they are in the hush of the dead

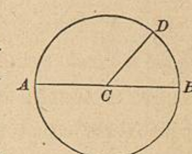
BOOK IV.

OF THE CIRCLE.

DEFINITIONS.

1. A CIRCLE is a plane bounded by a curve line, every point of which is equally distant from a point within, called the *centre*.

2. The CIRCUMFERENCE is the bounding line of a circle. An ARC is any part of the circumference; as, *BD*.

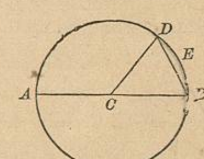


3. The RADIUS is a straight line drawn from the centre to any point of the circumference; thus, *CD* is a radius.

4. The DIAMETER is a straight line passing through the centre and terminating at both extremities in the circumference; as, *AB*.

5. A CHORD is a straight line joining the extremities of an arc; thus, *BD* is a chord.

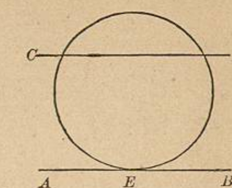
6. A SEGMENT is a portion of the circle included between an arc and its chord; as, *DBE*.



7. A SECTOR is a portion of the circle included by an arc and the radii drawn to its extremities; as, *DCBE*.

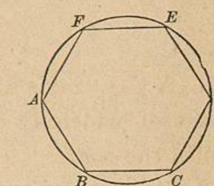
8. A TANGENT is a straight line which touches the circumference in one point; thus, *AB* is a tangent. The point *E* is called the *point of tangency*.

9. A SECANT is a straight line which cuts the circumference in two points; thus, *CD* is a secant.



10. An INSCRIBED ANGLE is an angle whose vertex is in the circumference and whose sides are chords; as, *ABC* in the next figure.

11. An INSCRIBED POLYGON is a polygon whose sides are chords, the vertices of the angles being in the circumference; as, *ABCDEF*.



12. A POLYGON is *circumscribed about a circle* when all of its sides are tangents to the circumference. The circle is at the same time *inscribed in a polygon.*

AXIOMS.

1. The radii, and also the diameters, of a circle, or of equal circles, are equal.

2. Every diameter is double the radius, or is equal to the sum of two radii.

3. A straight line can cut a circumference in only two points.

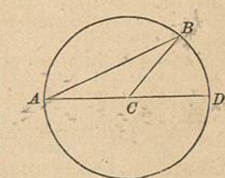
ANALYSIS.—This book treats of the nature of the circle, the measurement of angles, the finding of the circumference, the measurement of the area of a circle, and the relation of the circumferences, and also of the areas of circles. The method of treatment in finding the circumference and area, and also their relations, is to regard the circle as a polygon of an infinite number of sides, and derive the principles from those of polygons. By a simplification of the subject, we embrace in one book what is usually given in two.

NATURE OF THE CIRCLE.

THEOREM I.

The diameter of a circle is greater than any other chord.

Let AB be any chord; then will it be less than any diameter.

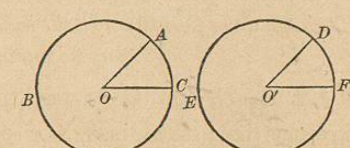


For, from the point A draw the diameter AD, and draw also the radius CB. Then, in the triangle ACB, the sum of the sides AC and CB is greater than AB (B. I. A. 10. C.). But $AC+CB$ equals AD (Ax. 2); hence, AD is greater than AB. Therefore, etc.

THEOREM II.

In the same circle or equal circles, equal angles at the centre intercept equal arcs on the circumference.

In the equal circles ABC and DEF let the angle AOC equal $DO'F$; then will the arc AC be equal to the arc DF.



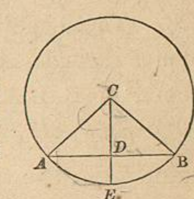
For, apply the circle ABC to the circle DEF so that the angle AOC shall coincide with the angle $DO'F$. Then, since $OC=O'F$ and $OA=O'D$, the point C will fall on F and the point A will fall on D, and the arc AC will coincide with the arc DF, since every point of each arc is equally distant from the centre of the circle. Therefore, etc.

Cor. Conversely.—*In the same circle or equal circles, equal arcs subtend equal angles at the centre.* For, if we apply the equal arcs AC and DF, placing the point C on F, they will coincide, and the point A will fall on D; hence, the line OC will coincide with $O'F$ and OA with $O'D$, and the angle AOC will be equal to $DO'F$.

THEOREM III.

Any radius which is perpendicular to a chord bisects the chord and also the arc subtended by the chord.

Let AB be the chord, and CD the radius perpendicular to it; then will $AD=DB$ and $AE=EB$.



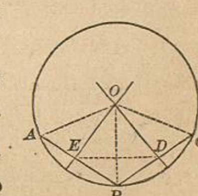
First. Draw the radii CA and CB; then the angle ACD equals DCB (B. I. Th. X. C. 1), and the triangles ACD and DCB are equal (B. I. Th. VI.); hence, the side AD equals DB.

Second. Since the triangles ACD and DCB are equal, the angle ACE equals ECB; hence, the arc AE equals the arc EB (Th. II. C.).

THEOREM IV.

Through three points not in the same straight line a circumference may be made to pass.

Let A, B, and C be any three points not in the same straight line; then may a circumference be described through them.



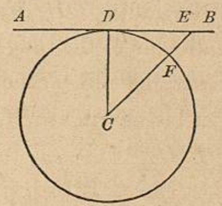
Draw AB and BC, and at E and D, the middle points of AB and BC, draw perpendiculars, and unite the points E and D. Now, since $OED+ODE$ is less than two right angles, the perpendiculars will meet in some point, as O (B. I. Th. IV. C. 3). Draw OA, OB, and OC; then $OA = OB$ (B. I. Th. XIV.), and, for the same reason, $OB = OC$; hence, a circumference described from O as a centre will pass through the three points A, B, and C.

Cor. It may also be readily shown that but one circumference can be made to pass through three points.

THEOREM V.

If a straight line is perpendicular to a radius at its extremity, it will be tangent to the circle at that point.

Let the straight line AB be perpendicular to the radius CD at D; then will it be tangent to the circle at the point D.



For, take any point of AB, as E, and draw the line CE. Now, CE is greater than CD (B. I. Th. XIV.); consequently, the point E will be without the circle, and hence the line AB touches the circumference in only one point: it is therefore tangent to it at the point D (D. 8). Therefore, etc.

Cor. Conversely.—*A tangent to the circle is perpendicular to the radius drawn to the point of contact.*

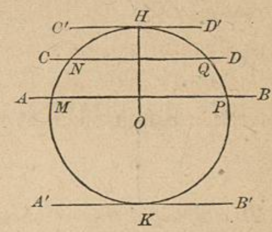
For any line, as CE, is greater than CF, or its equal CD; hence, CD, being the shortest line from C to the tangent, is perpendicular to the tangent at D (B. I. Th. XIV.). Therefore, etc.

THEOREM VI.

Two parallel lines intercept equal arcs on the circumference.

There may be three cases: first, when both lines are secants; second, when one is a secant and the other a tangent; third, when both are tangents.

First. Let AB and CD be two lines cutting the circle; then will the arcs MN and PQ be equal.



For, draw the radius OH perpendicular to the chord NQ; it will be perpendicular to MP (B. I. Th. III. C.), and will bisect the arcs NHQ and MHP at the point H (Th. III.); hence, NH equals HQ and MH equals HP; and, therefore, $MH - NH = PH - QH$, or MN equals PQ. Therefore, etc.

Second. If one of the lines, as $C'D'$, is a tangent. Then the radius OH, drawn to the point of contact, H, is perpendicular to the tangent $C'D'$ (Th. V.), and consequently to its parallel AB. Since OH is perpendicular to the chord MP, it bisects its arc MHP (Th. III.); hence, arc MH equals arc PH.

Third. If both lines, as $C'D'$ and $A'B'$, are tangents. Draw any secant, as AB, parallel to $A'B'$; it will be parallel to $C'D'$ (B. I. Th. IV. C. 2). By the second case we have arc $MK =$ arc PK, and arc $MH =$ arc PH; adding, we have $MH + MK = PH + PK$, or arc $HMK =$ arc HPK.

Cor. 1. In the case of parallel tangents it is evident that each arc is a semi-circumference.

Cor. 2. The straight line joining the points of contact of two parallel tangents is a diameter.

Scholium. Regarding a tangent as a secant whose two points of intersection coincide, the demonstration of the first case of the theorem may be regarded as including the other two cases.

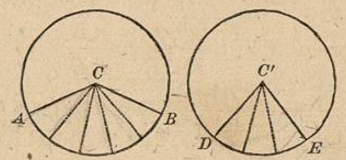
MEASUREMENT OF ANGLES.

THEOREM VII.

In the same circle or in equal circles, two angles at the centre have the same ratio as their intercepted arcs.

Let ACB and $DC'E$ be two angles at the centre of equal circles, and AB and DE their intercepted arcs; then will

$ACB : DC'E :: AB : DE.$



First. Suppose some common unit is contained 5 times in the arc AB and 3 times in the arc DE; then

$$\text{arc } AB : \text{arc } DE :: 5 : 3.$$

Draw radii to the several points of division of the arcs; the angles thus formed will be equal, since their arcs are equal (Th. II. C.); hence the angle ACB will consist of 5 equal parts and the angle $DC'E$ of 3 such equal parts; therefore

$$\text{angle } ACB : \text{angle } DC'E :: 5 : 3.$$

Comparing the two proportions, we have

$$\text{angle } ACB : \text{angle } DC'E :: \text{arc } AB : \text{arc } DE.$$

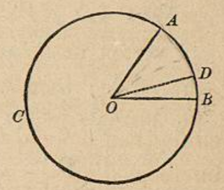
Second. Now this is true whatever the size of the unit of measure; hence it is true when the unit of measure becomes indefinitely or infinitely small, as it must when the two arcs are incommensurable. Therefore, *any two angles at the centre of the same or equal circles are to each other as their intercepted arcs.*

THEOREM VIII.

An angle having its vertex at the centre of a circle is measured by the arc intercepted between its sides.

Let AOB be an angle at the centre of the circle ACB, and AB its intercepted arc; then will AB be the measure of the angle AOB.

For, let BOD be the *unit of measure* of the angle AOB, and the arc BD be the unit of measure of the arc BA; then, by Theorem VII., we have



$$AOB : DOB :: AB : DB,$$

or, $$\frac{AOB}{DOB} = \frac{AB}{DB}.$$

Now, AOB divided by DOB equals the number of units in the angle AOB, and AB divided by DB equals the number of units in the arc AB; hence the number of units in the angle is equal to the number of units in the arc; therefore the arc may be used as the measure of the angle.

Scholium 1. This theorem is usually expressed thus: *An angle at the centre is measured by its intercepted arc.* The statement is, however, rather conventional, "measured by" meaning "having the same numerical measure." Both angle and arc have the same numerical measure; hence the arc may be assumed as the measure of the angle.

Scholium 2. It would seem more natural to measure an angle by a quantity of the same kind, and for this purpose the right angle would naturally be taken as the unit of measure. It has been found more convenient, however, to use the arc of a circle as the measure of an angle, and for this purpose the circumference has been divided into degrees, minutes, and seconds, as before explained.

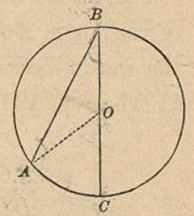
Look on th XIII B. I. angle O = exterior angle

THEOREM IX.

An angle having its vertex at the circumference of a circle is measured by half the arc intercepted between its sides.

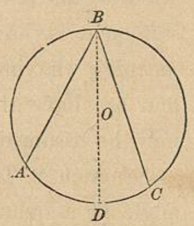
There may be three cases; first, when the centre of the circle is on one of the sides of the angle; second, when it is within the angle; third, when it is without the angle.

First. Let ABC be the angle, having its vertex at B, and O be the centre of the circle; then will ABC be measured by one-half of AC.



For, draw the radius AO; then the exterior angle AOC is equal to the sum of the opposite interior angles ABO and OAB (B. I. Th. XIII. C. 5). But, the triangle AOB being isosceles, the angles A and B are equal; and, consequently, the angle AOC is double the angle ABC. But AOC, being at the centre, is measured by the arc AC (Th. VIII.); hence, the angle ABC is measured by one-half of the arc AC.

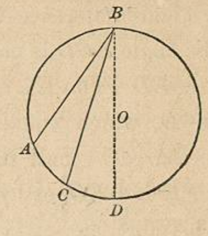
Second. Let ABC be the angle, and O the centre of the circle; then will ABC be measured by one-half of ADC.



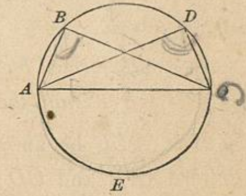
For, draw the diameter BD; then, from what we have just shown, the angle ABD is measured by one-half of AD, and the angle DBC by one-half of DC; hence, their sum, or the angle ABC, is measured by one-half of the sum of AD and DC, or one-half of ADC.

Third. Let ABC be the angle, and O the centre being without the angle; then will ABC be measured by one-half of AC.

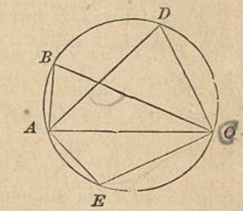
For, draw the diameter BD; then, ABD is measured by one-half of AD, and CBD is measured by one-half of CD; hence, ABC, their difference, is measured by one-half of AD minus CD, or one-half of AC. Therefore, etc.



Cor. 1. All the angles ABC, ADC, inscribed in a semicircle are right angles, being measured by one-half of the semi-circumference AEC (Th. IX.).



Cor. 2. All the angles ABC, ADC, etc., inscribed in a segment greater than a semicircle are less than right angles, being measured by less than one-half of a semi-circumference. Any angle AEC inscribed in less than a semicircle is greater than a right angle, being measured by more than one-half of a semi-circumference.



Cor. 3. All the angles inscribed in the same segment are equal, being measured by one-half of the same arc.

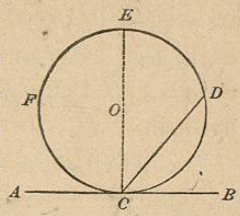
Scholium. A right angle is measured by one-half a semi-circumference, or a quadrant.

THEOREM X.

The angle formed by a tangent and a chord is measured by half the arc intercepted between its sides.

Let AB be a tangent to the circle at C, and CD a chord meeting the tangent at C; then will the angle ACD be measured by one-half the arc CED.

For, draw the diameter CE. The angle ACE is a right angle, and is measured by half the semi-circumference CFE (Th. IX. S.); the angle ECD is measured by half the arc ED (Th. IX.); hence, the angle ACD, which equals $ACE+ECD$, is measured by half the sum of the arcs CFE and ED, or by half the arc CFD. Therefore, etc.

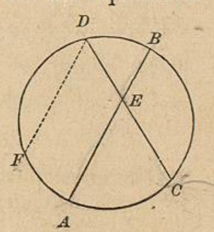


THEOREM XI.

An angle formed by two chords which intersect is measured by half the sum of the intercepted arcs.

Let AEC be an angle formed by the intersection of the chords AB and CD; then will it be measured by half the sum of AC and DB.

For, draw DF parallel to AB; then the arc AF equals the arc DB (Th. VI.), and the angle FDC equals the angle AEC (B. I. Th. III.). Now, the angle FDC is measured by one-half the arc FC (Th. VII.); hence, the angle AEC is measured by one-half of FC, or $\frac{1}{2}(AC+AF)$, or $\frac{1}{2}(AC+DB)$. Therefore, etc.



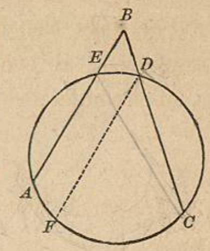
THEOREM XII.

The angle formed by two secants is measured by half the difference of the intercepted arcs.

Let the angle ABC be formed by the two secants AB and CB; then will it be measured by one-half the difference of the arcs AC and ED.

For, draw DF parallel to AB; then the arc AF is equal to the arc ED, and the angle FDC equal to ABC. Now, the angle FDC is measured by one-half of the arc FC; hence, ABC is measured by one-half of FC; that is, by

$\frac{1}{2}(AC-AF)$ or $\frac{1}{2}(AC-ED)$.

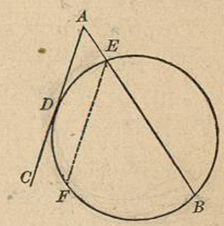


THEOREM XIII.

The angle formed by a secant and a tangent is measured by half the difference of the intercepted arcs.

Let AB be a secant cutting the circle in E, and AC a tangent at the point D; then will the angle BAC be measured by one-half of the difference of the arcs DB and DE.

For, draw EF parallel to AC; then the angle FEB equals CAB, and the arc DE equals arc DF. Now, the angle FEB is measured by one-half of FB; hence CAB is measured by one-half of FB; but $FB=DB-DF$ or $DB-DE$; therefore CAB is measured by $\frac{1}{2}(DB-DE)$.

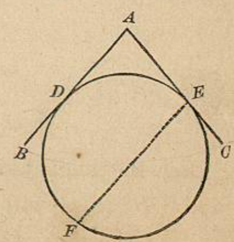


THEOREM XIV.

The angle formed by two tangents is measured by half the difference of the intercepted arcs.

Let AB be a tangent at D, and AC a tangent at E; then will the angle BAC be measured by half the difference of the arcs DFE and DE.

For, draw EF parallel to AB; then the angle FEC equals BAC, and the arc DE equals the arc DF. Now, the angle FEC is measured by half the



arc FE (Th. X.); hence BAC is measured by half the arc FE; but arc $FE = DFE - DF$, or $DFE - DE$; hence BAC is measured by $\frac{1}{2}(DFE - DE)$. Therefore, etc.

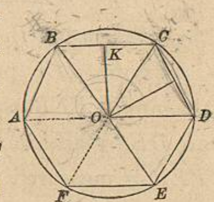
polygonal

THE CIRCUMFERENCE AND AREA.

THEOREM XV.

The circumference of a circle may be circumscribed about a regular polygon, and it may also be inscribed within it.

Let $ABCD$ be a regular polygon; then can the circumference of a circle be circumscribed about it.



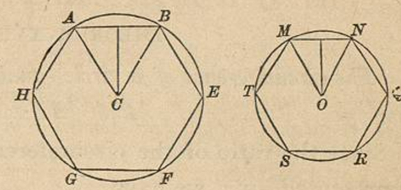
Through the three vertices A, B, and C, describe a circumference; its centre O will be in OK drawn perpendicular to BC at its middle point K. The triangle BOC being isosceles, the angles OBC and OCB are equal, which, being subtracted from the equal angles ABC and BCD, leave ABO and OCD equal; hence, the triangles OBA and OCD have two sides and an included angle respectively equal, and are equal (B. I. Th. VI.), and OD equals OA; hence, the circumference passing through A also passes through D; and in the same way it may be shown to pass through all the vertices.

Second. Since the triangles AOB, BOC, etc. are all equal, their altitudes are equal; hence, a circumference described from O as a centre with the radius OK will touch all the chords at their middle points, and, consequently, be inscribed within the polygon. Therefore, etc.

THEOREM XVI.

The circumferences of circles are as their radii, and their areas are as the squares of their radii.

Let C and O be the centres of two circles whose radii are CA and OM; then will their circumferences be to each other as their radii, and their areas as the squares of their radii.



Inscribe in the circles regular polygons of the same number of sides. These polygons being similar figures, their perimeters are to each other as any two homologous lines CA and OM, and their areas are as the squares of those lines (B. III. Th. XVIII. C.); and this is true whatever the number of sides; hence, it is true if the number of sides is infinite, and the polygon becomes the circle. Hence, we have,

$$circ.\ CA : circ.\ OM :: CA : OM; \text{ and, also,}$$
$$area\ CA : area\ OM :: \overline{CA}^2 : \overline{OM}^2.$$

Cor. 1. Since the radii of circles are to each other as the diameters, we have the circumferences to each other as the diameters, and the areas as the squares of the diameters.

Cor. 2. From this we see that the circumference of a circle is to its diameter as the circumference of another circle to its diameter; hence, the ratio of the circumference to the diameter is a constant quantity. This constant ratio mathematicians represent by π, the Greek letter

p, called *pi*. Letting C represent the *circumference* and D the *diameter* we have $\pi = \frac{C}{D}$.

NOTE.—This symbol π is of great importance in mathematics: the pupil should be very careful to thoroughly understand its signification and use.

THEOREM XVII.

The circumference of a circle equals the diameter multiplied by π.

Since the ratio of the circumference to the diameter is represented by π, we have,

$$\frac{C}{D} = \pi; \text{ and, multiplying by } D,$$

we have, $$C = \pi . D.$$

Therefore, etc.

Cor. Since the diameter is twice the radius, if we substitute $2\,R$ for D, we will have,

$$C = \pi \times 2\,R, \text{ or } C = 2\,\pi R.$$

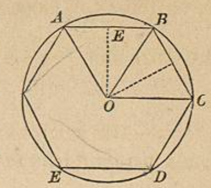
Hence, the circumference equals the radius multiplied by $2\,\pi$.

REMARK.—The value of π cannot be exactly expressed in numbers. The number generally used is 3.1416, which is sufficiently accurate for practical purposes.

THEOREM XVIII.

The area of a circle is equal to the circumference multiplied by one-half the radius.

Let O be the centre of a circle whose radius is OA, and circumference $ABCD$, etc.; then will its area be equal to $circ.\ OA \times \frac{1}{2}\,OA$.



Inscribe in the circle a regular polygon $ABCD$, etc., and draw the radii OA, OB, etc., and the perpendicular OE. The area of each triangle of the polygon is equal to its base multiplied by one-half its altitude, and since the altitudes are equal being radii of the inscribed circle, the area of the polygon is equal to the sum of the bases, or its perimeter multiplied by one-half of OE. Now, this is true whatever the number of sides; hence, it is true when the number of sides is infinite and the polygon becomes a circle. In this case the perimeter becomes the circumference, and the line $O\,E$, the radius. Therefore, the area of a circle is equal to the circumference multiplied by one-half of the radius.

Cor. The area of a circle is equal to the circumference multiplied by one-fourth of the diameter.

THEOREM XIX.

The area of a circle equals the square of the radius multiplied by π.

Let C be the centre of a circle; denote its radius CA by R, and its area by *area CA;* then from the previous theorem we have,

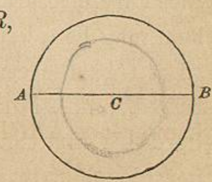
$$area\ CA = circ.\ CA \times \tfrac{1}{2}\,R; \quad \text{Th XVIII}$$

but, $$circ.\ CA = 2\,\pi . R \text{ (Th. XVII. C.)};$$

hence, $$area\ CA = 2\,\pi\,R \times \tfrac{1}{2}\,R,$$

which, reduced, gives,

$$area\ CA = \pi\,R^2.$$



Therefore, etc.

Cor. In a similar manner, we find that *area* $CA = \pi \frac{1}{4} D^2$, or *area* $CA = \frac{1}{4} \pi D^2$.

Scholium. The finding the exact length of the circumference of a circle is called the *rectification* of the circle. The finding of the area of a circle is called the *quadrature of* or *squaring the circle.* Both of these are celebrated problems, and can only be solved approximately, as may be shown by Calculus.

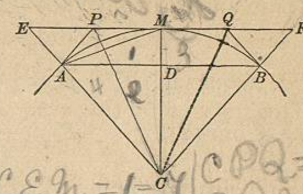
It was stated in Theorem XVII. that the value of π is about 3.1416. This value is generally determined by finding a numerical expression for the area of a circle whose radius is unity, which area may be shown equal to the ratio of the circumference to the diameter. The solution is given in the following proposition.

THEOREM XX.

PROBLEM.—*To find the numerical value of π, the ratio of the circumference to the diameter.*

The area of a circle equals πR^2; but when $R = 1$, the area of the circle equals π; hence, we may find the value of π by finding the area of a circle whose radius is 1. As a circle is a polygon of an infinite number of sides, by constructing successive similar inscribed and circumscribed polygons of double the number of sides, two may be found whose areas so nearly approach each other that either of them may be taken for the area of the circle.

Let C be the centre of the circle, AB the side of an inscribed, and EF of a circumscribed, polygon. Draw the chord AM, and the tangents AP and BQ; then AM will be the side of an inscribed, and PQ of a circumscribed polygon of double the number of sides. Draw CE, CP, CM and CF.



Let P represent the area of the given circumscribed polygon; p, the area of the given inscribed polygon; P', the area of a circumscribed polygon of double the number of sides; and p', the area of an inscribed polygon of double the number of sides. Also, represent the triangles CEM, CAD, CPQ and CAM, which are respectively like parts of P, p, P' and p', by T, t, T' and t'.

1. The triangle CAM is a mean proportional between CAD and CEM (B. III. Th. IX. C. 3), hence,

$$T : t' :: t' : t;$$

whence, $\quad P : p' :: p' : p \quad$ (B. II. Th. X.);

therefore, $\quad p' = \sqrt{p \times P}. \quad$ (1)

2. Because of a common altitude, CAM and CAD are to each other as CM to CD, and CEM to CPM as EM to PM;

hence, $\quad t' : t :: CM : CD,$

and, $\quad T : \frac{1}{2}T' :: EM : PM,$

by division, $T - \frac{1}{2}T' : \frac{1}{2}T' :: EP : PM$, since $EM - PM = EP$.

The triangles CAD and AEP are similar; hence,

$AC : CD :: EP : AP$, or, since $AC = CM$ and $AP = PM$,
$CM : CD :: EP : PM.$

Hence, from the first and third proportions, we have,

$$t' : t :: T - \frac{1}{2}T' : \frac{1}{2}T';$$

whence, $\quad p' : p :: 2P - P' : P'. \quad$ (B. II. Th. X.)

and $\quad p' + p : p :: 2P : P'; \quad$ (B. II. Th. VI.)

whence, $\quad P' = \dfrac{2p \times P}{p' + p}. \quad$ (2)