of a polygonal figure having m geometrical equations of condition, and x for the most probable value of the error of any observed angle, we have

$$e_2^2 = \frac{[wx^2]}{W}\frac{1}{m} = \frac{a}{m} \text{ for a single figure,}$$
$$= \frac{[a]}{[m]} \text{ for a group of figures,}$$

the brackets [] in each case denoting the sum of all the quantities involved. e_3 usually gives the best value of the theoretical error, then e_2. As a rule the value by e_1 is too small; but to this there are notable exceptions, in which it was found to be much too great. The instrument with which the angles were measured in these instances gave very discrepant results at different settings of the circle; but this was caused by large periodic errors of graduation which did not affect the "concluded angles," because they were eliminated by the systematic changes of setting, so the results were really more precise than was apparent.

When weights were determined for the final simultaneous reduction of triangulations executed by different instruments, it became necessary to find a factor ρ to be applied as a modulus to each group of angles measured with the same instrument and under similar conditions, to convert the as yet relative weights into absolute measures of precision. ρ was made $= e_1 \div e_3$ whenever data were available, if not to $e_1 \div e_2$; then the absolute weight of an observed angle in any group was taken as $w\rho^2$ and the *e.m.s.* of the angle as $1 \div \rho\sqrt{w}$. The average values of the *e.m.s.* thus determined for large groups of angles, measured with the 36-inch and the 24-inch theodolites, ranged from $\pm 0''\cdot24$ to $\pm 0''\cdot67$, the smaller values being usually obtained at hill stations, where the atmospheric conditions were most favourable.

Harmonizing angles.

13. *Harmonizing Angles of Trigonometrical Figures.*—Every figure, whether a single triangle or a polygonal network, was made consistent by the application of corrections to the observed angles to satisfy its geometrical conditions. The three angles of every triangle having been observed, their sum had to be made = 180° + the spherical excess; in networks it was also necessary that the sum of the angles measured round the horizon at any station should be exactly = 360°, that the sum of the parts of an angle measured at different times should equal the whole, and that the ratio of any two sides should be identical, whatever the route through which it was computed. These are called the *triangular*, *central*, *toto-partial*, and *side* conditions; they present n geometrical equations, which contain t unknown quantities, the errors of the observed angles, t being always $> n$. When these equations are satisfied and the deduced values of errors are applied as corrections to the observed angles, the figure becomes consistent. Primarily the equations were treated by a method of successive approximations; but afterwards they were all solved simultaneously by the so-called method of minimum squares, which leads to the most probable of any system of corrections; it is demonstrated under EARTH, FIGURE OF THE (vol. vii. p. 599). The following is a general outline of the process:—

Let x be the most probable value of the error and u the reciprocal of the weight of any observed angle X, and let $a, b, \ldots n$ be the coefficients of x in successive geometrical equations of condition whose absolute terms are $e_a, e_b, \ldots e_n$; then we have the following group of n equations containing t unknown quantities to be satisfied, the significant coefficients of x being 1 in the triangular, toto-partial, and central, and $\pm \cot X$ in the side equations:—

$$\left.\begin{aligned} a_1x_1 + a_2x_2 + \ldots + a_tx_t &= e_a \\ b_1x_1 + b_2x_2 + \ldots + b_tx_t &= e_b \\ \ldots\ldots\ldots\ldots \\ n_1x_1 + n_2x_2 + \ldots + n_tx_t &= e_n \end{aligned}\right\} \ldots(3).$$

The values of x will be the most probable when $\left[\frac{x^2}{u}\right]$ is a minimum, a condition which introduces n indeterminate factors $\lambda_a \ldots \lambda_n$ whose values are obtained by the solution of the following equations:—

$$\left.\begin{aligned} [aa.u]\lambda_a + [ab.u]\lambda_b + \ldots + [an.u]\lambda_n &= e_a \\ [ab.u]\lambda_a + [bb.u]\lambda_b + \ldots + [bn.u]\lambda_n &= e_b \\ \ldots\ldots\ldots\ldots \\ [an.u]\lambda_a + [bn.u]\lambda_b + \ldots + [nn.u]\lambda_n &= e_n \end{aligned}\right\} \ldots(4),$$

the brackets indicating summations of t terms as to left of (3). Then the value of any, the pth, x is

$$x_p = u_p\{a_p\lambda_a + b_p\lambda_b + \ldots + n_p\lambda_n\} \ldots(5).$$

$$\text{The minimum or } \left[\frac{x^2}{u}\right] \text{ is } = [e\lambda] \ldots(6)$$

In the application to a single triangle we have $x_1 + x_2 + x_3 = e$, $\lambda = e \div (u_1 + u_2 + u_3)$; $x_1 = u_1\lambda$; $x_2 = u_2\lambda$; $x_3 = u_3\lambda$.

In the application to a simple polygon, by changing symbols and putting X and Y for the exterior and Z for the central angles, with errors x, y, and z and weight reciprocals u, v, and w, a for $\cot X$ and b for $\cot Y$, e for any triangular error, e_c and e_s for the central and side errors, λ_c and λ_s for the factors for the central and side equations, and W for $u+v+w$, the equations for obtaining the factors become

$$\left.\begin{aligned} \left[w - \frac{w^2}{W}\right]\lambda_c - \left[\frac{w(au-bv)}{W}\right]\lambda_s &= e_c - \left[\frac{we}{W}\right] \\ -\left[\frac{w(au-bv)}{W}\right]\lambda_c + \left[a^2u + b^2v - \frac{(au-bv)^2}{W}\right]\lambda_s &= e_s - \left[\frac{(au-bv)e}{W}\right] \end{aligned}\right\} (7),$$

and the general expressions for the errors of the angles are—

$$\left.\begin{aligned} x &= \frac{u}{W}\{e + (aW - au + bv)\lambda_s - w\lambda_c\} \\ y &= \frac{v}{W}\{e - (bW + au - bv)\lambda_s - w\lambda_c\} \\ z &= \frac{w}{W}\{e - (au - bv)\lambda_s + (u+v)\lambda_c\} \end{aligned}\right\} \ldots(8).$$

Sides of triangles.

14. *Calculation of Sides of Triangles.*—The angles having been made geometrically consistent *inter se* in each figure, the side-lengths are computed from the base-line onwards by Legendre's theorem, each angle being diminished by one-third of the spherical excess of the triangle to which it appertains. The theorem is applicable without sensible error to triangles of a much larger size than any that are ever measured.

Latitude and longitude of stations and azimuth of sides.

15. *Calculation of Latitudes and Longitudes of Stations and Azimuths of Sides.*—A station of origin being chosen of which the latitude and longitude are known astronomically, and also the azimuth of one of the surrounding stations, the differences of latitude and longitude and the reverse azimuths are calculated in succession, for all the stations of the triangulation, by Puissant's formulæ (*Traité de Géodésie*, Paris, 1842, 3d ed.).

Problem.—Assuming the earth to be spheroidal, let A and B be two stations on its surface, and let the latitude and longitude of A be known, also the azimuth of B at A, and the distance between A and B at the mean sea-level; we have to find the latitude and longitude of B and the azimuth of A at B.

The following symbols are employed:—a the major and b the minor semi-axis; e the excentricity, $= \left\{\frac{a^2-b^2}{a^2}\right\}^{\frac{1}{2}}$; ρ the radius of curvature to the meridian in latitude λ, $= \frac{a(1-e^2)}{\{1-e^2\sin^2\lambda\}^{\frac{3}{2}}}$; ν the normal to the meridian in latitude λ, $= \frac{a}{\{1-e^2\sin^2\lambda\}^{\frac{1}{2}}}$; λ and L the given latitude and longitude of A; $\lambda + \Delta\lambda$ and $L + \Delta L$ the required latitude and longitude of B; A the azimuth of B at A; B the azimuth of A at B; $\Delta A = B - (\pi + A)$; c the distance between A and B. Then, all azimuths being measured from the south, we have

$$\Delta\lambda'' = \left\{\begin{aligned} &-\frac{c}{\rho}\cos A \operatorname{cosec} 1'' \\ &-\frac{1}{2}\frac{c^2}{\rho\nu}\sin^2 A \tan\lambda \operatorname{cosec} 1'' \\ &-\frac{3}{4}\frac{c^2}{\rho\nu}\frac{e^2}{1-e^2}\cos^2 A \sin 2\lambda \operatorname{cosec} 1'' \\ &+\frac{1}{6}\frac{c^3}{\rho\nu^2}\sin^2 A \cos A(1+3\tan^2\lambda) \operatorname{cosec} 1'' \end{aligned}\right\} (9)$$

$$\Delta L'' = \left\{\begin{aligned} &-\frac{c}{\nu}\frac{\sin A}{\cos\lambda} \operatorname{cosec} 1'' \\ &+\frac{1}{2}\frac{c^2}{\nu^2}\frac{\sin 2A \tan\lambda}{\cos\lambda} \operatorname{cosec} 1'' \\ &-\frac{1}{6}\frac{c^3}{\nu^3}\frac{(1+3\tan^2\lambda)\sin 2A\cos A}{\cos\lambda} \operatorname{cosec} 1'' \\ &+\frac{1}{3}\frac{c^3}{\nu^3}\frac{\sin^3 A \tan^2\lambda}{\cos\lambda} \operatorname{cosec} 1'' \end{aligned}\right\} (10).$$

$$\begin{aligned}\Delta A'' &\text{ or} \\ B-(\pi+A) &= \end{aligned}\left\{\begin{aligned} &-\frac{c}{\nu}\sin A \tan\lambda \operatorname{cosec} 1'' \\ &+\frac{1}{4}\frac{c^2}{\nu^2}\left\{1 + 2\tan^2\lambda + \frac{e^2\cos^2\lambda}{1-e^2}\right\}\sin 2A \operatorname{cosec} 1'' \\ &-\frac{c^3}{\nu^3}\left(\frac{5}{6} + \tan^2\lambda\right)\frac{\tan\lambda}{2}\sin 2A\cos A \operatorname{cosec} 1'' \\ &+\frac{1}{6}\frac{c^3}{\nu^3}\sin^3 A \tan\lambda(1+2\tan^2\lambda) \operatorname{cosec} 1'' \end{aligned}\right\} (11).$$

Each Δ is the sum of four terms symbolized by δ_1, δ_2, δ_3, and δ_4; the calculations are so arranged as to produce these terms in the order $\delta\lambda$, δL, and δA, each term entering as a factor in calculating the following term. The arrangement is shown below in equations in which the symbols $P, Q, \ldots Z$ represent the factors which depend on the adopted geodetic constants, and vary with the latitude; the logarithms of their numerical values are tabulated in the *Auxiliary Tables to Facilitate the Calculations of the Indian Survey.*

$$\left.\begin{aligned} \delta_1\lambda &= -P.\cos A.c & \delta_1 L &= +\delta_1\lambda.Q.\sec\lambda.\tan A & \delta_1 A &= +\delta_1 L.\sin\lambda \\ \delta_2\lambda &= +\delta_1 A.R.\sin A.c & \delta_2 L &= -\delta_2\lambda.S.\cot A & \delta_2 A &= +\delta_2 L.T \\ \delta_3\lambda &= -\delta_2 A.V.\cot A & \delta_3 L &= +\delta_3\lambda.U.\sin A.c & \delta_3 A &= +\delta_3 L.W \\ \delta_4\lambda &= -\delta_3 A.X.\tan A & \delta_4 L &= +\delta_4\lambda.Y.\tan A & \delta_4 A &= +\delta_4 L.Z \end{aligned}\right\} (12).$$

By this artifice the calculations are rendered less laborious and made susceptible of being readily performed by any persons who are acquainted with the use of logarithm tables.

Limits of geodetic formulæ.

16. *Limits within which Geodetic Formulæ may be employed without Sensible Error.*—Each Δ is expressed as a series of ascending differentials in which all terms above the third order are neglected; for the side length c in no case exceeded 70 miles, nor was the latitude ever higher than 36°, and for these extreme values the maximum magnitudes of the fourth differential are only $0''\cdot002$ in latitude and $0''\cdot004$ in longitude and azimuth.

Far greater error may arise from uncertainties regarding the elements of the earth's figure, which was assumed to be spheroidal, with semi-axes $a = 20{,}922{,}932$ feet and $b = 20{,}853{,}375$ feet. The changes in $\Delta\lambda$, ΔL, and ΔA which would arise from errors da and db in a and b are indicated by the following formulæ:—

$$\left.\begin{aligned} d.\Delta\lambda &= -\Delta\lambda.\frac{d\rho}{\rho} - \delta_2\lambda.\frac{d\nu}{\nu} - \delta_3\lambda\left(\frac{d\nu}{\nu} - \frac{2de}{(1-e^2)e}\right) - 2\delta_4\lambda.\frac{d\nu}{\nu} \\ d.\Delta L &= -\Delta L.\frac{d\nu}{\nu} - \delta_2 L.\frac{d\nu}{\nu} - (\delta_3 L + \delta_4 L)2\frac{d\nu}{\nu} \\ d.\Delta A &= -\Delta A.\frac{d\nu}{\nu} - \delta_2 A\left\{\frac{d\nu}{\nu} - \frac{\nu}{\rho}\left(\frac{d\nu}{\nu} - \frac{d\rho}{\rho}\right)\frac{1}{2\tan^2\lambda + \frac{\nu}{\rho}}\right\} \\ &\qquad - (\delta_3 A + \delta_4 A)2\frac{d\nu}{\nu} \end{aligned}\right\} \ldots(13),$$

in which

$$\left.\begin{aligned} \frac{d\rho}{\rho} &= -\cdot000{,}000{,}0478\{da - 2db - 3(da - db)\sin^2\lambda\} \\ \frac{d\nu}{\nu} &= +\cdot000{,}000{,}0478\{da + (da - db)\sin^2\lambda\} \\ \frac{2de}{(1-e^2)e} &= +\qquad \cdot000{,}0145\{da - db\} \end{aligned}\right\} \ldots(14).$$

The adopted values of the semi-axes were determined by Colonel Everest in an investigation of the figure of the earth from such data as were available in 1826. Forty years afterwards an investigation was made by Captain (now Colonel) A. R. Clarke with additional data, which gave new values, both exceeding the former.[1] Accepting these as exact, the errors of the first values are $da = -3130$ feet and $db = -1746$ feet, the former being 150, the latter 84 millionth parts of the semi-axis. The corresponding changes in arcs of 1° of latitude and longitude, expressed in seconds of arc and in millionth parts (μ) of arc-length, are as follows:—

In lat. 5° $d.\Delta\lambda = -''\cdot069$ or 19 μ and $d.\Delta L = -''\cdot540$ or 150 μ;
,, 15° ,, $-''\cdot113$,, 31 ,, ,, ,, $-''\cdot554$,, 154 ,,;
,, 25° ,, $-''\cdot195$,, 54 ,, ,, ,, $-''\cdot581$,, 161 ,,;
,, 35° ,, $-''\cdot303$,, 84 ,, ,, ,, $-''\cdot617$,, 171 ,,.

These assumed errors in the geodetic latitudes and longitudes are of service when comparisons are made between independent astronomical and geodetic determinations at any points for which both may be available: they indicate the extent to which differences may be attributable to errors in the adopted geodetic constants, as distinct from errors in the trigonometrical or the astronomical operations.

17. *Final Reduction of Principal Triangulation.*—The calculations described so far suffice to make the angles of the several trigonometrical figures consistent *inter se*, and to give preliminary values of the lengths and azimuths of the sides and the latitudes and longitudes of the stations. The results are amply sufficient for the requirements of the topographer and land surveyor, and they are published in preliminary charts, which give full numerical details of latitude, longitude, azimuth, and side-length, and of height also, for each portion of the triangulation—secondary as well as principal—as executed year by year. But on the completion of the several chains of triangles further reductions became necessary, to make the triangulation everywhere consistent *inter se* and with the verificatory base lines, so that the lengths and azimuths of common sides and the latitudes and longitudes of common stations should be identical at the junctions of chains, and that the measured and computed lengths of the base-lines should also be identical.

How this was done will now be set forth. But first it must be noted that the triangulation might at the same time have been made consistent with any values of latitude, longitude, or azimuth which had been determined by astronomical observations at either of the trigonometrical stations. This, however, was undesirable, because such observations are liable to errors from deflexion of the plumb-line from the true normal under the influence of local attraction, and these errors are of a much greater magnitude than those that would be generated in triangulating between astronomical stations which are not a great distance apart. The trigonometrical elements could not be forced into accordance with the astronomical without altering the angles by amounts much larger than their probable errors, and the results would be useless for investigations of the figure of the earth. The only independent facts of observation which could be legitimately combined with the angular adjustments were the base-lines, and all these were employed, while the several astronomical determinations—of latitude, differential longitude, and azimuth—were held in reserve for future geodetic investigations.

As an illustration of the problem for treatment, suppose a combination of three meridional and two longitudinal chains comprising seventy-two single triangles, with a base-line at each corner, as shown in the accompanying diagram (fig. 2); suppose the three angles of every triangle to have been measured and made consistent. Let A be the origin, with its latitude and longitude given, and also the length and azimuth of the adjoining base-line. With these data processes of calculation are carried through the triangulation to obtain the lengths and azimuths of the sides and the latitudes and longitudes of the stations, say in the following order:—from A through B to E, through F to E, through F to D, through F and E to C, and through F and D to C. Then there are two values of side, azimuth, latitude, and longitude at E,—one from the right-hand chains *via* B, the other from the left-hand chains *via* F; similarly there are two sets of values at C; and each of the base-lines at B, C, and D has a calculated as well as a measured value. Thus eleven absolute errors are presented for dispersion over the triangulation by the application of the most appropriate correction to each angle, and, as a preliminary to the determination of these corrections, equations must be constructed between each of the absolute errors and the unknown errors of the angles from which

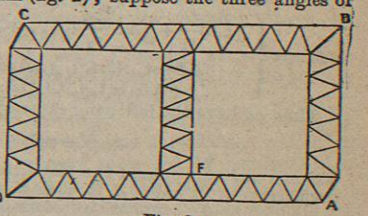

Fig. 2.

[1] See *Account of the Principal Triangulation of the Ordnance Survey*, 1858, and *Comparisons of Standards of Length*, 1866.

they originated. For this purpose assume X to be the angle opposite the flank side of any triangle, and Y and Z the angles opposite the sides of continuation; also let x, y, and z be the most probable values of the errors of the angles which will satisfy the given equations of condition. Then each equation may be expressed in the form $[ax+by+cz]=E$, the brackets indicating a summation for all the triangles involved. We have first to ascertain the values of the coefficients a, b, and c of the unknown quantities. They are readily found for the side equations on the circuits and between the base-lines, for x does not enter them, but only y and z, with coefficients which are the cotangents of Y and Z, so that these equations are simply $[\cot Y.y-\cot Z.z]=E$. But three out of four of the circuit equations are geodetic, corresponding to the closing errors in latitude, longitude, and azimuth, and in them the coefficients are very complicated. They are obtained as follows. The first term of each of the three expressions for $\Delta\lambda$, ΔL, and B is differentiated in terms of c and A, giving

$$\left.\begin{aligned} d.\Delta\lambda &= \Delta\lambda\left\{\frac{dc}{c}-dA\tan A\sin 1''\right\} \\ d.\Delta L &= \Delta L\left\{\frac{dc}{c}+dA\cot A\sin 1''\right\} \\ dB &= dA+\Delta A\left\{\frac{dc}{c}+dA\cot A\sin 1''\right\} \end{aligned}\right\}\text{.........(15),}$$

in which dc and dA represent the errors in the length and azimuth of any side c which have been generated in the course of the triangulation up to it from the base-line and the azimuth station at the origin. The errors in the latitude and longitude of any station which are due to the triangulation are $d\lambda_{,}=[d.\Delta\lambda]$, and $dL_{,}=[d.\Delta L]$. Let station 1 be the origin, and let 2, 3, ... be the succeeding stations taken along a predetermined line of traverse, which may either run from vertex to vertex of the successive triangles, zigzagging between the flanks of the chain, as in fig. 3 (1), or be carried directly along one of the flanks, as in fig. 3 (2). For the general symbols of the differential equations substitute $\Delta\lambda_n$, ΔL_n, ΔA_n, c_n, A_n, and B_n, for the side between stations n and $n+1$ of the traverse; and let δc_n and δA_n be the errors generated between the sides c_{n-1} and c_n; then

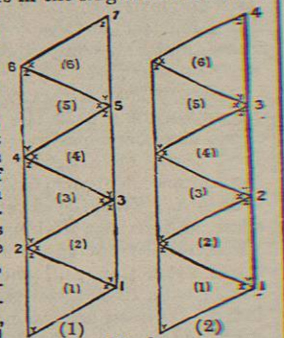

Fig. 3.

$$\frac{dc_1}{c_1}=\frac{\delta c_1}{c_1};\quad \frac{dc_2}{c_2}=\frac{\delta c_1}{c_1}+\frac{\delta c_2}{c_2};\quad \ldots\quad \frac{dc_n}{c_n}={}_1^n\left[\frac{\delta c}{c}\right];$$
$$dA_1=\delta A_1;\quad dA_2=dB_1+\delta A_2;\quad \ldots\quad dA_n=dB_{n-1}+\delta A_n.$$

Performing the necessary substitutions and summations, we get

$$dB_n=\begin{cases} {}_1^n[\Delta A]\frac{\delta c_1}{c_1}+{}_2^n[\Delta A]\frac{\delta c_2}{c_2}+\ldots+\Delta A_n\frac{\delta c_n}{c_n} \\ +(1+{}_1^n[\Delta A\cot A]\sin 1'')\delta A_1+(1+{}_2^n[\Delta A\cot A]\sin 1'')\delta A_2 \\ \quad +\ldots+(1+\Delta A_n\cot A_n\sin 1'')\delta A_n \end{cases}$$

$$d\lambda_{n+1}=\begin{cases} {}_1^n[\Delta\lambda]\frac{\delta c_1}{c_1}+{}_2^n[\Delta\lambda]\frac{\delta c_2}{c_2}+\ldots+\Delta\lambda_n\frac{\delta c_n}{c_n} \\ -\{{}_1^n[\Delta\lambda\tan A]\delta A_1+{}_2^n[\Delta\lambda\tan A]\delta A_2+\ldots \\ \quad +\Delta\lambda_n\tan A_n\delta A_n\}\sin 1'' \end{cases}$$

$$dL_{n+1}=\begin{cases} {}_1^n[\Delta L]\frac{\delta c_1}{c_1}+{}_2^n[\Delta L]\frac{\delta c_2}{c_2}+\ldots+\Delta L_n\frac{\delta c_n}{c_n} \\ +\{{}_1[\Delta L\cot A]\delta A_1+{}_2^n[\Delta L\cot A]\delta A_2+\ldots \\ \quad +\Delta L_n\cot A_n\delta A_n\}\sin 1'' \end{cases}$$

Thus we have the following expression for any geodetic error—

$$\mu_1\frac{\delta c_1}{c_1}+\ldots+\mu_n\frac{\delta c_n}{c_n}+\phi_1\delta A_1+\ldots+\phi_n\delta A_n=E,\quad\text{...(16),}$$

where μ and ϕ represent the respective summations which are the coefficients of δc and δA in each instance but the first, in which 1 is added to the summation in forming the coefficient of δA.

The angular errors x, y, and z must now be introduced, in place of δc and δA, into the general expression, which will then take different forms, according as the route adopted for the line of traverse was the zigzag or the direct. In the former, the number of stations on the traverse is ordinarily the same as the number of triangles, and, whether or no, a common numerical notation may be adopted for both the traverse stations and the collateral triangles; thus the angular errors of every triangle enter the general expression in the form $\pm\phi x+\cot Y.\mu' y-\cot Z.\mu' z$, in which $\mu'=\mu\sin 1''$, and the upper sign of ϕ is taken if the triangle lies to the left, the lower if to the right, of the line of traverse. When the direct traverse is adopted, there are only half as many traverse stations as triangles, and therefore only half the number of μ's and ϕ's to determine; but it becomes necessary to adopt different numberings for the stations and the triangles, and the form of the coefficients of the angular errors alternates in successive triangles. Thus, if the pth triangle has no side on the line of the traverse but only an angle at the lth station, the form is

$$+\phi_l.x_p+\cot Y_p.\mu'_l.y_p-\cot Z_p.\mu'_l.z_p.$$

If the qth triangle has a side between the lth and the $(l+1)$th stations of the traverse, the form is

$$\cot X_q(\mu'_l-\mu'_{l+1})x_q+(\phi_l+\mu'_{l+1}\cot Y_q)y_q-(\phi_{l+1}-\mu'_l\cot Z_q)z_q.$$

As each circuit has a right-hand and a left-hand branch, the errors of the angles are finally arranged so as to present equations of the general form

$$[ax+by+cz]_r-[ax+by+cz]_l=E.$$

The eleven circuit and base-line equations of condition having been duly constructed, the next step is to find values of the angular errors which will satisfy these equations, and be the most probable of any system of values that will do so, and at the same time will not disturb the existing harmony of the angles in each of the seventy-two triangles. Harmony is maintained by introducing the equation of condition $x+y+z=0$ for every triangle. The most probable results are obtained by the method of minimum squares, which may be applied in two ways.

(i.) A factor λ may be obtained for each of the eighty-three equations under the condition that $\left[\frac{x^2}{u}+\frac{y^2}{v}+\frac{z^2}{w}\right]$ is made a minimum, u, v, and w being the *reciprocals* of the weights of the observed angles. This necessitates the simultaneous solution of eighty-three equations to obtain as many values of λ. The resulting values of the errors of the angles in any, the pth, triangle, are

$$x_p=u_p[a_p\lambda];\ y_p=v_p[b_p\lambda];\ z_p=w_p[c_p\lambda]\text{......... (17).}$$

(ii.) One of the unknown quantities in every triangle, as x, may be eliminated from each of the eleven circuit and base-line equations by substituting its equivalent $-(y+z)$ for it, a similar substitution being made in the minimum. Then the equations take the form $[(b-a)y+(c-a)z]=E$, while the minimum becomes

$$\left[\frac{(y+z)^2}{u}+\frac{y^2}{v}+\frac{z^2}{w}\right].$$

Thus we have now to find only eleven values of λ by a simultaneous solution of as many equations, instead of eighty-three values from eighty-three equations; but we arrive at more complex expressions for the angular errors as follows:—

$$\left.\begin{aligned} y_p&=\frac{v_p}{u_p+v_p+w_p}\{(u_p+w_p)[(b_p-a_p)\lambda]-w_p[(c_p-a_p)\lambda]\} \\ z_p&=\frac{w_p}{u_p+v_p+w_p}\{(u_n+v_p)[(c_p-a_p)\lambda]-v_p[(b_p-a_p)\lambda]\} \end{aligned}\right\}\text{... (18).}$$

The second method has invariably been adopted, originally because it was supposed that, the number of the factors λ being reduced from the total number of equations to that of the circuit and base-line equations, a great saving of labour would be effected. But subsequently it was ascertained that in this respect there is little to choose between the two methods; for, when x is not eliminated, and as many factors are introduced as there are equations, the factors for the triangular equations may be readily eliminated at the outset. Then the really severe calculations will be restricted to the solution of the equations containing the factors for the circuit and base-line equations, as in the second method.

In the preceding illustration it is assumed that the base-lines are errorless as compared with the triangulation. Strictly speaking, however, as base-lines are fallible quantities, presumably of different weight, their errors should be introduced as unknown quantities of which the most probable values are to be determined in a simultaneous investigation of the errors of all the facts of observation, whether linear or angular. When they are connected together by so few triangles that their ratios may be deduced as accurately, or nearly so, from the triangulation as from the measured lengths, this ought to be done; but, when the connecting triangles are so numerous that the direct ratios are of much greater weight than the trigonometrical, the errors of the base-lines may be neglected. In the reduction of the Indian triangulation it was decided, after examining the relative magnitudes of the probable errors of the linear and the angular measures and ratios, to assume the base-lines to be errorless (see § 19, p. 704 below).

The chains of triangles being largely composed of polygons or other networks, and not merely of single triangles, as has been assumed for simplicity in the illustration, the geometrical harmony to be maintained involved the introduction of a large number of "side," "central," and "toto-partial" equations of condition, as well as the triangular. Thus the problem for attack was the simultaneous solution of a number of equations of condition = that of all the geometrical conditions of every figure + four times the number of circuits formed by the chains of triangles + the number of base-lines − 1, the number of unknown quantities contained in the equations being that of the whole of the observed angles; the method of procedure, if rigorous, would be precisely similar to that

already indicated for "harmonizing the angles of trigonometrical figures," of which it is merely an expansion from single figures to great groups.

The rigorous treatment would, however, have involved the simultaneous solution of about 4000 equations between 9230 unknown quantities, which was quite impracticable. The triangulation was therefore divided into sections for separate reduction, of which the most important were the five between the meridians of 67° and 92° (see fig. 1, p. 696), consisting of four quadrilateral figures and a trigon, each comprising several chains of triangles and some base-lines. This arrangement had the advantage of enabling the final reductions to be taken in hand as soon as convenient after the completion of any section, instead of being postponed until all were completed. It was subject, however, to the condition that the sections containing the best chains of triangles were to be first reduced; for, as all chains bordering contiguous sections would necessarily be "fixed" as a part of the section first reduced, it was obviously desirable to run no risk of impairing the best chains by forcing them into adjustment with others of inferior quality. It happened that both the north-east and the south-west quadrilaterals contained several of the older chains; their reduction was therefore made to follow that of the collateral sections containing the modern chains.

But the reduction of each of these great sections was in itself a very formidable undertaking, necessitating some departure from a purely rigorous treatment. For the chains were largely composed of polygonal networks and not of single triangles only as assumed in the illustration, and therefore cognizance had to be taken of a number of "side" and other geometrical equations of condition, which entered irregularly and caused great entanglement. Equations 17 and 18 of the illustration are of a simple form because they have a single geometrical condition to maintain, the triangular, which is not only expressed by the simple and symmetrical equation $x+y+z=0$, but—what is of much greater importance—recurs in a regular order of sequence that materially facilitates the general solution. Thus, though the calculations must in all cases be very numerous and laborious, rules can be formulated under which they can be well controlled at every stage and eventually brought to a successful issue. The other geometrical conditions of networks are expressed by equations which are not merely of a more complex form but have no regular order of sequence, for the networks present a variety of forms; thus their introduction would cause much entanglement and complication, and greatly increase the labour of the calculations and the chances of failure. Wherever, therefore, any compound figure occurred, only so much of it as was required to form a chain of single triangles was employed. The figure having previously been made consistent, it was immaterial what part was employed, but the selection was usually made so as to introduce the fewest triangles. The triangulation for final simultaneous reduction was thus made to consist of chains of single triangles only; but all the included angles were "fixed" simultaneously. The excluded angles of compound figures were subsequently harmonized with the fixed angles, which was readily done for each figure *per se*.

This departure from rigorous accuracy was not of material importance, for the angles of the compound figures excluded from the simultaneous reduction had already, in the course of the several independent figural adjustments, been made to exert their full influence on the included angles. The figural adjustments had, however, introduced new relations between the angles of different figures, causing their weights to increase *cæteris paribus* with the number of geometrical conditions satisfied in each instance. Thus, suppose w to be the average weight of the t observed angles of any figure, and n the number of geometrical conditions presented for satisfaction; then the average weight of the angles after adjustment may be taken as $w.\frac{t}{t-n}$, the factor thus being 1·5 for a triangle, 1·8 for a hexagon, 2 for a quadrilateral, 2·5 for the network around the Sironj base-line, &c.

In framing the normal equations between the indeterminate factors λ for the final simultaneous reduction, it would have greatly added to the labour of the subsequent calculations if a separate weight had been given to each angle, as was done in the primary figural reductions; this was obviously unnecessary, for theoretical requirements would now be amply satisfied by giving equal weights to all the angles of each independent figure. The mean weight that was finally adopted for the angles of each group was therefore taken as

$$w.\rho\frac{t}{t-n},$$

ρ being the modulus already indicated in section 12.

The second of the two processes for applying the method of minimum squares having been adopted, the values of the errors y and z of the angles appertaining to any, the pth, triangle were finally expressed by the following equations, which are derived from (18) by substituting u for the reciprocal final mean weight as above determined:—

$$\left.\begin{aligned} y_p&=\frac{u_p}{3}[(2b_p-a_p-c_p)\lambda] \\ z_p&=\frac{u_p}{3}[(2c_p-a_p-b_p)\lambda] \end{aligned}\right\}\text{..................(19).}$$

The most laborious part of the calculations was the construction and solution of the normal equations between the factors λ. On this subject a few hints are desirable, because the labour involved is liable to be materially influenced by the order of sequence adopted in the construction. The normal equations invariably take the form of (4), the coefficients on the diagonal containing summations of squares of the coefficients in the primary equations, while those above and below contain summations of products of the primary quantities, such that the coefficient of the pth λ in the qth equation is the same as that of the qth λ in the pth equation. In practice, as any single angular error only enters a few of the primary equations of condition, many of the coefficients vanish, both in the primary and in the normal equations; and it is an object of great importance so to arrange the normal equations that most blanks shall occur *above* and fewest blanks *between* the significant values on each vertical line of coefficients; in other words, the significant values above and below the diagonal should lie as closely as possible to the diagonal, every value on which is always significant. This advantage is secured when the primary equations are arranged in groups in which each contains a number of angular errors in common and as many as possible of those entering the group on each side. Thus the arrangement must follow the natural succession of the chains of triangles rather than the characteristics of the primary equations; if, for example, all the side equations were grouped together, and all the latitude equations, and so on, great entanglement would arise in the solution of the normal equations, enormously increasing the labour and the chances of failure. The best arrangement was found to be to group the side and the three geodetic equations of each circuit together in the order of sequence of the meridional chains of triangles, and then to introduce the side equations connecting base-lines between the groups with which they had most in common.

The following table (II.) gives the number of equations of condition and unknown quantities—the angular errors—in the five great sections of the triangulation, which were respectively included in the simultaneous general reductions and relegated to the subsequent adjustments of each figure *per se*:—

Section.	Simultaneous.			External Figural.					
	Equations.		Angular Errors.	Equations.				Angular Errors.	No. of Figures.
	Circuit and Base-line.	Tri-angular.		Tri-angular.	Central.	Side.	Toto-partial.		
1. N.W. Quad. ..	23	550	1650	267	104	152	6	761	110
2. S.E. Quad. ..	15	277	831	164	64	92	2	476	68
3. N.E. Quad. ..	49	573	1719	112	56	69	·0	341	50
4. Trigon........	22	303	909	192	79	101	2	547	77
5. S.W. Quad. ..	24	172	516	83	32	52	1	237	40

The magnitudes of the 2481 angular errors determined simultaneously in the first two sections were very small, 2240 being under 0″·1, 205 between 0″·1 and 0″·2, 33 between 0″·2 and 0″·3, 2 between 0″·3 and 0″·4, and 1 between 0″·4 and 0″·5. In the third section, which contained a number of old chains, executed with instruments inferior to the 2 and 3 foot theodolites, they were larger: 780 were under 0″·1, 911 between 0″·1 and 1″·0, 27 between 1″·0 and 2″·0, and 1 between 2″·0 and 2″·1. Thus the corrections to the angles were generally very minute, rarely exceeding the theoretical probable errors of the angles, and therefore applicable without taking any liberties with the facts of observation.

18. *Theoretical Error of any Function of Angles of a Geometrically corrected Triangulation.*—The investigation of such theoretical errors was no easy matter. When first essayed it was generally assumed by mathematicians in England that any attempt to exhibit the theoretical error by a purely algebraical process soon led to results of intolerable complexity, so that it was desirable to introduce numbers as soon as possible for every symbol except the absolute terms of the geometrical or primary equations of condition. But on continuing the algebraical process certain relations were found to exist between the coefficients of the indeterminate factors in the normal equations of the minimum square method and the coefficients of the unknown quantities in the primary equations of condition, which enormously simplified the process and led to a general algebraical expression of no great complexity; it was also found that, the number of primary equations being n, the

labour of calculation by the formula was reduced to an nth of that involved by resorting at once to numbers.

Let F be any function whatever of the corrected angles (X_1-x_1), (X_2-x_2), . . . of a trigonometrical figure; let

$$f_1=\frac{dF}{dX_1},\ f_2=\frac{dF}{dX_2}\ldots;$$

also let $u_1, u_2, \ldots$, symbols hitherto employed to represent the relative reciprocal weights of the observed angles $X_1, X_2, \ldots$, in future represent absolute measures of precision, the $p.e.^2$ of the observed angles; then the following formula expresses the $p.e.$ of any function of the corrected angles rigorously:—

$$\begin{array}{l} p.e.^2 \text{ of } F \\ =f^2.u- \end{array}\left\{\begin{array}{l} +[fa.u]\{[fa.u]A_a+[fb.u]A_b+\ldots+[fn.u]A_n\} \\ +[fb.u]\{[fa.u]B_a+[fb.u]B_b+\ldots+[fn.u]B_n\} \\ +\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ +[fn.u]\{[fa.u]N_a+[fb.u]N_b+\ldots+[fn.u]N_n\} \end{array}\right\}\ (20).$$

The symbols $a, b, \ldots n$ have the same signification as in (3) to (6) of section 13. $A, B, \ldots N$ are coefficients which must be determined in the process of solving the normal equations as follows:—

$$\left.\begin{array}{l} \lambda_a=A_ae_a+A_be_b+\ldots+A_ne_n \\ \lambda_b=B_ae_a+B_be_b+\ldots+B_ne_n \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \lambda_n=N_ae_a+N_be_b+\ldots+N_ne_n \end{array}\right\}\ldots\ldots\ldots\ldots\ldots\ldots(21),$$

where the coefficient represented by any two letters in one order is identical with that represented by the same letters in the reverse order; thus $A_n=N_a$. Hence to find the $p.e.$ of any angle, as (X_1-x_1), in a single triangle we have

$$f_1=1, \text{ and } A_a=\frac{1}{[aa.u]}=\frac{1}{u_1+u_2+u_3};$$

all the other factors vanish, and

$$p.e.^2 \text{ of } (X_1-x_1)=u_1-\frac{u_1}{u_1+u_2+u_3}=p.e.^2 \text{ of } X_1-p.e.^2 \text{ of } x_1.$$

To find the $p.e.$ of the ratio R of either side to the base,—if

$$R=\sin(X_1-x_1)\div\sin(X_2-x_2),$$

then $\quad f_1=R\cot X_1\sin 1'',\ f_2=R\cot X_2\sin 1'',\ f_3=0,$

and $p.e.^2$ of R

$$=R^2\sin^2 1''\left\{u_1\cot^2X_1+u_2\cot^2X_2-\frac{(u_1\cot X_1-u_2\cot X_2)^2}{u_1+u_2+u_3}\right\}\ (22).$$

When the function of the corrected angles is the ratio of the terminal to the initial side of an equilateral triangle or a regular quadrilateral or polygon (either of two sides being taken if the figure has an odd number of exterior sides), then, assuming all the angles to be of equal weight, we have the following values of the $p.e.$'s and the relative weights of the ratios:—

Figure.	p.e.	Weight.	Figure.	p.e.	Weight.
Triangle......	$\pm\cdot 82\sqrt{u}\sin 1''$	1·49	Pentagon	$\pm 1\cdot 21\sqrt{u}\sin 1''$	0·68
Quadrilateral	1·00	1·00	Hexagon	1·29	0·60
Trigon	1·05	0·90	Heptagon	1·41	0·50
Tetragon	1·15	0·75	Octagon	1·57	0·41

In ordinary ground seven single triangles will span about as much as two hexagons and the weights of the terminal sides would be as twenty-one by the former to thirty by the latter. In a flat country two quadrilaterals would not span more than one hexagon, giving terminal side weights as five to six; but in hills a quadrilateral may span as much as any polygon and give a more exact side of continuation. Thus in the Indian Survey polygons predominate in the plains and quadrilaterals in the hills.

The theoretical errors of the lengths and azimuths of the sides, and of the latitudes and longitudes of the stations, at the termini of the chains of triangles or at the circuit closings, might be calculated with the coefficients a, b, and c of x, y, and z in the circuit and base-line equations as the f's, and the known $p.e.$'s of X, Y, and Z and the other data of the figural reductions. Such calculations are, however, much too laborious to be ordinarily undertaken. Thus the exactitude of a triangulation is very generally estimated merely on the evidence of the magnitudes of the differences between the trigonometrical and the measured lengths of the base-lines; for, though the combined influence of angular precision and geometrical configuration is what really governs the precision of the results, it is not readily ascertainable, and is therefore generally ignored. But, when questions as to the intrinsic value of a triangulation arise, the theory of errors should always be appealed to, and its intimations accepted rather than the evidence of base-line discrepancies, which if very small are certainly accidental, and if seemingly large may be no greater than what we should be prepared to expect. Good work has occasionally been redone unnecessarily, and inferior work upheld, because their merits were erroneously estimated. The following formulæ will be found useful in acquiring a fairly approximate knowledge of the magnitude of the errors which theory would lead us to expect, not only in side, but in latitude, longitude, and azimuth also, at the close of any chain of triangles. They indicate rigorously the $p.e.$'s at the terminal end of a chain of equilateral triangles of which all the angles have been measured and corrected and are of equal weight; the results may be made to serve for less symmetrical chains, including networks of varying weight, by the application of certain factors which can be estimated with fair precision in each instance.

Let c be the side length, ϵ the $p.e.$ of the angles, n the number of triangles, and R the ratio (here=1) of the terminal to the initial side, then

$$\left.\begin{array}{r} p.e. \text{ of } R=\epsilon\sin 1''R\sqrt{\tfrac{2}{3}n} \\ p.e. \text{ of azimuth}=\epsilon\sqrt{\tfrac{2}{3}n} \\ p.e. \text{ of either coordinate}=\epsilon c\dfrac{\sin 1''}{6}\sqrt{2n^3+3n^2+10n} \end{array}\right\}\ldots\ldots(23).[1]$$

When the form of the triangles deviates much from the equiangular, the $p.e.$ must be multiplied by a factor increasing up to 1·4 as the angles diminish from 60° to 30°, and a mean value of c must be adopted. When the chain is double throughout, the $p.e.$ must be diminished by a factor taking cognizance of the greater weight of compound figures than of single triangles. When the chain is composed of groups of angles measured with different instruments, a separate value of ϵ must be employed for each group, and the final result obtained from $\sqrt{[p.e.^2]}$. The $p.e.$ of R may be determined rigorously for any chain of single triangles, with angles of varying magnitudes and weights, by (22), with little labour of calculation.

19. *Relations between Theoretical Errors of Base-lines and those of a Triangulation.*—These relations have to be investigated in order to ascertain whether the base-lines may be assumed to be errorless in the general reduction of the triangulation; being fallible quantities, their errors must be included among the unknown quantities to be investigated simultaneously, if their respective $p.e.$'s differ sensibly, or if the $p.e.$'s of their ratios are not materially smaller than those of the corresponding trigonometrical ratios. By (23) the $p.e.$ of the ratio of any two sides of an equilateral triangle is $\epsilon\sin 1''\sqrt{2\div 3}$; but the $p.e.$ of the ratio of two base-lines of equal length and weight is $\eta\sqrt{2}$, where η is the $p.e.$ of either base-line; thus weight of trigonometrical ratio : weight of base-line ratio : $3\eta^2 : \epsilon^2\sin^2 1''$, or as $3:1$ when $\epsilon=\pm 0''\cdot 3$ and $\eta=\pm 1\cdot 5$ millionth parts, which happens generally in the Indian triangulation. But the chains between base-lines were always composed of a large number of triangles, and the average weight of the base-line ratios was about eleven times greater by the direct linear measurements than by the triangulation, even when all the unascertainable constant or accidental errors—as from displacements of mark-stones—which might be latent in the latter were disregarded. Moreover, the base-lines were practically all of the same precision; they were therefore treated as errorless, and the triangulation was made accordant with them.

If a base-line AD be divided at B and C into three equal sections connected together by equilateral triangles, and every angle has been measured with a $p.e.=\epsilon$, the $p.e.$ of any trigonometrical ratio may be put $=\kappa.\epsilon\sin 1''$, κ being a coefficient which has two values for each ratio,—the greater value when the triangulation has been carried along one flank of the line, the smaller when along both flanks, as follows:—for ratio $\frac{BC}{AB}$, $\kappa=1\cdot 41$ and 1; for $\frac{CD}{AB}$, 1·83 and 1·25; for $\frac{AD}{AB}$, 2·94 and 1·99; for $\frac{AD}{BC}$, 2·16 and 1·46. The values for the last two ratios show that, when the length of a base-line is determined partly by measurement and partly by triangulation, the $p.e.$ is smallest if the central section rather than an end section is measured.

If, with linear and angular $p.e.$'s as in the Indian operations, a single section is measured once only, and the lengths of the other sections are derived from it by triangulation, the $p.e.$ of the entire length will be greater than that of the whole line once measured; it will be less if the section is measured oftener than once and the mean taken.

20. *Azimuth Observations in connexion with Principal Triangulation.*—These were invariably determined by measuring the horizontal angle between a referring mark and a circumpolar star, shortly before and after elongation, and usually at both elongations in order to eliminate the error of the star's place. Systematic changes of "face" and of the zero settings of the azimuthal circle were made as in the measurement of the principal angles (§ 9); but the repetitions on each zero were more numerous; the azimuthal levels were read and corrections applied to the star observations for dislevelment. As already mentioned (§ 17), the triangulation was not adjusted, in the course of the final simultaneous reduction, to the astronomically determined azimuths, because they are liable to be vitiated by local attractions; but the azimuths observed at about fifty stations around the primary azimuthal station, which was adopted as the origin of the geodetic calculations, were referred to that station, through the triangulation, for comparison with the primary azimuth. A table was prepared of the differences (observed at the origin − computed from a distance) between the primary and the geodetic azimuths; the differences were assumed to be mainly due to the local deflexions of the plumb-line and only partially to error in the triangulation, and each was multiplied by the factor

$$p=\frac{\text{tangent of latitude of origin}}{\text{tangent of latitude of comparing station}},$$

in order that the effect of the local attraction on the azimuth observed at the distant station—which varies with the latitude and is = the deflexion in the prime vertical × the tangent of the latitude—might be converted to what it would have been had the station been situated in the same latitude as the origin. Each deduction was given a weight, w, inversely proportional to the number of triangles connecting the station with the origin, and the most probable value of the error of the observed azimuth at the origin was taken as

$$x=\frac{[(\text{observed}-\text{computed})\ p\ w]}{[w]}\ldots\ldots\ldots\ldots(24);$$

the value of x thus obtained was $-1''\cdot 1$.

The formulæ employed in the reduction of the azimuth observations were as follows. In the spherical triangle PZS, in which P is the pole, Z the zenith, and S the star, the co-latitude PZ and the polar distance PS are known, and, as the angle at S is a right angle at the elongation, the hour angle and the azimuth at that time are found from the equations

$$\cos P=\tan PS\cot PZ,$$
$$\cos Z=\cos PS\sin P.$$

The interval, δP, between the time of any observation and that of the elongation being known, the corresponding azimuthal angle, δZ, between the two positions of the star at the times of observation and elongation is given rigorously by the following expression—$\tan\delta Z$

$$=-\frac{2\sin^2\frac{1}{2}\delta P}{\cot PS\sin PZ\sin P\{1+\tan^2 PS\cos\delta P+\sec^2 PS\cot P\sin\delta P\}}\ (25),$$

which is expressed as follows for logarithmic computation—

$$\delta Z=-\frac{m\tan Z\cos^2 PS}{1-n+l},$$

where $m=2\sin^2\frac{\delta P}{2}\operatorname{cosec}1''$, $n=2\sin^2 PS\sin^2\frac{\delta P}{2}$, and $l=\cot P\sin\delta P$; l, m, and n are tabulated.

21. *Calculation of Height and Refraction.*—Let A and B (fig. 4) be any two points the normals at which meet at C, cutting the sea-level at p and q; take $Dq=Ap$, then BD is the difference of height; draw the tangents Aa and Bb at A and B, then aAB is the depression of B at A and bBA that of A at B; join AD, then BD is determined from the triangle ABD. The triangulation gives the distance between A and B at the sea-level, whence $pq=c$; thus, putting Ap, the height of A above the sea-level, $=H$, and $pC=r$,

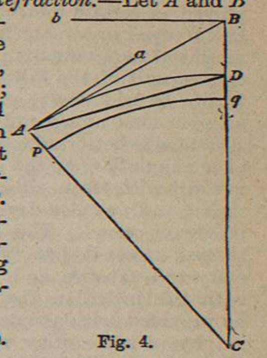

Fig. 4.

$$AD=c\left(1+\frac{H}{r}-\frac{c^2}{24r^2}\right)\ \ldots\ldots\ldots\ldots\ldots(26).$$

Putting D_a and D_b for the actual depressions at A and B, S for the angle at A, usually called the "subtended angle," and h for BD—

$$S=\tfrac{1}{2}(D_b-D_a)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(27),$$

and

$$h=AD\frac{\sin S}{\cos D_b}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(28).$$

The angle at C being $=D_b+D_a$, S may be expressed in terms of a single vertical angle and C when observations have been taken at only one of the two points. C, the "contained arc," $=c\frac{\rho+\nu}{2\rho\nu}\operatorname{cosec}1''$ in seconds. Putting D'_a and D'_b for the observed vertical angles, and ϕ_a, ϕ_b for the amounts by which they are affected by refraction, $D_a=D'_a+\phi_a$ and $D_b=D'_b+\phi_b$; ϕ_a and ϕ_b may differ in amount (see § 10), but as they cannot be separately ascertained they are always assumed to be equal; the hypothesis is sufficiently exact for practical purposes when both verticals have been measured under similar atmospheric conditions. The refractions being taken equal, the observed verticals are substituted for the true in (27) to find S, and the difference of height is calculated by (28); the third term within the brackets of (26) is usually omitted. The mean value of the refraction is deduced from the formula

$$\phi=\tfrac{1}{2}\{C-(D'_a+D'_b)\}\ldots\ldots\ldots\ldots\ldots\ldots\ldots(29).$$

An approximate value is thus obtained from the observations between the pairs of reciprocating stations in each district, and the corresponding mean "coefficient of refraction," $\phi\div C$, is computed for the district, and is employed when heights have to be determined from observations at a single station only. When either of the vertical angles is an elevation $-E$ must be substituted for D in the above expressions.[1]

II. Traversing, as a basis for Survey.—Rectangular Spherical Coordinates.

Traversing is a combination of linear and angular measures in equal proportions: the surveyor proceeds from point to point, measuring the lines between them and at

[1] For an investigation of these formulæ, see Appendix No. 3, vol. vii. of *Account of Operations of Great Trigonom. Survey*, 1882.

[1] In topographical and levelling operations it is sometimes convenient to apply small corrections to observations of the height for curvature and refraction simultaneously. Putting d for the distance, r for the earth's radius, and κ for the coefficient of refraction, and expressing the distance and radius in miles and the correction to height in feet, then correction for curvature $=\frac{2}{3}d^2$; correction for refraction $=-\frac{4}{3}\kappa d^2$; correction for both $=\frac{2-4\kappa}{3}.d^2$.