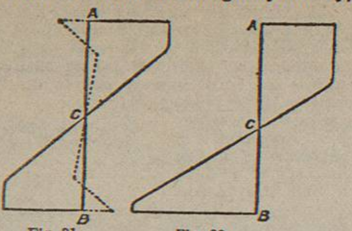

Fig. 31. Fig. 32.

59. The strain produced by bending stress in a bar or beam is, as regards any imaginary filament taken along the length of the piece, sensibly the same as if that filament were directly pulled or compressed by itself. The resulting deformation of the piece consists, in the first place and chiefly, of curvature in the direction of the length, due to the longitudinal extension and compression of the filaments, and, in the second place, of transverse flexure, due to the lateral compression and extension which go along with their longitudinal extension and compression (see ELASTICITY, § 57). Let l_1, fig. 33, be a short portion of the length of a beam strained by a bending moment M (within the limits of elasticity). The beam, which we assume to be originally straight, bends in the direction of its length to a curve of radius R, such that $R/l = y_1/\delta l$, δl being the change of l by extension or compression, at a distance y_1 from the neutral axis. But $\delta l = lp_1/E$ by § 10, and $p_1 = My_1/I$. Hence $R = EI/M$. The transverse flexure is not, in general, of practical importance. The centre of curvature for it is on the opposite side from the centre for longitudinal flexure, and the radius is $R\sigma$, where σ is the ratio of longitudinal extension to lateral contraction under simple pull.

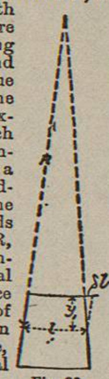

Fig. 33.

60. Bending combined with shearing is the mode of stress to which beams are ordinarily subject, the loads, or externally applied forces, being applied at right angles to the direction of the length. Let AB, fig. 34, be any cross-section of a beam in equilibrium. The portion V of the beam, which lies on one side of AB, is in equilibrium under the joint action of the external forces F_1, F_2, F_3, &c., and the forces which the other portion U exerts on V in consequence of the state of stress at AB. The forces F_1, F_2, F_3, &c. may be referred to AB by introducing couples whose moments are F_1x_1, F_2x_2, F_3x_3, &c. Hence the stress at AB must equilibrate, first, a couple whose moment is ΣFx, and, second, a force whose value is ΣF, which tends to shear V from U. In these summations regard must of course be had to the sign of each force; in the diagram the sign of F_3 is opposite to the sign of F_1 and F_2. Thus the stress at AB may be regarded as that due to a bending moment M equal to the sum of the moments about the section of the externally applied forces on one side of the section (ΣFx), and a shearing force equal to the sum of the forces about one side of the section (ΣF). It is a matter of convenience only whether the forces on V or on U be taken in reckoning the bending moment and the shearing force. The bending moment causes a uniformly varying normal stress on AB of the kind already discussed in § 56; the shearing force causes a shearing stress in the plane of the section, the distribution of which will be investigated later. This shearing stress in the plane of the section is (by § 6) accompanied by an equal intensity of shearing stress in horizontal planes parallel to the length of the beam.

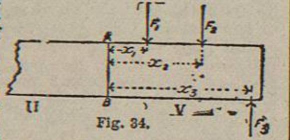

Fig. 34.

61. The stress due to the bending moment, consisting of longitudinal push in filaments above the neutral axis and longitudinal pull in filaments below the neutral axis, is the thing chiefly to be considered in practical problems relating to the strength of beams. The general formula $p_1 = My_1/I$ becomes, for a beam of rectangular section of breadth b and depth h, $p_1 = 6M/bh^2 = 6M/Sh$, S being the area of section. For a beam of circular section it becomes $p_1 = 32M/\pi h^3 = 8M/Sh$. The material of a beam is disposed to the greatest advantage as regards resistance to bending when the form is that of a pair of flanges or booms at top and bottom, held apart by a thin but stiff web or by cross-bracing, as in I beams and braced trusses. In such cases sensibly the whole bending moment is taken by the flanges; the intensity of stress over the section of each flange is very nearly uniform, and the areas of section of the tension and compression flanges (S_1 and S_2 respectively) should be proportioned to the value of the ultimate strengths f_t and f_c, so that $S_1 f_t = S_2 f_c$. Thus for cast-iron beams Hodgkinson has recommended that the tension flange should have six times the sectional area of the compression flange. The intensity of longitudinal stress on the two flanges of an I beam is approximately M/S_1h and M/S_2h, h being the depth from centre to centre of the flanges.

62. In the examination of loaded beams it is convenient to represent graphically the bending moment and the shearing force at various sections by setting up ordinates to represent the values of these quantities. Curves of bending moment and shearing force for a number of important practical cases of beams supported at the ends will be found in the article BRIDGES, with expressions for the maximum bending moment and maximum shearing force under various distributions of load. The subject may be briefly illustrated *Diagrams of bending moment and shearing force.*

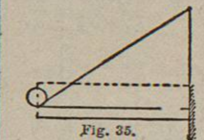
Fig. 35.

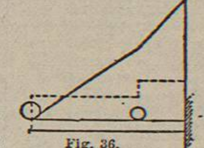
Fig. 36.

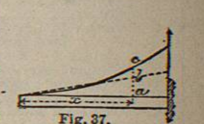

Fig. 37.

here by taking the case of a cantilever or projecting bracket—(1) loaded at the end only (fig. 35); (2) loaded at the end and at another point (fig. 36); (3) loaded over the whole length with a uniform load per foot run. Curves of bending moment are given in full lines and curves of shearing force in dotted lines in the diagrams.

The area enclosed by the curve of shearing force, up to any ordinate, such as ab (fig. 37), is equal to the bending moment at the same section, represented by the ordinate ac. For let x be increased to $x + \delta x$, the bending moment changes to $\Sigma F(x + \delta x)$, or $\delta M = \delta x \Sigma F$. Hence the shearing force at any section is equal to the rate of change of the bending moment there per unit of the length, and the bending moment is the integral of the shearing force with respect to the length. In the case of a continuous distribution of load, it should be observed that, when x is increased to $x + \delta x$, the moment changes by an additional amount which depends on $(\delta x)^2$ and may therefore be neglected.

63. To examine the distribution of shearing stress over any vertical section of a beam, we may consider two closely adjacent sections AB and DE (fig. 38), on which the bending moments are M and $M + \delta M$ respectively. The resultant horizontal force due to the bending stresses on a piece ADHG enclosed between the adjacent sections, and bounded by the horizontal plane GH at a distance y_0 from the neutral axis, is shown by the shaded figure. This must be equilibrated by the horizontal shearing stress on GH, which is the only other horizontal force acting on the piece. At any height y the intensity of resultant horizontal stress due to the difference of the bending moments is $y\delta M/I$, and the whole horizontal force on GH is $\frac{\delta M}{I}\int_{y_0}^{y_1} yzdy$, z being the breadth. If q be the intensity of horizontal shearing stress on the section GH, whose breadth is z_0, we have *Distribution of shearing stress.*

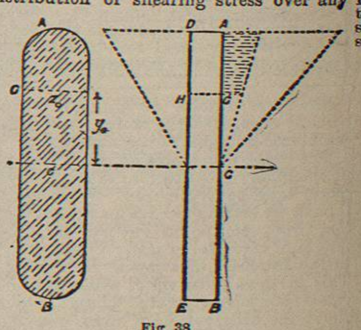

Fig. 38

$$qz_0\delta x = \frac{\delta M}{I}\int_{y_0}^{y_1} yzdy\,.$$

But $\delta M/\delta x$ is the whole shearing force Q on the section of the beam. Hence

$$q = \frac{Q}{z_0 I}\int_{y_0}^{y_1} yzdy\,;$$

and this is also the intensity of vertical shearing stress at the distance y_0 from the neutral axis. This expression may conveniently be written $q = QAy/z_0I$, where A is the area of the surface AG and y the distance of its centre of gravity from the neutral axis. The intensity q is a maximum at the neutral axis and diminishes to zero at the top and bottom of the beam. In a beam of rectangular section the value of the shearing stress at the neutral axis is q max. $= \frac{3}{2}Q/bh$. In other words, the maximum intensity of shearing stress on any section is $\frac{3}{2}$ of the mean intensity. Similarly, in a beam of circular section the maximum is $\frac{4}{3}$ of the mean. This result is of some importance in application to the pins of pin-joints, which may be treated as very short beams liable to give way by shearing.

In the case of an I beam with wide flanges and a thin web, the above expression shows that in any vertical section q is nearly constant in the web, and insignificantly small in the flanges. Practically all the shearing stress is borne by the web, and its intensity is very nearly equal to Q divided by the area of section of the web.

64. The foregoing analysis of the stresses in a beam, which resolves them into longitudinal pull and push, due to bending moment, along with shear in longitudinal and transverse planes, is generally sufficient in the treatment of practical cases. If, however, it is desired to find the direction and greatest intensity of stress at any point in a beam, the planes of principal stress passing through the point must be found by an application of the general method given in the article ELASTICITY, chapter iii. In the present case the problem is exceptionally simple, from the fact that the stresses on two planes at right angles are known, and the stress on one of these planes is wholly tangential. Let AC (fig. 39) be an indefinitely small portion of the horizontal section of a beam, on which there is only shearing stress, and let AB be an indefinitely small portion of the vertical section at the same place, on which there is shearing and normal stress. Let q be the intensity of the shearing stress, which is the same on AB and AC, and let p be the intensity of normal stress on AB: it is required to find a third plane BC, such that the stress on it is wholly normal, and to find r, the intensity of that stress. Let θ be the angle (to be determined) which BC makes with AB. Then the equilibrium of the triangular wedge ABC requires that

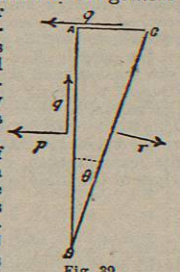

Fig. 39.

$$r\text{BC}\cos\theta = p\,.\,\text{AB} + q\,.\,\text{AC}, \text{ and } r\text{BC}\sin\theta = q\,.\,\text{AB}\,;$$

or

$$(r-p)\cos\theta = q\sin\theta, \text{ and } r\sin\theta = q\cos\theta\,.$$

Hence,

$$q^2 = r(r-p),$$
$$\tan 2\theta = 2q/p,$$
$$r = \tfrac{1}{2}p \pm \sqrt{q^2 + \tfrac{1}{4}p^2}\,.$$

The positive value of r is the greater principal stress, and is of the same sign as p. The negative value is the lesser principal stress, which occurs on a plane at right angles to the former. The equation for θ gives two values corresponding to the two planes of principal stress. The greatest intensity of shearing stress occurs on the pair of planes inclined at 45° to the planes of principal stress, and its value is $\sqrt{q^2 + \frac{1}{4}p^2}$ (by § 5).

65. The above determination of r, the greatest intensity of stress due to the combined effect of simple bending and shearing, is of some practical importance in the case of the web of an I beam. We have seen that the web takes practically the whole shearing force, distributed over it with a nearly uniform intensity q. If there were no normal stress on a vertical section of the web, the shearing stress q would give rise to two equal principal stresses, of pull and push, each equal to q, in directions inclined at 45° to the section. But the web has further to suffer normal stress due to bending, the intensity of which at points near the flanges approximates to the intensity on the flanges themselves. Hence in these regions the greater principal stress is increased, often by a considerable amount, which may easily be calculated from the foregoing formula. What makes this specially important is the fact that one of the principal stresses is a stress of compression, which tends to make the web yield by buckling, and must be guarded against by a suitable stiffening of the web.

The equation for θ allows the lines of principal stress in a beam to be drawn when the form of the beam and the distribution of loads are given. An example has been shown in the article BRIDGES (§ 13, fig. 12), vol. iv. p. 290.

66. The deflexion of beams is due partly to the distortion caused by shearing, but chiefly to the simple bending which occurs at each vertical section. As regards the second, which in most cases is the only important cause of deflexion, we have seen (§ 59) that the radius of curvature R at any section, due to a bending moment M, is EI/M, which may also be written Ey_1/p_1. Thus beams of uniform strength and depth (and, as a particular case, beams of uniform section subjected to a uniform bending moment) bend into a circular arc. In other cases the form of the bent beam, and the resulting slope and deflexion, may be determined by integrating the curvature throughout the span, or by a graphic process (see BRIDGES, § 25), which consists in drawing a curve to represent the beam with its curvature greatly exaggerated, after the radius of curvature has been determined for a sufficient number of sections. In all practical cases the curvature is so small that the arc and chord are of sensibly the same length. Calling i the angle of slope, and u the dip or deflexion from the chord, the equation to the curve into which an originally straight beam bends may be written

$$\frac{du}{dx} = i;\ \frac{d^2u}{dx^2} = \frac{di}{dx} = \frac{EI}{M}\,.$$

Integrating this for a beam of uniform section, of span L, supported at its ends and loaded with a weight W at the centre, we have, for the greatest slope and greatest deflexion, respectively, $i_1 = WL^2/16EI$, $u_1 = WL^3/48EI$. If the load W is uniformly distributed over L, $i_1 = WL^2/24EI$ and $u_1 = 5WL^3/384EI$. For other cases, see BRIDGES, § 24.

The additional slope which shearing stress produces in any originally horizontal layer is q/C, where q is, as before, the intensity of shearing stress and C is the modulus of rigidity. In a round or rectangular bar the additional deflexion due to shearing is scarcely appreciable. In an I beam, with a web only thick enough to resist shear, it may be a somewhat considerable proportion of the whole.

67. *Torsion* occurs in a bar to which equal and opposite couples are applied, the axis of the bar being the axis of the couples, and gives rise to shearing stress in planes perpendicular to the axis. Let AB (fig. 40) be a uniform circular shaft held fast at the end A, and twisted by a couple applied in the plane BB. Assuming the strain to be within the limits of elasticity, a radius CD turns round to CD′, and a line AD drawn at any distance r from the axis, and originally straight, changes into the helix AD′. Let θ be the angle which this helix makes with lines parallel to the axis, or in other words the angle of shear at the distance r from the axis, and let i be the angle of twist DCD′. Taking two sections at a distance dx from one another, we have the arc $\theta dx = rdi$. Hence q, the intensity of shearing stress in a plane of cross-section, varies as r, since $q = C\theta = Cr\frac{di}{dx}$. The resultant moment of the whole shearing stress on each plane of cross-section is equal to the twisting moment M. Thus

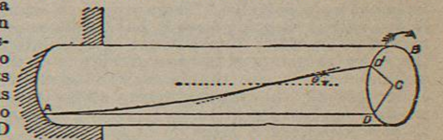

Fig. 40.

$$\int 2\pi r^2 q dr = M\,.$$

Calling r_1 the outside radius (where the shearing stress is greatest) and q_1 its intensity there, we have $q = rq_1/r_1$, and hence, for a solid shaft, $q_1 = 2M/\pi r_1^3$. For a hollow shaft with a central hole of radius r_2 the same reasoning applies: the limits of integration are now r_1 and r_2, and

$$q_1 = \frac{2Mr_1}{\pi(r_1^4 - r_2^4)}$$

The lines of principal stress are obviously helices inclined at 45° to the axis.

If the shaft has any other form of section than a solid or symmetrical hollow circle, an originally straight radial line becomes warped when the shaft is twisted, and the shearing stress is no longer proportional to the distance from the axis. The twisting of shafts of square, triangular, and other sections has been investigated by M. de St Venant (see ELASTICITY, § 66–71, where a comparison of torsional rigidities is given). In a square shaft (side $= h$) the stress is greatest at the middle of each side, and its intensity there[1] is $q_1 = M/0{\cdot}281h^3$.

For round sections the angle of twist per unit of length is $i = \frac{q_1}{Cr_1} = \frac{2M}{\pi Cr_1^4}$ in solid and $\frac{2M}{\pi C(r_1^4 - r_2^4)}$ in hollow shafts.

68. In what has been said above it is assumed that the stress is within the limit of elasticity. When the twisting couple is increased so that this limit is passed, plastic yielding begins in the outermost layer, and a larger proportion of the whole stress falls to be borne by layers nearer the centre. The case is similar to that of a beam bent beyond the elastic limit, described in § 57. If we suppose the process of twisting to be continued, and that after passing the limit of elasticity the material is capable of much distortion without further increase of shearing stress, the distribution of stress on any cross section will finally have an approximately uniform value q', and the moment of torsion will be $\int_{r_2}^{r_1} 2\pi r^2 q' dr = \frac{2}{3}\pi q'(r_1^3 - r_2^3)$. In the case of a solid shaft this gives for M a

[1] Rankine, *Applied Mechanics*, § 321

value greater than it has when the stress in the outermost layer only reaches the intensity q', in the ratio of 4 to 3.[1] It is obvious from this consideration that the ultimate strength of a shaft to resist torsion is no more deducible from a knowledge of the ultimate shearing strength of the material than the ultimate strength of a beam to resist bending is deducible from a knowledge of f_t and f_c. It should be noticed also that as regards ultimate strength a solid shaft has an important advantage over a hollow shaft of the same elastic strength, or a hollow shaft so proportioned that the greatest working intensity of stress is the same as in the solid shaft.

69. *Twisting combined with Longitudinal Stress.*—When a rod is twisted and pulled axially, or when a short block is twisted and compressed axially, the greatest intensity of stress (the greater principal stress) is to be found by compounding the longitudinal and shearing stresses as in § 64. In a circular rod of radius r_1, a total longitudinal force P in the direction of the axis gives a longitudinal normal stress whose intensity $p_1 = P/\pi r_1^2$. A twisting couple M applied to the same rod gives a shearing stress whose greatest intensity $q_1 = 2M/\pi r_1^3$. The two together give rise to a pair of principal stresses of intensities $r = P/2\pi r_1^2 \pm \sqrt{(2M/\pi r_1^3)^2 + (P/2\pi r_1^2)^2}$, their inclinations to the axis being defined by the equation $\tan 2\theta = 2M/r_1 P$, and the term under the square root is the greatest intensity of shearing stress.

70. *Twisting combined with Bending.*—This important practical case is realized in a crank-shaft (fig. 41). Let a force P be applied at the crank-pin A at right angles to the plane of the crank. At any section of the shaft C (between the crank and the bearing) there is a twisting moment $M_1 = P \cdot AB$, and a bending moment $M_2 = P \cdot BC$. There is also a direct shearing force P, but this does not require to be taken into account in calculating the stress at points at the top or bottom of the circumference (where the intensity is greatest), since (by § 63) the direct shearing stress is distributed so that its intensity is zero at these points. The stress there is consequently made up of longitudinal normal stress (due to bending), $p_1 = 4M_2/\pi r_1^3$, and shearing stress (due to torsion), $q_1 = 2M_1/\pi r_1^3$. Combining these, as in § 64, we find for the principal stresses $r = 2(M_2 \pm \sqrt{M_1^2 + M_2^2})/\pi r_1^3$, or $r = 2P(BC \pm AC)/\pi r_1^3$. The greatest shearing stress is $2P \cdot AC/\pi r_1^3$, and the axes of principal stress are inclined so that $\tan 2\theta = M_1/M_2 = AB/BC$. The axis of greater principal stress bisects the angle ACB.

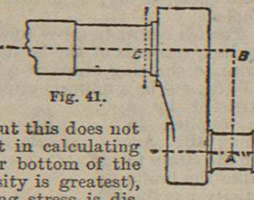
Fig. 41.

71. *Long Columns and Struts—Compression and Bending.*—A long strut or pillar, compressed by forces P applied at the ends in the direction of the axis, becomes unstable as regards flexure when P exceeds a certain value. Under no circumstances can this value of P be exceeded in loading a strut. But it may happen that the intensity of stress produced by smaller loads exceeds the safe compressive strength of the material, in which case a lower limit of load must be chosen. If the applied load is not strictly axial, if the strut is not initially straight, if it is subject to any deflexion by transverse forces, or if the modulus of elasticity is not uniform over each cross-section,—then loads smaller than the limit which causes instability will produce a certain deflexion which increases with increase of load, and will give rise to a uniformly varying stress of the kind illustrated in figs. 26 and 28. We shall first consider the ideal case in which the forces at the ends are strictly axial, the strut perfectly straight and free from transverse loads and perfectly symmetrical as to elasticity. Two conditions have to be distinguished—that in which the ends are left free to bend, and that in which the ends are held fixed. In what follows, the ends are supposed free to bend. The value of the load which causes instability will be found by considering what force P applied to each end would suffice to hold an originally straight strut in a bent state, supposing it to have received a small amount of elastic curvature in any way. Using u as before to denote the deflexion at any part of the length, the bending moment is Pu, and (taking the origin at the middle of the chord) the equation to the elastic curve is

$$\frac{d^2u}{dx^2} = \frac{-Pu}{EI},$$

from which, for a strut of uniform section, $u = u_1 \cos x\sqrt{P/EI}$, u_1 being the deflexion at the centre. Now $u = 0$ when $x = \frac{1}{2}L$ (the half length), and therefore $\frac{1}{2}L\sqrt{P/EI} = \frac{1}{2}\pi$ or an integral multiple of $\frac{1}{2}\pi$. The smallest value ($\frac{1}{2}\pi$) corresponds to the least force P. Thus the force required to maintain the strut in its curved state is $P = \pi^2 EI/L^2$, and is independent of u_1. This means that with this particular value of P (which for brevity we shall write P_1) the strut will be in neutral equilibrium when bent; with a value of P less than P_1 it will be stable; with a greater value it will be unstable. Hence a load exceeding P_1 will certainly cause rupture. The value $\pi^2 EI/L^2$ applies to struts with round ends, or ends free to turn. If the ends are fixed the effective length for bending is reduced by one half, so that P_1 then is $4\pi^2 EI/L^2$. When one end is fixed and the other is free P_1 has an intermediate value, probably about $9\pi^2 EI/4L^2$.

72. The above theory, which is Euler's, assigns P_1 as a limit to the strength of a strut on account of flexural instability; but a stress less than P_1 may cause direct crushing. Let S be the area of section, and f_c the strength of the material to resist crushing. Thus a strut which conforms to the ideal conditions specified above will fail by simple crushing if $f_c S$ is less than P_1, but by bending if $f_c S$ is greater than P_1. Hence with a given material and form of section the ideal strut will fail by direct crushing if the length is less than a certain multiple of the least breadth (easily calculated from the expression for P_1), and in that case its strength will be independent of the length; when the length is greater than this the strut will yield by bending, and its strength diminishes rapidly as the length is increased.

But the conditions which the above theory assumes are never realized in practice. The load is never strictly axial, nor the strut absolutely straight to begin with, nor the elasticity uniform. The result is that the strength is in all cases less than either $f_c S$ or P_1. The effect of deviations from axiality, from straightness, and from uniformity of elasticity may be treated by introducing a term expressing an imaginary initial deflexion, and in this way Euler's theory may be so modified as to agree well with experimental results on the fracture of struts,[2] and may be reconciled with the observed fact that the deflexion of a strut begins gradually and passes through stable values before the stage of instability is reached. In consequence of this stable deflexion the stress of compression on the inside edge becomes greater than P/S, the stress on the outside edge becomes less than P/S, and may even change into tension, and the strut may yield by one or the other of these stresses becoming greater than f_c or f_t respectively. As regards the influence of length and moment of inertia of section on the deflexion of struts, analogy to the case of beams suggests that the greatest deflexion consistent with stability will vary as L^2/b, b being the least breadth, and the greatest and least stress, at opposite edges of the middle section, will consequently be

$$p_1 = \frac{P}{S}\left(1 \pm \frac{aL^2}{b^2}\right),$$

where a is a coefficient depending on the material and the form of the section. This gives, for the breaking load, $P = Sf_c/(1 + aL^2/b^2)$ or $-Sf_t/(1 - aL^2/b^2)$, the smaller of the two being taken.

This formula, which is generally known as Gordon's, can be made to agree fairly with the results of experiments on struts of ordinary proportions, when the values of f as well as a are treated as empirical constants to be determined by trial with struts of the same class as those to which the formula is to be applied. Gordon's formula may also be arrived at in another way. For very short struts we have seen that the breaking load is $f_c S$, and for very long struts it is $\pi^2 EI/L^2$. If we write $P = f_c S/(1 + f_c SL^2/\pi^2 EI)$, we have a formula which gives correct values in these two extreme cases, and intermediate values for struts of medium length. By writing this $P = fS/(1 + cSL^2/I)$, and treating f and c as empirical constants, we have Gordon's formula in a slightly modified shape. Gordon's formula is largely used; it is, however, essentially empirical, and it is only by adjustment of both constants that it can be brought into agreement with experimental results.[3] For values of the constants, see BRIDGES. In the case of fixed ends, c is to be divided by 4.

73. *Bursting Strength of Circular Cylinders and Spheres.*—Space remains for the consideration of only one other mode of stress, of great importance from its occurrence in boilers, pipes, hydraulic and steam cylinders, and guns. The material of a hollow cylinder, subjected to pressure from within, is thrown into a stress of circumferential pull. When the thickness t is small compared with the radius R, we may treat this stress as uniformly distributed over the thickness. Let p be the intensity of fluid pressure within a hollow circular cylinder, and let f be the intensity of circumferential stress. Consider the forces on a small rectangular plate (fig. 42), with its sides parallel and perpendicular to the direction of the axis, of length l and width $R\delta\theta$, $\delta\theta$ being the small angle it subtends at the axis. Whatever forces act on this plate in the direction of the axis are equal and opposite. The remaining forces, which are in equilibrium, are P, the total pressure from within, and a force T at each side due to the circumferential stress. $P = plR\delta\theta$ and $T = ft$. But by the triangle of forces (fig. 43) $P = T\delta\theta$. Hence $f = pR/t$.

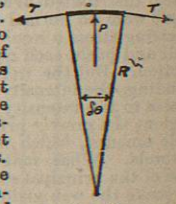
Fig. 42.

The ends of the cylinder may or may not be held together by longitudinal stress in the cylinder sides; if they are, then, whatever be the form of the ends, a transverse section, the area of which is $2\pi Rt$, has to bear a total force $p\pi R^2$. Hence, if f' be the intensity of longitudinal stress, $f' = pR/2t = \frac{1}{2}f$.

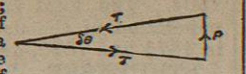
Fig. 43.

74. A thin hollow sphere under internal pressure has equal circumferential pull in all directions. To find its value consider the plate of fig. 42. There are now four equal forces T, on each of the four sides, to equilibrate the radial force P. Hence $P = 2T\delta\theta$ and $f = pR/2t$.

75. When the thickness is not small compared with the radius, the radial pressure is transmitted from layer to layer with reduced intensity, and the circumferential pull diminishes towards the outside. In the case of a thick cylinder with free ends[1] we have to deal at any point with two principal stresses, radial and circumferential, which may be denoted by p and p' respectively. Supposing (as we may properly do in dealing with a cylinder which is not very short) that a transverse section originally plane remains plane, the longitudinal strain is uniform. Since there is no longitudinal stress this strain is due entirely to the lateral action of the stresses p and p', and its amount is $(p + p')/\sigma E$. Hence at all points $p + p' = \text{constant}$.[2] Further, by considering the equilibrium of any thin layer, as we have already considered that of a thin cylinder, we have $\frac{d}{dr}(pr) = p'$.

These two equations give by integration, $p = C + C'/r^2$, and $p' = C - C'/r^2$.

If r_1 be the external and r_2 the internal radius, and p_0 the pressure on the inner surface, the conditions that $p = p_0$ when $r = r_2$ and $p = 0$ when $r = r_1$ give $C = -p_0 r_2^2/(r_1^2 - r_2^2)$ and $C' = -Cr_1^2$. Hence the circumferential stress at any radius r is $p' = -p_0 r_2^2(1 + r_1^2/r^2)/(r_1^2 - r_2^2)$. At the inside, where this is greatest, its value is $-p_0(r_1^2 + r_2^2)/(r_1^2 - r_2^2)$,—a quantity always greater than p_0, however thick the cylinder is.

In the construction of guns various devices have been used to equalize the circumferential tension. With cast guns a chilled core has been employed to make the inner layers solidify and cool first, so that they are afterwards compressed by the later contraction of the outer layers. In guns built up of wrought-iron or steel hoops the hoops are bored small by a regulated amount and are shrunk on over the barrel or over the inner hoops. In Mr Longridge's system, now under trial, the gun is made by winding steel wire or ribbon, with suitable initial tension, on a central barrel.

76. The circumferential stress at any point of a thick hollow sphere exposed to internal fluid pressure is found, by a process like that of the last paragraph, to be $-p_0 r_2^3(1 + r_1^3/2r^3)/(r_1^3 - r_2^3)$, which gives, for the greatest tension, the value

$$-p_0(r_1^3 + 2r_2^3)/2(r_1^3 - r_2^3).$$

(J. A. E.)

[1] See ELASTICITY, §§ 10–20.

[2] See papers by Profs. Ayrton and Perry, *The Engineer*, Dec. 10 and 24, 1886, and by T. C. Fidler, *Min. Proc. Inst. C.E.*, vol. lxxxvi. p. 261.

[3] For experiments on the breaking strength of struts, see papers by Hodgkinson, *Phil. Trans.*, 1840; Berkeley, *Min. Proc. Inst. C.E.*, vol. xxx.; Christie, *Trans. Amer. Soc. Civ. Eng.*, 1884.

[1] This condition is realized in practice when the fluid causing internal pressure is held in by a piston, and the stress between this piston and the other end of the cylinder is taken by some other part of the structure than the cylinder sides.

[2] The solution which follows in the text is applicable even when there is longitudinal stress, provided that the longitudinal stress is uniformly distributed over each transverse section. If we call this stress p'', the longitudinal strain is $p''/E + (p + p')/\sigma E$. Since the whole strain is uniform, and p'' is uniform, the sum of p and p' is constant at all points, as in the case where the ends are free.

STRICKLAND, AGNES (1806–1874), a popular historical writer, was born in 1806, the third daughter of Thomas Strickland, of Roydon Hall, Suffolk. Her first literary efforts were historical romances in verse in the style of Walter Scott,—*Worcester Field* (published without date), *Demetrius and other Poems* (1833). From this she passed to prose histories, written in a simple style for the young. A picturesque sketch of the *Pilgrims of Walsingham* appeared in 1835, two volumes of *Tales and Stories from History* in the following year. Then with the assistance of her sister she projected a more ambitious work, *The Lives of the Queens of England*, from Mathilda of Flanders to Queen Anne. The first volume appeared in 1840, the twelfth and last in 1849. Miss Strickland was a warm partisan on the side of royalty and the church, but she made industrious study of "official records and other public documents," gave copious extracts from them, and drew interesting pictures of manners and customs. While engaged on this work she found time to edit (in 1843) the *Letters of Mary, Queen of Scots*, whose innocence she championed with enthusiasm. In 1850 she followed up her *Queens of England* with the *Lives of the Queens of Scotland*, completing the series in eight volumes in 1859. Unresting in her industry, she turned next to the *Batchelor Kings of England*, about whom she published a volume in 1861. The *Lives of the Seven Bishops* followed in 1866—after a longer interval, part of which was employed in producing an abridged version of her *Queens of England*. Her last work was the *Lives of the Last Four Stuart Princesses*, published in 1872. In 1871 she obtained a civil list pension of £100 in recognition of her merits. She died at Roydon Hall on the 8th of July 1874.

A *Life* by her sister, Jane Margaret Strickland, appeared in 1887.

STRIEGAU, an industrial town of Prussia, in the province of Silesia, is situated on a small tributary of the Weistritz, 30 miles to the south-west of Breslau. In 1880 it contained 11,470 inhabitants, 6928 of whom were Protestants and 4379 Roman Catholics. Their chief occupations are tanning and the manufacture of albums, portfolios, and other articles in leather. Granite is quarried in the neighbourhood, and a trade is carried on in grain. It was near Striegau that Frederick the Great gained the important victory usually named after the village of Hohenfriedberg (June 4, 1745).

STROMBOLI. See LIPARI ISLANDS.

STRONTIUM, a metallic chemical element intermediate in its character between barium and calcium, with which it forms a natural "triad." Though widely diffused as a frequent companion of calcium (including oceanic), it occurs nowhere in abundance. Its most important mineral forms are the sulphate, $SrSO_4$, known as *Cœlestine* (from the sky-blue colour of certain varieties), and the carbonate, $SrCO_3$, called *Strontianite* because it was discovered first at Strontian, in Argyllshire, Scotland. Crawford and (independently of him) Cruickshanks in 1790 were the first to recognize the latter mineral as a thing of its own kind and different from witherite ($BaCO_3$). Hope, in 1793, proved it to be the carbonate of a new earth, which discovery was confirmed by Klaproth.

Regarding metallic strontium, see CHEMISTRY, vol. v. pp. 525–6. For the making of strontium preparations strontianite, of course, is the handier raw material, being readily convertible into (for instance) nitrate by treatment with dilute nitric acid. From the nitrate the oxide, SrO, is obtained by prolonged calcination at ultimately a bright red heat, as a greyish-white absolutely infusible and non-volatile mass, which acts violently on water with formation of the hydrate, $Sr(OH)_2$, which latter readily takes up $8H_2O$ of water to form crystals soluble in fifty parts of cold and far less of boiling water. An impure oxide is obtainable directly from strontianite by strong ignition with charcoal; and from such crude oxide pure crystals of the hydrate are easily produced by obvious operations.

In the working up of cœlestine the first step is to reduce it to sulphide, SrS, by means of charcoal at a red heat. The sulphide when boiled with water is decomposed thus:—

$$2SrS + 2H_2O = \underset{\text{Hydrate.}}{SrOH_2O} + \underset{\text{Sulph-hydrate.}}{SrSH_2S}.$$

Both products dissolve in the hot water; from the solution the S of the SrH_2S_2 is easily eliminated, by treatment with oxide of copper or oxide of zinc, as insoluble metallic sulphide; the filtrate on cooling gives crystals of pure hydrate. From it any strontia salt of course is easily made by means of the respective acid; in many cases the salt wished for can be obtained similarly from the sulphide.

Nitrate of strontia from hot solutions crystallizes in anhydrous octahedra, SrN_2O_6, soluble in about ½ part of boiling and in 5 parts of cold water. From colder solutions hydrated crystals, $SrN_2O_6 + 4H_2O$, separate out. The anhydrous salt is used largely by pyrotechnists for the making of "red fire."

The hydroxide some years ago promised to play an important part in the sugar industry as a precipitant for the cane-sugar known to be present largely in uncrystallizable molasses (see SUGAR), but the process so far has failed to take root in industry.