$A : B :: E : F$; we have

$A \times D = B \times C$, and

$A \times F = B \times E$; adding

to these, $A \times B = A \times B$, we have,

$$A \times B + A \times D + A \times F = A \times B + B \times C + B \times E,$$

or, $A \times (B + D + F) = B(A + C + E)$;

whence, $A : B :: A + C + E : B + D + F$.

PRACTICAL EXERCISES.

1. If the first three terms of a proportion are 12, 14, and 18, what is the fourth term? *Ans.* 21.

2. Given the proportion $3 : 12 :: 5 : 20$; what proportion have we by composition?

3. Find a mean proportional to 12 and 27; to m and n. *Ans.* 18; $\sqrt{m \times n}$.

4. If the ratio of A to B is $\frac{1}{3}$, what is the ratio of $3A$ to $2B$? *Ans.* $\frac{2}{3}$.

5. If the ratio of $3A$ to $2B$ is $\frac{3}{4}$, what is the ratio of A to B? *Ans.* $\frac{1}{2}$.

6. What proportion is deducible from the equation $M \times N = A^2 - B^2$. *Ans.* $M : A + B :: A - B : N$.

7. What proportion is deducible from the equation $(C + D) \times A = (A + B) \times C$? *Ans.* $A : B :: C : D$

THEOREMS FOR ORIGINAL THOUGHT.

1. If $a : b :: c : d$, prove that $am : bn :: cm : dn$.
2. If $a : b :: c : d$, prove that $\frac{a}{m} : \frac{b}{n} :: \frac{c}{m} : \frac{d}{n}$.
3. If $a : b :: c : d$, prove that $a : a + b :: c : c + d$.
4. If $a : b :: c : d$, prove that $a + b : a - b :: c + d : c - d$.
5. If $a : b :: c : d$ and $m : c :: n : d$, prove that $a : b :: m : n$.

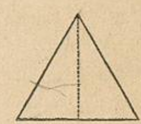
BOOK III.

AREAS AND RELATIONS OF POLYGONS.

1. THIS book treats of the area of polygons and their relation to each other.

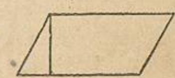
2. The AREA of a polygon is its quantity of surface: it is expressed by the number of times which the polygon contains some other area assumed as a *unit of measure.*

3. The ALTITUDE OF A TRIANGLE is the perpendicular distance from the vertex of either angle to the opposite side, or the opposite side produced.



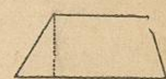
The vertex of the angle from which the altitude is drawn is called the *vertex of the Triangle;* the opposite side is called the *base* of the triangle.

4. The ALTITUDE OF A PARALLELOGRAM is the perpendicular distance between two opposite sides.



These opposite sides are called *bases*, one is the *upper base*, the other the *lower base.*

5. The ALTITUDE OF A TRAPEZOID is the perpendicular distance between its parallel sides.



These sides are called bases; one is called the *upper base*, the other the *lower base.*

6. SIMILAR POLYGONS are those which are mutually equiangular, and in which the corresponding sides are proportional.

Corresponding sides or *angles* are those which are like placed. They are sometimes called *homologous.*

7. EQUIVALENT POLYGONS are those which are equal in *area.* Polygons which, being applied to each other, coincide throughout their whole extent, are said to be *equal in all their parts,* or simply *equal.*

The term *equal* is often used in geometry for *equivalent,* meaning equal in area. The sign of equality, =, is used in comparing equivalent figures, and is read "equals," or "is equal to."

8. A REGULAR POLYGON is a polygon which is both equilateral and equiangular.

ANALYSIS.—The first object of this book is to find the area of polygons. It begins with the area of a rectangle, assuming as a unit of measure a square whose side is a measure of the sides of the given rectangle. From the area of the rectangle we pass to the area of any parallelogram, thence to the area of a triangle, and from this to the area of any plane figure.

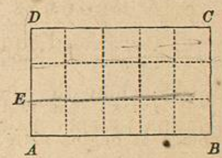
The book also treats of the relations of the squares on the sides of triangles, and the relation of the angles, sides, and area of similar polygons, to each other. It is one of the most interesting and practical books of Geometry.

AREA OF POLYGONS.

THEOREM I.

The area of a rectangle is equal to the product of its base and altitude.

Let $ABCD$ be a rectangle; then will its area be equal to the product of its base and altitude.



For, let the line AE be a unit of measure of the base and altitude, and suppose it contained any number as 5 times in the base and 3 times in the altitude; then, divide AB into 5 equal parts and AD into 3 equal parts, and through the points of division draw lines parallel, respectively, to the sides AB and AD; then will the rectangle be divided into equal squares. For, their sides are equal (B. I. Th. XV. C. 2); their angles are right (B. I. Th. III.); hence, the figures are equal squares (B. I. Th. XV. C. 3).

Now, the whole number of these squares is equal to the number in one row multiplied by the number of rows, which is the same as the number of linear units in the base multiplied by the number of linear units in the altitude; and the same is evidently true for any other numbers than 3 and 5. Hence, the area of $ABCD$ equals $AB \times AD$.

Since this is true when the linear unit of measure is any length, it is true when it becomes exceedingly small, and is, therefore, true when it becomes infinitely small, as it must when the two sides are incommensurable. There fore, the area of a rectangle is equal to the product of its base and altitude.

Cor. 1. *Rectangles are to each other as the products of their bases and altitudes.* For, let AB and AD represent the base and altitude of one rectangle, and EF and EH the base and altitude of another; then we will have the identical proportion, $ABCD : EFGH :: AB \times AD : EF \times EH$.

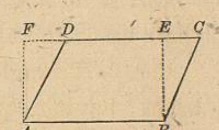
Cor. 2. *Rectangles having equal bases are to each other as their altitudes.* For, suppose the bases AB and EF equal; then, cancelling the equal factor in the second couplet, we have, $ABCD : EFGH :: AD : EH$.

Cor. 3. *Rectangles having equal altitudes are to each other as their bases.* For, suppose the altitudes AD and EH are equal; then, by cancelling the equal factor in the second couplet of Cor. 1, we have, $ABCD : EFGH :: AB : EF$.

THEOREM II.

The area of a parallelogram is equal to the product of its base and altitude.

Let $ABCD$ be a parallelogram, AB its base, and EB its altitude; then will its area be equal to $AB \times EB$.



For, at the points A and B draw the two perpendiculars AF and BE, and complete the rectangle $ABEF$. Then, the angle ADF equals the angle BCE, and FAD equals CBE (B. I. Th. V.); hence, the two triangles are equal (B. I. Th. VII.); therefore, $ABED + BCE$ is equal to $ABED + ADF$, or the parallelogram $ABCD$ is equal to the rectangle $ABEF$. But the area of the rectangle is equal to $AB \times BE$; hence, the area of the parallelogram is equal to $AB \times BE$. Therefore, etc.

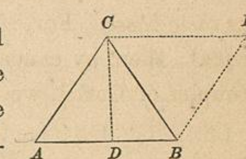
Cor. 1. Parallelograms are to each other as the products of their bases and altitudes.

Cor. 2. Parallelograms having equal altitudes are to each other as their bases; and parallelograms having equal bases are to each other as their altitudes.

THEOREM III.

The area of a triangle is equal to half the product of its base and altitude.

Let ABC be a triangle, AB its base, and CD its altitude; then will its area be equal to half the product of its base and altitude.



For, draw BE parallel to AC, and CE parallel to AB, completing the parallelogram $ABEC$; then will the triangle ABC be one-half the parallelogram $ABEC$ (B. I. Th. XV. C. 1). But the area of the parallelogram is equal to $AB \times CD$; hence, the area of the triangle is equal to $\frac{1}{2} AB \times CD$. Therefore, etc.

Cor. 1. Triangles are to each other as the products of their bases and altitudes.

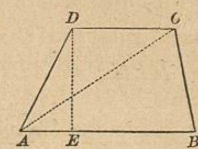
Cor. 2. Triangles having equal altitudes are as their bases; having equal bases, they are as their altitudes.

THEOREM IV.

The area of a trapezoid is equal to one-half the sum of the parallel sides multiplied by the altitude.

Let $ABCD$ be a trapezoid, AB and DC its parallel sides, and DE its altitude; then will its area equal $\frac{1}{2}(AB + DC) \times DE$.

For, draw the diagonal AC, dividing the trapezoid into the two triangles ABC and ADC, the altitude of each being DE. The area of ABC is $\frac{1}{2}AB \times DE$, the area of ADC is $\frac{1}{2}DC \times DE$; hence, the area of $ABCD$, the sum of these triangles, is $\frac{1}{2}AB \times DE$ plus $\frac{1}{2}DC \times DE$, which is $\frac{1}{2}(AB + DC) \times DE$. Therefore, etc.



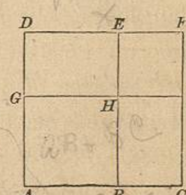
SQUARES ON LINES.

THEOREM V.

The square described on the sum of any two lines is equal to the sum of the squares described on the lines, plus twice the rectangle of the lines.

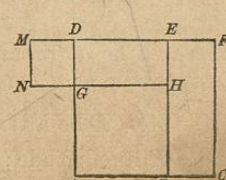
Let AB and BC be two lines, and AC their sum; then will

$$\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2 + 2\,AB \times BC.$$



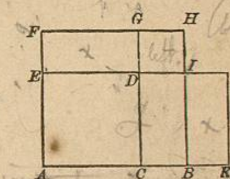
For, on AC construct the square $ACFD$ and on AB construct the square $ABHG$; prolong BH to E and GH to I. Now, it is readily seen that $HIFE$ is the square of BC, also that $BCIH$ equals the rectangle on AB and BC, and $GHED$ equals the rectangle on AB and BC; therefore, the square $ACFD$ consists of the square on the two lines plus twice the rectangle of the two lines.

Cor. 1. *The square of the difference of two lines equals the sum of the squares of the lines, minus twice the rectangle of the lines.* For, construct a square on AC and on AB, prolong BH to E and HG to N, making $GN = BC$, and construct the square GM; then the rectangles BF and HM are each equal to $AC \times BC$. Now, $ACFD + NGDM - BCFE - HEMN = ABHG$; or,



$$\overline{AC}^2 + \overline{BC}^2 - 2AC \times BC = \overline{AB}^2.$$

Cor. 2. *The rectangle contained by the sum and difference of two lines equals the difference of their squares.* For, construct a square on AB and on AC, take $BK = BC$, and construct the rectangle AL; then $AK = AB + BC$, $AC = AB - BC$, $BKLI = DGFE$, and $AKLE = (AB + BC)\ (AB - BC)$. Now, $AKLE = ABIE + DF$, which equals $ABHF - DIHG$; hence,



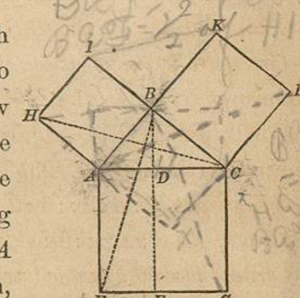
$$(AB + BC)(AB - BC) = \overline{AB}^2 - \overline{BC}^2.$$

THEOREM VI.

The square described on the hypothenuse of a right-angled triangle is equal to the sum of the squares described on the other two sides.

Let ABC be a triangle, right-angled at B; then will $\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2$.

For, construct squares on each of the sides, draw BD parallel to AF and produce it to E, and draw the diagonals BF and HC. The two triangles HAC and BAF are equal; for, AC equals AF, being sides of the same square, HA equals AB, for the same reason, and the angle HAC equals the angle BAF, both being equal to a right angle plus BAC; hence, the triangle HAC equals BAF.



The triangle BAF is one-half of the rectangle $AFED$, since it has the same base and the same altitude (Th. III.);

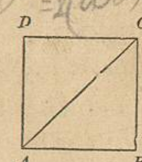
also, since IBC is a straight line, the triangle HAC and square $ABIH$ have the same altitude; hence, the triangle is one-half of the square (Th. III.). But these two triangles BAF and HAC are equal; hence, the rectangle $AFED$ is equal to the square $ABIH$. In the same manner we may prove that the rectangle $EGCD$ is equal to the square $BCLK$; hence, the sum of the two rectangles, or the square on AC is equal to the sum of the two squares HB and BL. Therefore, etc.

Cor. 1. *The square of either side about the right angle is equal to the square of the hypothenuse diminished by the square of the other side.*

For, since $\overline{AB}^2 + \overline{BC}^2 = \overline{AC}^2$, we have, by transposing, $\overline{AB}^2 = \overline{AC}^2 - \overline{BC}^2$.

Cor. 2. *The square of the diagonal of a square is equal to twice the square of the side of the square.*

Let $ABCD$ be a square, then will $\overline{AC}^2 = 2\,\overline{AB}^2$. For, we have, by the theorem, $\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2$; but $\overline{AB}^2$ equals $\overline{BC}^2$; hence, by substitution, we have $\overline{AC}^2 = \overline{AB}^2 + \overline{AB}^2$, or, $\overline{AC}^2 = 2\,\overline{AB}^2$.



Cor. 3. *The side of a square is to its diagonal as* 1 *is to the square root of* 2.

For, since $2\,\overline{AB}^2 = \overline{AC}^2$, or, $2 \times \overline{AB}^2 = \overline{AC}^2 \times 1$, we have the proportion (B. II. Th. II.),

$AB^2 : AC^2 :: 1 : 2$; extracting the square root,

we have, $AB : AC :: 1 : \sqrt{2}$.

Cor. 4. *Two right-angled triangles are equal in all their parts when they have two corresponding sides respectively equal.*

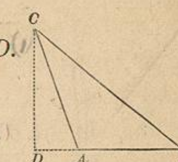
NOTE.—This is the celebrated Pythagorean proposition, so called because it was discovered by Pythagoras. It is also known as the 47th of Euclid, that being its number in the first book of Euclid's Elements.

THEOREM VII.

In any obtuse-angled triangle, the square of the side opposite the obtuse angle is equal to the sum of the squares of the other two sides, plus twice the product of the base into the distance from the vertex of the obtuse angle to the foot of the perpendicular drawn from the vertex of the angle opposite the base to the base produced.

Let ABC be a triangle, of which A is an obtuse angle, AB its base, and CD the perpendicular drawn to the base produced; then will

$$\overline{BC}^2 = \overline{AC}^2 + \overline{AB}^2 + 2\,AB \times AD.$$



For, in the right-angled triangle DBC,

we have, $\overline{BC}^2 = \overline{DC}^2 + \overline{DB}^2$;

but $DB = AB + AD$;

hence, $\overline{DB}^2 = \overline{AB}^2 + \overline{AD}^2 + 2\,AB \times AD$ (Th. V.).

Hence, $\overline{BC}^2 = \overline{DC}^2 + \overline{AB}^2 + \overline{AD}^2 + 2\,AB \times AD$.

But, $\overline{DC}^2 + \overline{AD}^2 = \overline{AC}^2$.

Hence, $\overline{BC}^2 = \overline{AB}^2 + \overline{AC}^2 + 2\,AB \times AD$.

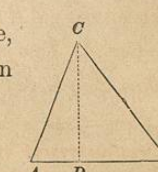
Cor. 1. If the angle CAB becomes a right angle, AD becomes zero, and we have, $\overline{BC}^2 = \overline{AB}^2 + \overline{AC}^2$.

THEOREM VIII.

In any triangle, the square of a side opposite an acute angle is equal to the sum of the squares of the other two sides, minus twice the product of the base and the distance from the vertex of the acute angle to the foot of the perpendicular let fall upon the base or the base produced.

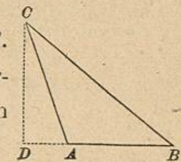
Let ABC be any triangle, B an acute angle, AB its base, and CD the perpendicular; then will

$$AC^2 = \overline{AB}^2 + \overline{BC}^2 - 2\,AB \times BD.$$



For, in the right-angled triangle ADC,

we have, $\overline{AC}^2 = \overline{DC}^2 + \overline{AD}^2;$
but $AD = AB - DB;$
hence, $\overline{AD}^2 = \overline{AB}^2 + \overline{DB}^2 - 2\,AB \times DB$ (Th. V. C. 1).
Hence, $\overline{AC}^2 = \overline{DC}^2 + \overline{AB}^2 + \overline{DB}^2 - 2\,AB \times DB.$
But, $\overline{DC}^2 + \overline{DB}^2 = \overline{BC}^2$, in BDC.
Hence, $\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2 - 2\,AB \times DB.$



The same may also be shown if the perpendicular meets the base produced, as in the second figure. Therefore, etc.

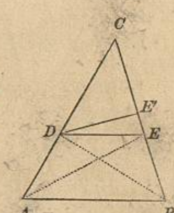
NOTE.—This 8th Proposition can be very prettily drawn from the 7th, by transposing the terms of the 7th, and reducing. Let the pupil try it.

THEOREM IX.

In any triangle, a straight line drawn parallel to the base divides the other sides proportionally.

Let ABC be a triangle, and DE a line parallel to the base; then will

$$CD : DA :: CE : EB.$$



For, draw AE and DB; then, since the two triangles AED and DEC have their bases in the same line and their vertices at the same point E, they have the same altitude; hence, they are to each other as their bases (Th. III. C. 2), or,

$$AED : DEC :: AD : DC.$$

For a similar reason, the triangles BED and DEC are to each other as their bases; hence, we have,

$$BED : DEC :: BE : EC.$$

But the triangles AED and BED have the same base DE and the same altitude, since their vertices are in the line AB parallel to DE; hence, they are equal (Th. III.), and

the two proportions have a couplet in each equal; hence, the remaining terms are proportional (B. II. Th. VIII.), and we have,

$$AD : DC :: BE : EC.$$

Therefore, etc.

Cor. 1. By composition, we have,

$$AD + DC : AD :: BE + EC : BE,$$

or, $AC : AD :: BC : BE;$ and, in the same way,

$$AC : DC :: BC : EC.$$

Cor. 2. Conversely, *If a line divides two sides of a triangle proportionally, it will be parallel to the third side.*

Let DE divide CA and CB proportionally; then, if DE is not parallel to AB, draw DE' parallel to AB. Now $CA : CD :: CB : CE'$ (Cor. 1), but $CA : CD :: CB : CE$ byhypothesis; hence, $CE' = CE$, which is absurd. Therefore, etc.

Cor. 3. Since $DEC : AEC :: DC : AC$ and $AEC : ABC :: EC : BC$, and also, $DC : AC :: EC : BC;$ therefore,

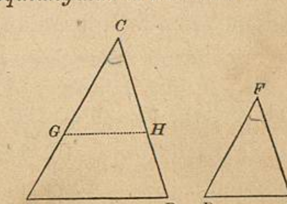
$$DEC : AEC :: AEC : ABC.$$

That is, *the triangle AEC is a mean proportional between DEC and ABC.*

SIMILAR TRIANGLES.

THEOREM X.

Triangles which are mutually equiangular are similar.



Let ABC and DEF be two triangles having the angle $A = D$, the angle $B = E$, and $C = F$; then will they be similar.

For, on AC take CG equal to FD, and on BC take CH equal to

FE, and draw *GH;* then the triangle *CGH* will be equal to *FDE* (B. I. Th. VI.) and the angle *CGH* will equal *FDE;* hence, the angle *CGH* equals *CAB*, and *GH* is parallel to *AB* (B. I. Th. IV.). Hence, we have (Th. IX. C. 1).

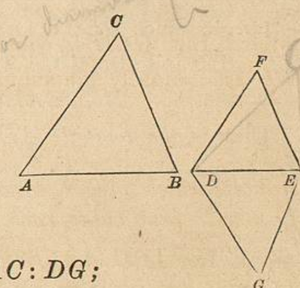
$$AC : BC :: GC : HC, \text{ or,}$$
$$AC : BC :: DF : EF;$$

and the same may be shown for the sides containing the other equal angles; hence, the triangles are similar (D. 6). Therefore, etc.

THEOREM XI.

Triangles which have their corresponding sides proportional are similar.

Let *ABC* and *DEF* be two triangles having their corresponding sides proportional; then will they be similar.



For, if they are not similar, suppose some other triangle, as *DEG*, to be constructed upon the side *DE*, similar to *ABC*. Then, by the preceding theorem, we have,

$$AB : DE :: AC : DG;$$

but, by hypothesis,

$$AB : DE :: AC : DF; \text{ hence,}$$

we have, $$DG = DF.$$

In the same way, it may be shown that

$$EG = EF.$$

Hence, the triangles *DEG* and *DEF* must be equal in all their parts (B. I. Th. IX.), and, therefore, mutually equiangular; hence *ABC* and *DEF* are mutually equiangular, and, consequently, similar. Therefore, etc.

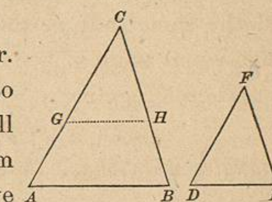
THEOREM XII.

Triangles which have an angle in each equal, and the sides including them proportional, are similar.

Let *ABC* and *DEF* be two triangles having the angle *C* equal to the angle *F*, and

$$AC : BC :: DF : EF;$$

then will the triangles be similar.



For, apply the angle *DFE* to *ACB*, and the triangle *DFE* will take the position *GCH*, and, from the proportion above, we shall have

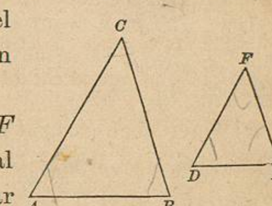
$$AC : BC :: GC : HC;$$

hence, *GH* is parallel to *AB* (Th. IX. C. 2), and the triangles *GCH* and *ACB* mutually equiangular, and therefore similar. But, *GCH* is equal to *DFE;* therefore, *ACB* and *DFE* are mutually equiangular, and similar.

THEOREM XIII.

Triangles which have their sides parallel, each to each, or perpendicular, each to each, are similar.

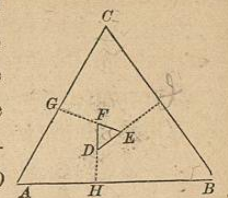
First. Let *ABC* and *DEF* be two triangles having the side *AB* parallel to *DE*, *AC* parallel to *DF*, and *CB* parallel to *FE;* then will they be similar.



For, since *AC* is parallel to *DF* and *AB* to *DE*, the angle *A* is equal to *D* (B. I. Th. V.); for a similar reason *C* is equal to *F* and *B* to *E;* hence, the triangles are mutually equiangular, and, consequently, similar.

Second. Let *ABC* and *DEF* be two triangles having their sides respectively perpendicular; then will they be similar.

For, produce the sides of DEF till they meet the sides of ABC. In the trapezium $GEIC$, the sum of the four angles equals four right angles (B. I. Th. XIX. C. 2), and since two of the angles are right angles, the sum of the angles C and GEI equals two right angles. But the sum of GEI and FED equals two right angles (B. I. Th. I.); hence, the angle FED equals the angle C. In the same way it may be shown that FDE equals B, and DFE equals A; hence, the two triangles are mutually equiangular, and, consequently, similar. Therefore, etc.



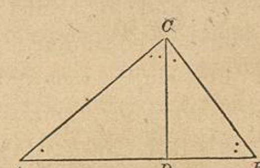
THEOREM XIV.

If, in a right-angled triangle, a line be drawn from the vertex of the right angle perpendicular to the hypothenuse;

1. *The two triangles thus formed will be similar to the given triangle and to each other.*
2. *Each side about the right angle will be a mean proportional between the hypothenuse and adjacent segment.*
3. *The perpendicular will be a mean proportional between the two segments of the hypothenuse.*

Let ABC be a right-angled triangle, C the right angle, and CD the perpendicular; then,

First. The triangles ACD and ABC have each a right angle, and the angle A common; hence, the remaining angles are equal, and the triangles are similar (Th. X.). In the same manner, we show BCD and ABC equiangular and similar; and then ADC and BDC, being both similar to ABC, are similar to each other.



Second. The two triangles being similar to the given one, we have,

$$AB : AC :: AC : AD,$$

and also,

$$AB : BC :: BC : BD.$$

Therefore, etc.

Third. The two triangles being similar, we have,

$$AD : DC :: DC : DB.$$

Therefore, etc.

RELATION OF POLYGONS.

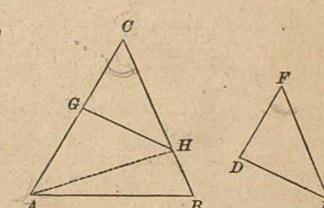
THEOREM XV.

Triangles which have an angle in each equal, are to each other as the products of the sides including those equal angles.

Let ABC and DEF be two triangles having the angle F equal to the angle C; then will

$$ABC : DEF :: AC \times BC : DF \times EF.$$

For, place the angle F on its equal C, and the triangle DEF will take the place GCH; then draw AH. Now, since the triangles AHC and GHC have their bases AC and GC in the same line AC, and vertices at H, they have the same altitude, and are to each other as their bases; hence,



$$AHC : GHC :: AC : GC.$$

Also, since AHC and ABC have their bases HC and BC in the same line, and vertices at the point A, they have the same altitude, and are as their bases; hence,

$$ABC : AHC :: BC : HC;$$

multiplying the corresponding terms of these two proportions together, and omitting the common factor AHC, we have,

$$ABC : GHC :: AC \times BC : GC \times HC,$$

or,

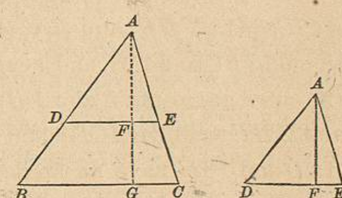
$$ABC : DEF :: AC \times BC : DF \times FE.$$

Therefore, etc.

THEOREM XVI.

Similar triangles are to each other as the squares of their homologous sides.

Let ABC and ADE be two similar triangles; then will they be to each other as the squares of any two homologous sides. Draw the altitudes AG and AF; then, since the triangles are as the product of their bases and altitudes (Th. III. C. 1), we have,



$$ABC : ADE :: BC \times AG : DE \times AF.$$

But, by similar triangles, we have,

$$BC : DE :: AB : AD,$$

and,

$$AG : AF :: AB : AD;$$

hence,

$$BC \times AG : DE \times AF :: \overline{AB}^2 : \overline{AD}^2.$$

Comparing this with the first proportion, we have,

$$ABC : ADE :: \overline{AB}^2 : \overline{AD}^2.$$

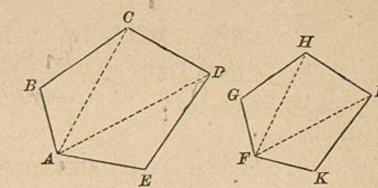
THEOREM XVII.

Similar polygons may be divided into the same number of triangles, similar each to each, and similarly situated.

Let $ABCDE$ and $FGHIK$ be two similar polygons, having the angle A equal to the angle F, B to G, C to H, etc.; then

can they be divided into the same number of similar triangles similarly situated.

From the homologous angles A and F draw the diagonals AC, AD, and FH, FI. Since the polygons are similar, the triangles ABC and FGH have the angles B and G equal, and the sides about these angles proportional; they are, therefore, similar (Th. XII.).

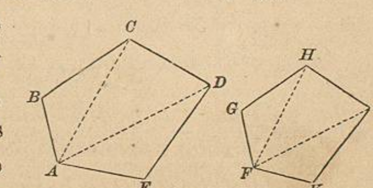


Since the triangles ABC and FGH are similar, the angle ACB equals FHG, and the sides AC and FH are proportional to BC and GH, and hence to CD and HI. If we take the equal angles ACB and FHG from the equal angles BCD and GHI, we have ACD equal to FHI; hence, the triangles ACD and FHI have an angle in each equal, and the sides including these angles proportional; they are, therefore, similar (Th. XII.). In a similar manner, it may be shown that ADE and FIK are similar. Therefore, etc.

THEOREM XVIII.

The perimeters of similar polygons are to each other as any two homologous sides; and the polygons are to each other as the squares of those sides.

Let $ABCDE$ and $FGHIK$ be two similar polygons; then will their perimeters be to each other as any two homologous sides, and their areas be as the squares of those sides.



First. Since the polygons are similar, we have,

$AB : FG :: BC : GH :: CD : HI$, etc.;

hence (B. II. Th. XII.),

$AB + BC + CD +$ etc. $: FG + GH + HI +$ etc. $:: AB : FG$;

or, the perimeter of the first to the perimeter of the second as any side of the first to the homologous side of the second.

Second. Since the triangles are respectively similar, we have, $ABC : FGH :: \overline{AC}^2 : \overline{FH}^2$;

and also, $ACD : FHI :: \overline{AC}^2 : \overline{FH}^2$;

hence, we have, $ABC : FGH :: ACD : FHI$.

In a similar manner, we find,

$$ACD : FHI :: ADE : FIK.$$

Hence (B. II. Th. XII.), the sum of the antecedents, $ABC + ACD + ADE$, is to the sum of the consequents, $FGH + FHI + FIK$, as any antecedent ABC is to its consequent FGH; and, since ABC is to FGH as $\overline{AB}^2$ to $\overline{FG}^2$, we have,

$$ABCDE : FGHIK :: \overline{AB}^2 : \overline{FG}^2.$$

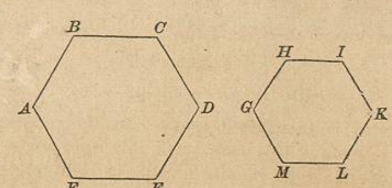
Therefore, etc.

Cor. The perimeters are to each other as any two *homologous lines*, and the polygons are as the squares of those lines.

THEOREM XIX.

Regular polygons of the same number of sides are similar figures.

Let $ABCDEF$ and $GHIKLM$ be two regular polygons of the same number of sides; then will they be similar.



For, the corresponding angles in each are equal (B. I. Th. XIX. C. 5), and the corresponding sides are proportional, since they

are equal; hence, the polygons are similar (D. 6). Therefore, etc.

Cor. Since regular polygons of the same number of sides are similar figures, their perimeters are proportional to any homologous lines, and their areas are as the squares of those lines.

PRACTICAL EXAMPLES.

1. Required the perimeter and area of a square whose sides are each 20 inches.

2. Required the perimeter and area of a rectangle whose sides are respectively 18 and 24 inches.

3. What is the area of a parallelogram whose base is 16 inches and altitude 12 inches?

4. A man has a board in the form of a triangle; what is its area if the base is 9 feet and the altitude 18 inches?

5. A farmer has a field in the form of a trapezoid; the two parallel sides are 40 and 60 rods, and the distance between them 32 rods; required its area.

*6. Required the hypothenuse of a right-angle triangle, the two sides being 3 and 4 inches respectively.

7. The sides of a triangle are 18 and 21, and the base 24; what are the sides of a similar triangle whose base is 8?

8. A man had a lot in the form of a right-angle triangle; the hypothenuse is 78 and one side 30; required the other side and the area.

9. A ladder 65 feet long is placed against a house, so that its foot is 25 feet from the house; how high does it reach?

10. A pole was broken 75 feet from the top, and fell so that the end struck 60 feet from the foot; required the length of the pole.

11. A has a triangular piece of ground, the base of the triangle being

* The numbers 3, 4, and 5 are the smallest integers which can express the relation of the three sides of a right-angle triangle. It is evident that we may have an infinite number of right-angle triangles with their sides in this ratio. Thus, 6, 8, 10; 9–12, 15, etc. Another integral relation of sides is 5, 12, 13.

20 rods; what is the base of a similarly-shaped lot containing 4 times as much land? *Ans.* 40 rods.

12. A man has a lot 40 rods long and 23 rods wide; what are the dimensions of a similar lot 9 times as large? *Ans.* 120; 69.

13. A ladder, whose length is 91 feet, stands close against a building; how far must it be drawn out at the bottom that the top may be lowered 7 feet? *Ans.* 35 feet.

14. A ladder 130 feet long, with its foot in the street, will reach on one side to a window 78 feet high, and on the other to a window 50 feet high; what is the width of the street? *Ans.* 224 feet.

15. There is a rectangular field whose sides are 25 yards and 16 yards respectively; what is the side of a square field of equal area? *Ans.* 20 yards.

16. If it cost $328 to put a fence around a farm 50 rods long and 32 rods wide, how much less will it cost to enclose a square farm of equal area with the same kind of fence? *Ans.* $8.

17. The gable ends of a house are each 48 feet wide, and the perpendicular height of the ridge above the eaves is 10 feet; how many feet of boards will it take to board up both gables? *Ans.* 480.

18. A man has a field in the form of a rectangle which contains 40 acres; what are its dimensions if the length is twice the breadth? *Ans.* Length, 113.136 rods; width, 56.568 rods.

19. A cemetery containing 60 acres is laid out in such a manner that its length is equal to three times its width; required the dimensions of the cemetery. *Ans.* Length, 169.704 rods; width, 56.568 rods.

20. A general wishing to draw up his corps in the form of a square, found by the first trial he had 100 men over; he then increased the side of the square by 2 men, and found he lacked 136 men to complete the square; how many men had he in the corps? *Ans.* 3464.

21. A man has a square yard containing $\frac{1}{10}$ of an acre; he makes a gravel walk around it which occupies $\frac{15}{64}$ of the whole yard; what is the width of the walk? *Ans.* 4 feet $1\frac{1}{2}$ inches.

22. In a triangle the two sides are 13 and 15, respectively, and the perpendicular from the vertex of the angle which they form to the opposite side, 12; required the third side. *Ans.* 14.

EXERCISES FOR ORIGINAL THOUGHT.

1. Two squares are to each other as the squares of their diagonals.

2. Two similar parallelograms are to each other as the squares of their diagonals

3. Prove that the diagonals of a rectangle are equal to each other.

4. Prove that the greater diagonal of a parallelogram is opposite the greater angle.

5. Show where a line from the vertex of a triangle must be drawn to divide the triangle into two equal parts.

6. Prove that the ratio of the side of a square to its diagonal is as 1 to the square root of 2.

7. The straight line joining the middle points of the oblique sides of a trapezoid will be parallel to the other sides, and equal to half their sum.

8. The four lines joining the middle points of the adjacent sides of a quadrilateral form a parallelogram.

9. The lines drawn from the vertices of the three angles of an equilateral triangle, perpendicular to the opposite sides of the triangle, will intersect each other in the same point.

10. The line which bisects the vertical angle of a triangle divides the base into two parts which are proportional to the adjacent sides.

11. If a line be drawn parallel to the base of a triangle, and lines be drawn from the vertex of the triangle to the base, these lines will divide the base and parallel proportionally.

12. Triangles which have an angle in each equal, are to each other as the rectangles of the sides including those angles.