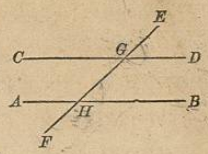
First. If the angles EGD and EHB are equal, the lines are parallel. For, since EGD and EHB are equal, the lines GD and HB must diverge equally from EF; hence, they have the same direction, and are, therefore, parallel (D. 14).



Second. If the alternate angles CGH and GHB are equal, the lines are parallel. For, since CGH equals EGD (Th. II.), when CGH equals GHB, EGD equals GHB; but then the lines are parallel, as has just been shown; hence, the lines are parallel when the alternate angles are equal.

Third. If the sum of the two interior angles GHB and HGD equals two right angles, the lines are parallel. For, $EGD + HGD =$ two right angles (Th. I.); hence, $EGD + HGD = HGD + GHB$ (A. 1). Taking HGD from each, we have $EGD = GHB$; but then the lines are parallel, according to the first part of the theorem; hence, they are parallel when $GHB + HGD$ equals two right angles.

Cor. 1. *If two lines are perpendicular to the same line, they are parallel.* For, if EGD and GHB are both right angles, they are equal, and the lines CD and AB are parallel.

Cor. 2. If two lines are parallel to the same line, they are parallel to each other.

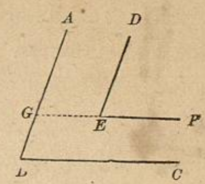
Cor. 3. If the sum of the two interior angles on the same side is less than two right angles, the lines will meet.

THEOREM V.

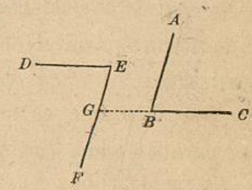
Two angles which have their sides respectively parallel, and lying in the same direction or in opposite directions, are equal.

First. Let the angles ABC and DEF have their sides parallel and lying in the same direction; then will ABC equal DEF. For, prolong FE to G. Then, since AB and

DE are parallel, DEF equals AGE (Th. III.); and since GF and BC are parallel, AGE equals ABC (Th. III.); hence, DEF equals ABC (A. 1).



Second. Let the angles ABC and DEF have their sides parallel and lying in opposite directions; then will ABC equal DEF. For, prolong CB to G. Then ABC equals EGB (Th. III.); but EGB equals DEF, being alternate; hence, ABC equals DEF. Therefore, etc.

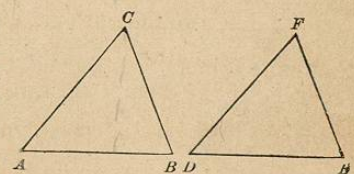


TRIANGLES.

THEOREM VI.

If two triangles have two sides and the included angle of the one equal to two sides and the included angle of the other, each to each, the triangles will be equal in all their parts.

Let the triangles ABC and DEF have the side AB equal to DE, AC to DF, and the angle A equal to the angle D; then will the triangle ABC be equal to the triangle DEF.



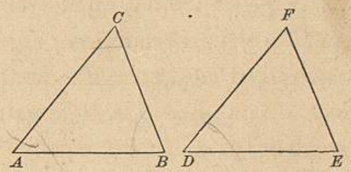
For, apply the triangle ABC to the triangle DEF, placing the side AB upon the equal side DE; then, since the angle A equals the angle D, the side AC will take the direction of DF, and the point C will coincide with F, since the two lines are equal; and the side CB will coincide with the side

FE (A. 9). Therefore, the triangles coincide and are equal in all their parts. Therefore, etc.

THEOREM VII.

If two triangles have two angles and the included side of the one equal to two angles and the included side of the other, each to each, the triangles will be equal in all their parts.

Let ABC and DEF be two triangles having the angle A equal to the angle D, the angle B equal to the angle E, and the included side AB equal to the included side DE; then will the two triangles be equal in all their parts.



For, apply the triangle ABC to the triangle DEF, placing the side AB upon DE, the point A upon D, and the point B upon E; then, since the angle A equals the angle D, the side AC will take the direction DF, and the point C will be found somewhere in the line DF; and since the angle B equals the angle E, the side BC will take the direction EF, and the point C will be found somewhere in EF. Hence, the point C being in the two lines DF and EF, must be at their intersection; consequently, the triangles coincide and are equal in all their parts. Therefore, etc.

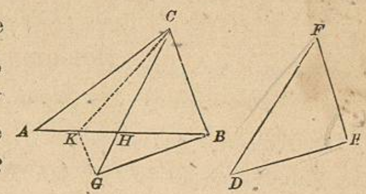
THEOREM VIII.

If two triangles have two sides of the one equal to two sides of the other, each to each, and the included angles unequal, the third side will be greater in the triangle having the greater included angle.

Let ABC and DEF be two triangles in which $AC = DF$,

$BC = EF$ and $ACB > DFE$; then will AB be greater than DE.

For, at the point C make the angle $BCG = EFD$, make $CG = FD$, and draw BG; then will the triangle CGB equal DFE and GB equal DE (Th. VI.). Draw CK, bisecting the angle ACH, and draw also GK; the two triangles ACK and KCG are equal (Th. VI.), and $AK = KG$. Now, $KG + KB > GB$; hence $AK + KB$, or AB, is greater than GB or its equal DE.



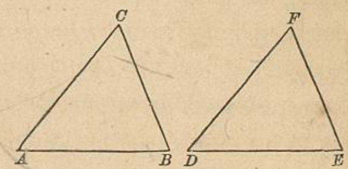
The same demonstration will apply when the point G falls within AB. If it falls on AB, the theorem is true by A. 7.

Cor. The converse of this theorem is also true.

THEOREM IX.

If two triangles have the three sides of the one equal to the three sides of the other, each to each, the triangles will be equal in all their parts.

Let ABC and DEF be two triangles, having AB equal to DE, AC to DF, and BC to EF; then will the triangles be equal in all their parts.

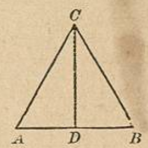


For, since AC and AB are respectively equal to DF and DE, if the angle A were greater than D, BC would be greater than EF (Th. VIII.); and if A were less than D, BC would be less than EF, for the same reason. But BC is equal to EF, therefore the angle A must be equal to D.

In the same way it may be shown that the angle C equals F, and the angle B equals E. Therefore, etc.

THEOREM X.

In an isosceles triangle the angles opposite the equal sides are equal.



Let ABC be an isosceles triangle, having the side AC equal to the side BC; then will the angle A be equal to the angle B.

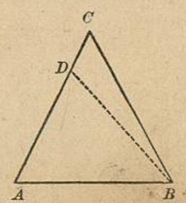
Join the vertex C and the middle point of the base AB; then in the two triangles ADC and CDB, AC equals BC, DC is common, and AD equals DB; hence, the two triangles are equal in all their parts (Th. IX.), and the angle A is equal to the angle B.

Cor. 1. A line drawn from the vertex of an isosceles triangle to the middle point of the base, bisects the vertical angle and is perpendicular to the base; also, a line bisecting the vertical angle is perpendicular to the base and bisects it; also, a line drawn from the vertex perpendicular to the base bisects both the base and the vertical angle.

Cor. 2. Hence, also, an equilateral triangle is equiangular; that is, it has all its angles equal.

THEOREM XI.

Conversely.—*If two angles of a triangle are equal, the sides opposite them are also equal, and the triangle is isosceles.*



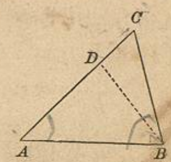
Let ABC be a triangle, having the angle A equal to the angle B; then will the side AC be equal to BC.

For, if AC and CB are not equal, suppose one of them, as AC, to be the greater. Then, take AD equal to BC, and draw DB. Now, in the triangles ABC and ABD, we have the side AD

equal to BC, by construction, the side AB common, and the included angle ABC equal to the included angle DAB, by hypothesis; hence, the two triangles ABD and ABC are equal (Th. VI.); that is, a part is equal to the whole, which is impossible (A. 7). Hence, AC cannot be greater than BC; in the same way it may be shown that AC cannot be less than BC; hence, AC and BC are equal, and the triangle is isosceles. Therefore, etc.

THEOREM XII.

In any triangle the greater side is opposite the greater angle, and, conversely, the greater angle is opposite the greater side.



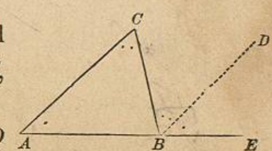
In the triangle ABC, let the angle ABC be greater than CAB; then will AC be greater than BC.

For, draw BD, making the angle $ABD = DAB$; then will $AD = DB$ (Th. XI.). To each add DC and we have $AD + DC = DB + DC$; but $DB + DC > BC$ (A. 10); hence, $AD + DC$, or AC, is greater than BC.

Conversely. Let the side $AC > BC$; then will the angle $ABC > CAB$. For, if $ABC < CAB$, $AC < BC$, from what has just been proved; and if $ABC = CAB$, $AC = BC$ (Th. XI.); but both of these results are contrary to the hypothesis; hence, ABC must be greater than CAB. Therefore, etc.

THEOREM XIII.

In every triangle the sum of the three angles is equal to two right angles.



Let ABC be a triangle; then will the sum of its three angles, A, B, C, be equal to two right angles.

For, prolong AB, and draw BD

parallel to AC; then, since the parallels AC and BD are cut by AE, the angle A is equal to the opposite exterior angle DBE (Th. III.). In like manner, since the parallels are cut by BC, the alternate angles C and CBD are equal; hence, the sum of the three angles of the triangle is equal to the sum of the angles ABC, CBD, DBE; but this latter sum equals two right angles (Th. I. C. 2); therefore, the sum of the three angles of the triangle equals two right angles. Therefore, etc.

Cor. 1. If two angles of a triangle are given, the third will be found by subtracting their sum from two right angles, or 180°; hence, if two triangles have two angles of the one equal to two angles of the other, the third angles will be equal.

Cor. 2. A triangle cannot have more than one right angle; for if there were two the third angle would be zero.

Cor. 3. A triangle can have only one obtuse angle, but must have at least two acute angles.

Cor. 4. In a right-angled triangle the sum of the two acute angles equals one right angle, or 90°.

Cor. 5. In every triangle the exterior angle is equal to the sum of the two interior opposite angles.

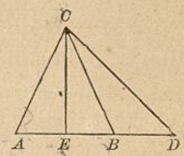
Scholium. This theorem may be demonstrated by drawing a line parallel to either of the other sides of the triangle. Let the pupils be required to do it.

THEOREM XIV.

If from a point without a straight line a perpendicular be let fall on the line and oblique lines be drawn;

1. *The perpendicular will be shorter than any oblique line.*
2. *Any two oblique lines which terminate at equal distances from the foot of the perpendicular are equal.*
3. *The oblique line which terminates at the greater distance from the foot of the perpendicular is the greater.*

Let C be a given point, and AD a given line, CE a perpendicular, and CA, CB, and CD, oblique lines; then,



First. In the triangle AEC, the angle AEC is a right angle, and, consequently, greater than A; therefore, the side CE is shorter than CA (Th. XII.).

Second. Let $AE = EB$; then, since CE is common and the angle $AEC = CEB$, the triangles AEC and CEB are equal (Th. VI.), and AC equals BC.

Third. Let ED be greater than EB; then, since CBE is an acute angle, CBD must be obtuse and BDC acute, and in the triangle CBD, CD is greater than BC, being opposite the greater angle (Th. XII.). Therefore, etc.

Cor. 1. Only one perpendicular can be drawn from the same point to the same straight line.

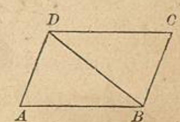
Cor. 2. Two equal oblique lines terminate at equal distances from the foot of the perpendicular.

Cor. 3. A line having two points, each equally distant from the extremities of another line, is perpendicular to that line and bisects it.

QUADRILATERALS.

THEOREM XV.

In any parallelogram, the opposite sides and angles are equal, each to each.



Let $ABCD$ be a parallelogram; then will AB be equal to DC, and AD to BC.

For, draw the diagonal DB. Then, since AB and DC are parallel, the alternate angles ABD and BDC are equal (Th. III.); and since AD and

BC are parallel, the alternate angles ADB and DBC are equal. Hence, the two triangles ABD and DBC have two angles and the included side, DB, of one, equal to two angles and the included side, DB, of the other, each to each; therefore, the triangles are equal (Th. VII.); and the side AB opposite the angle ADB is equal to the side DC opposite the equal angle DBC: hence, also, the side AD equals BC; therefore, the opposite sides of a parallelogram are equal.

Again, since the triangles are equal, the angle A is equal to the angle C; and the angle ADC, which is made up of the two angles ADB and BDC, is equal to the angle ABC, which is made up of the equal angles DBC and ABD. Therefore, etc.

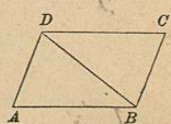
Cor. 1. The diagonal divides the parallelogram into two equal triangles.

Cor. 2. Two parallels included between two other parallels are equal.

Cor. 3. Two parallelograms are equal when they have two sides and the included angle of one, equal to two sides and included angle of the other.

THEOREM XVI.

If the opposite sides of a quadrilateral are equal, each to each, the equal sides are parallel, and the figure is a parallelogram.



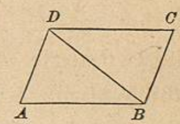
Let $ABCD$ be a quadrilateral, in which AB equals DC, and AD equals BC; then will it be a parallelogram.

For, draw the diagonal DB. Then the triangles ABD and DBC have all the sides of the one equal to all the sides of the other, each to each; therefore the two triangles are equal (Th. IX.); and the

angle ABD opposite the side AD is equal to the angle BDC opposite the equal side BC; therefore, the side AB is parallel to the side DC (Th. IV.). For a like reason, AD is parallel to BC; therefore, the figure $ABCD$ is a parallelogram. Therefore, etc.

THEOREM XVII.

If two sides of a quadrilateral are equal and parallel, the figure is a parallelogram.

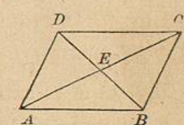


Let $ABCD$ be a quadrilateral, having the sides AB and DC equal and parallel; then will $ABCD$ be a parallelogram.

For, draw the diagonal DB. Then, since AB is parallel to DC, the alternate angles ABD and BDC are equal (Th. III.). Now, the triangles ABD and DBC have the side AB equal to DC, by hypothesis, the side DB common, and the included angles ABD and BDC equal; hence, the triangles are equal (Th. VI.), and the alternate angles ADB and DBC are equal; hence, the sides AD and BC are parallel (Th. IV.), and the figure is a parallelogram. Therefore, etc.

THEOREM XVIII.

The diagonals of a parallelogram bisect each other; that is, divide each other into equal parts.



Let $ABCD$ be a parallelogram, and AC and DB its diagonals; then will AE be equal to EC, and DE to EB.

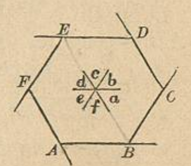
For, since AB and DC are parallel, the angle CDE equals ABE (Th. III.); and also DCE equals EAB; and since AB equals DC, the triangles AEB and DEC have two angles and the included side of the one equal to two angles and the included side of the other, hence, the triangles are equal (Th. VII.), AE equals CE, and DE equals BE; therefore, the diagonals are bisected at E.

ANGLES OF POLYGONS.

THEOREM XIX.

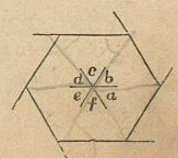
If each side of a convex polygon be produced so as to form one exterior angle at each vertex, the sum of the exterior angles will be equal to four right angles.

Let $ABCDEF$ be a convex polygon, with each side produced so as to form one exterior angle at each vertex; then will the sum of the exterior angles be equal to four right angles.



For, from any point within the polygon, draw lines respectively parallel to the sides of the polygon; the angles contained by the lines about this point will be equal to the exterior angles of the polygon (Th. V.). But the sum of the angles formed about a point equals four right angles (Th. II. C. 2); hence, the sum of the exterior angles of a polygon equals four right angles. Therefore, etc.

Cor. 1. The sum of the interior angles of a polygon is equal to twice as many right angles as the polygon has sides, less four right angles.



The sum of each exterior and interior angle equals two right angles, and there are as many of each as the polygon has sides; hence, the sum of all the exterior and interior angles equals two right angles taken as many times as there are sides of the polygon. But the sum of the exterior angles equals four right angles; hence, the sum of the interior angles equals two right angles taken as many times as the polygon has sides, minus four right angles.

Cor. 2. The sum of the interior angles of a quadrilateral equals 2 right angles multiplied by 4, minus 4 right angles, which is $8 - 4$, or 4 right angles. In a rectangle each angle is a right angle.

Cor. 3. The sum of the angles of a pentagon equals $2 \times 5 - 4 = 6$ right angles. Each angle of an equiangular pentagon is $\frac{1}{5}$ of 6 or $\frac{6}{5}$ of a right angle, or 108°.

Cor. 4. The sum of the angles of a hexagon equals $2 \times 6 - 4 = 8$ right angles. Each angle of an equiangular hexagon is $\frac{4}{3}$ of a right angle, or 120°.

Cor. 5. In polygons of the same number of sides, the sum of the angles is the same. In equiangular polygons, each angle equals the sum divided by the number of sides.

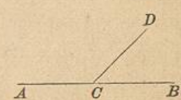
Scholium. This theorem is true at whichever extremity the sides are produced.

PRACTICAL EXAMPLES.

A common deficiency of pupils in the study of Geometry, is their inability to make a practical application of their knowledge. To remedy this, practical examples should be given, either in connection with the theorems or at the close of each book. The following problems may be used in either of these ways which the teacher may prefer.

1. If one line meet another line at an angle of 60°, what is the value of the adjacent angle?

SOLUTION.—If the line DC meets AB, making the angle DCB equal to 60°, the angle ACD will equal $180° - 60°$, or 120°, since $ACD + DCB = 180°$.



2. If two lines meet a third at the same point, making angles equal to 30° and 80° respectively, required the angle between the two lines.

3. How many degrees in each angle of a rectangle?

4. How many degrees in each angle of an equilateral triangle?

5. If two angles of a triangle are 43° and 75° respectively, what is the other angle?

6. If two angles of a triangle are each 45°, what is the other angle, and what is the kind of triangle?

7. If one angle of a triangle is 60°, what is each of the other two, if equal, and what is the kind of triangle?

8. If one of the two equal angles of a triangle is 30°, what is each of the other angles?

9. Required the number of degrees in each angle of an equiangular pentagon.

10. Required the number of degrees in each angle of an equiangular hexagon.

11. In a triangle whose angles are A, B, C, what is each angle if A is twice and B three times C?

12. In the preceding problem, what is the kind of triangle?

13. Required each angle of an isosceles triangle, if the unequal angle equals twice the sum of the other two.

14. Required the value of each exterior angle of an equiangular octagon.

EXERCISES FOR ORIGINAL THOUGHT.

We now give some theorems to exercise the pupil in original thought. The importance of such exercises cannot be overestimated. Much of the discipline of Geometry is lost by the pupil memorizing the demonstrations given in the book. One can become a good geometer only by trying his powers with new theorems and problems, and endeavoring to find out demonstrations and solutions for himself.

These theorems may be given upon review, one of them in connection with the regular lesson; or, if the teacher prefer, the lesson may consist wholly of them. With classes whose time for the study is limited, they may be omitted.

1. If the equal sides of an isosceles triangle be produced, the two obtuse angles below the base will be equal.

2. If the three sides of an equilateral triangle be produced, all the external acute angles will be equal, and all the obtuse angles will be equal.

3. Either side of a triangle is greater than the difference between the other two.

4. If a line be drawn bisecting an angle, any point of the bisecting line is equally distant from the sides of the angle.

5. Prove that the diagonals of a rectangle are equal.

6. If the diagonals of a quadrilateral bisect each other at right angles, the figure is a rhombus or square.

7. If a line joining two parallels be bisected, any other line through the point of bisection and joining the two parallels, is also bisected at that point.

8. If from any point within a triangle, two straight lines be drawn to the extremities of any side, their sum will be less than the sum of the other two sides of the triangle.

9. If a line is perpendicular to another line at its middle point,—1. Any point in the perpendicular will be equally distant from the extremities. 2. Any point out of the perpendicular will be unequally distant from the extremities.