Now if p and P are squares, the radius being 1, the area of P is 4; and the side of p is $\sqrt{2}$ (B. III. Th. VI. C. 3); hence the area of p is 2;

then, from (1), $p' = \sqrt{8} = 2.8284271$,

and, from (2), $P' = \dfrac{16}{2 + \sqrt{8}} = 3.3137085$;

which are the areas of the inscribed and circumscribed *octagons;* and in the same manner we may find the areas of polygons of 16, 32, etc. sides. For 8192 sides, the area of the inscribed polygon is 3.1415923 +, and of the circumscribed polygon, 3.1415928 +, either of which may be taken for the area of the circle whose radius is 1; and, since we have shown this to be the value of π, we have $\pi = 3.14159+$.

Scholium. The value of π is generally taken to be 3.1416.

NOTE.—We invite special attention to the method of treating the circumference and area of the circle, and also to the simple and concise method of presenting the derivation of the value of π, as given in the last proposition.

PRACTICAL EXERCISES.

1. The radius of a circle is 6 inches; what is its circumference?
2. The diameter of a circle is 8 inches; what is its area?
3. The circumference of a circle is 50.2656 feet; required the radius. *Ans.* 8 feet.
4. The area of a circle is 490.875 square inches; required the diameter and circumference. *Ans.* Diameter, 25; circumference, 78.54.
5. The distance around a circular park is 180 rods; required the area of the park. *Ans.* 16 A. 18.23 P.
6. What is the length of an arc of 75° on the circumference of a circle whose radius is 5 feet? *Ans.* 6.545 feet.
7. How many degrees in an arc 18 inches long, on a circumference whose radius is 5 feet? *Ans.* 17° 11′ 19″.

8. A circle 20 feet in diameter is circumscribed by another circle 30 feet in diameter; what is the area of the space included between them?
9. A has a circular garden whose diameter is 18 rods, and B has one whose area is $2\frac{7}{9}$ times as great; what is the diameter of B's garden? *Ans.* 30 rods.
10. Find the side of a square inscribed in a circle whose diameter is 5 feet. *Ans.* 3.535 feet.
11. Within a circular park 160 rods in circumference is a circular lake 80 rods in circumference; required the width of the ring of land surrounding the lake. *Ans.* 12.732 rods.
12. Deborah has a circular garden and John a square one, and the distance around each is 120 rods; which contains the most land, and how much? *Ans.* 245.95 square rods.
13. A man has a square garden and his wife a circular one, and each garden contains one acre; how much further around is one than the other? *Ans.* 5.756 rods.
14. The area of a circle is 314.16; if this circle be circumscribed by a square, required the area of the part between the circumference and the perimeter of the square. *Ans.* 85.84.
15. The area of a circle is 4 acres; required the side of the inscribed square, and the area of the part of the circle between the circumference and perimeter of the square. *Ans.* 1 A. 1 R. 32 P.

THEOREMS FOR ORIGINAL THOUGHT.

1. If two circumferences intersect, the distance between their centres will be less than the sum of their radii and greater than the difference.
2. If two circumferences intersect, the points of intersection will lie in a perpendicular to the line joining their centres, and at equal distances from it.
3. In equal circles the greater arc has the greater chord, and, conversely, the greater chord subtends the greater arc.
4. In equal circles, equal chords are equally distant from the centre, and the greater chord is nearer the centre.
5. If we inscribe a square in a circle, the radius is to the side of the inscribed square as 1 is to $\sqrt{2}$.

6. If a regular hexagon be inscribed in a circle, each side will be equal to the radius of the circle.

7. The area of a triangle is equal to the perimeter multiplied by one-half the radius of the inscribed circle.

8. In any inscribed quadrilateral, the sum of the opposite angles is equal to two right angles.

9. When a quadrilateral circumscribes a circle, the sums of its opposite sides are equal.

10. When the radius of a circle is unity, its area and semi-circumference are numerically equal.

PRACTICAL PROBLEMS IN GEOMETRICAL CONSTRUCTION,

INVOLVING THE PRINCIPLES OF BOOKS I., II., III., AND IV.

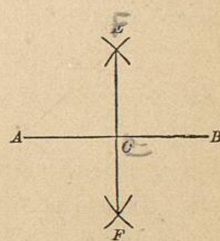
THE following problems are solved by the principles of the previous books. The solution of a few is given in full; in others, the construction is given, and the reason for the solution indicated by referring to the theorem or theorems upon which it depends. The pupil will give the explanation in full.

The object of these is to teach the pupil to draw accurately upon paper. They are of great use in drawing the notes of a survey, or in representing any geometrical figure upon paper. The pupils need two instruments, a rule and compasses; with these all the following problems may be readily solved

PROBLEM I.

To bisect a given straight line.

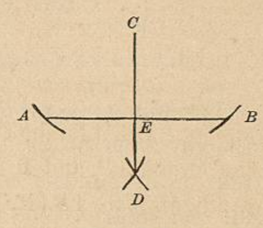
Let AB be the given straight line. From A and B, as centres, with a radius greater than one-half of AB, describe arcs intersecting at E and F; draw the line EF; then will C be the middle point of AB. For, E and F are each equally distant from A and B; hence, EC bisects AB (B. I. Th. XIV. C. 3).



PROBLEM II.

From a given point without a straight line to draw a perpendicular to the line.

Let AB be the given line, and C the given point. From C as the centre, with a radius sufficiently great, describe an arc cutting the line AB in the two points A and B; then from A and B as centres, with a radius greater than one-half of AB, describe two arcs cutting each other in D, and draw CD;

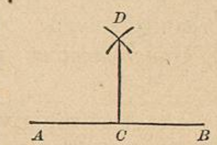


it will be the perpendicular required (B. I. Th. XIV. C. 3).

PROBLEM III.

At a given point in a straight line to erect a perpendicular to that line.

Let AB be the given line, and C the given point. Then, in the line AB take the points A and B equally distant from C, and with A and B as centres, and a radius greater than one-half of AB, describe two arcs cutting each other at D; draw DC; it will be the perpendicular required (B. I. Th. XIV. C. 3).

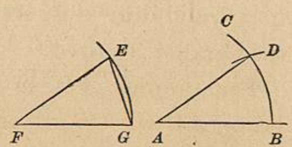


PROBLEM IV.

At a point on a given straight line to make an angle equal to a given angle.

Let A be the given point, AB the given line, and EFG the given angle.

From the point F as a centre,



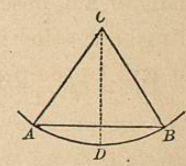
with any radius FG, describe the arc EG. From A as a centre, with the same radius, describe the arc CB; then, with a radius equal to the chord EG, describe an arc from B as a centre, cutting the arc CB in D, and draw AD; then will the angle DAB equal EFG (B. I. Th. IX.).

PROBLEM V.

To bisect a given arc, or a given angle.

First. Let ADB be the given arc, and C its centre. Draw the chord AB, and from C draw CD perpendicular to AB (P. II.); then will CD bisect AB (B. IV. Th. III.).

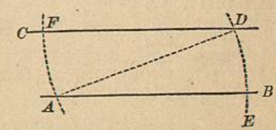
Second. Let ACB be the given angle. Then, with C as a centre and any radius CA, describe the arc AB, and bisect this arc by the line CD, as in the previous case; then will CD bisect ACB (B. IV. Th. III.).



PROBLEM VI.

Through a given point to draw a straight line parallel to a given straight line.

Let A be the given point and CD the given line. From A as a centre, with a radius greater than the shortest distance from A to CD, describe an indefinite arc DE; from D as a centre, with the same radius, describe the arc AF; take DE equal to AF, and draw AB; AB will be the parallel required.

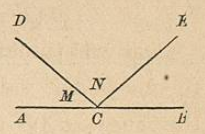


For, drawing AD, we have $ADF = DAE$ (Prob. IV.); hence, AE and CD are parallel (B. I. Th. IV.).

PROBLEM VII.

Two angles of a triangle being given, to find the third.

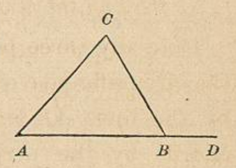
Let M and N be the given angles. Draw the indefinite line AB; at any point, as C, construct the angle ACD equal to M, and the angle DCE equal to N; then will ECB equal the third angle.



PROBLEM VIII.

Given two sides and the included angle of a triangle, to construct the triangle.

Draw the indefinite line AD; take AB equal to one of the given sides; at A construct the angle A equal to the given angle, and take AC equal to the other given side; draw BC; then will ABC be the required triangle (B. I. Th. VI.).

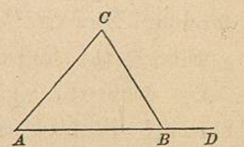


PROBLEM IX.

Given one side and two angles of a triangle, to construct the triangle.

If the angles are not adjacent, find the third angle by P. VII.; we then have two angles and the included side, and proceed thus:—

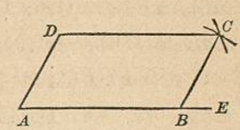
Draw the indefinite line AD; take AB equal to the given side; at A make the angle BAC equal to one of the angles; at B make the angle ABC equal the other angle; then produce AC and BC till they meet, and ABC will be the required triangle (B. I. Th. VII.).



PROBLEM X.

Given two adjacent sides of a parallelogram and the included angle, to construct the parallelogram.

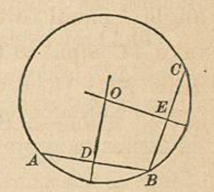
Draw the indefinite line AE, and upon it take AB equal to one of the sides. At A construct the angle BAD equal to the given angle, and take AD equal to the other given side. Draw DC parallel to AB, and BC parallel to AD; then will $ABCD$ be the parallelogram required (B. I. Th. XV. C. 3).



PROBLEM XI.

To find the centre of a given circumference or arc.

Take any three points, A, B, and C, on the circumference or arc, and unite them by the lines AB and BC. Bisect these chords by the perpendiculars DO and EO; then will their intersection O be the centre of the circle (B. IV. Th. IV.).

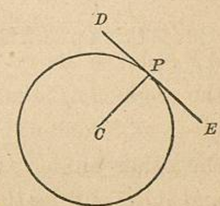


PROBLEM XII.

Through a given point to draw a tangent to a given circle.

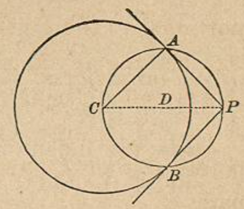
First. Suppose the given point P to be in the circumference.

Find C, the centre of the circle (P. XI.); draw the radius CP; and then through P draw the perpendicular DE; DE will be the tangent required (B. IV. Th. V.).



Second. Suppose the given point P

to be without the circle. Join P and the centre of the circle; bisect PC in D; with D as a centre, and a radius DC, describe the circumference intersecting the given circumference in A and B; draw PA or PB; then each of those will be the tangent required.



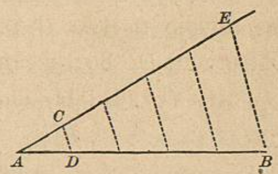
For, since CAP is a semicircle, the angle CAP is a right angle (B. IV. Th. IX. C. 1); hence, AP is a tangent (B. IV. Th. V.).

PROBLEM XIII.

To divide a given line into any number of equal parts.

Let AB be the given line, and suppose we wish to divide it into any number, say 5 equal parts.

Through A draw the indefinite line AE, making any angle with AB. Take AC of any convenient length, and apply it 5 times to AE; join B with the last point of the division; and through the other points of division draw lines parallel to EB; then will AB be divided into 5 equal parts.



For, since DC and BE are parallel, we have (B. III. Th. IX.),

$$AC : AE :: AD : AB.$$

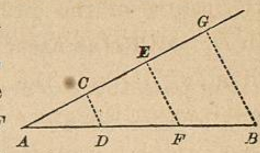
But AC is one-fifth part of AE; hence, AD is one-fifth part of AB.

PROBLEM XIV.

To divide a given line into parts proportional to given lines.

Let AB be the given line, to be divided into parts pro-

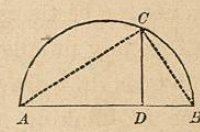
portional to the given lines P, Q, and R. Through A draw AG, making any angle with AB. On AG lay off AC equal P, CE equal Q, EG equal R; draw BG, and from the points C and E draw CD and EF parallel to GB; then will AD, DF, and FB be proportional to AC, CE, and EG (B. III. Th. IX.).



PROBLEM XV.

To construct a mean proportional to two given lines.

Let P and Q be the two given lines. Draw an indefinite line, and on it lay off AD equal to P, and DB equal to Q; on AB as a diameter describe a semicircle, and draw DC perpendicular to AB; then, in the triangle ACB, will DC be a mean proportional to AD and DB (B. III. Th. XIV.).

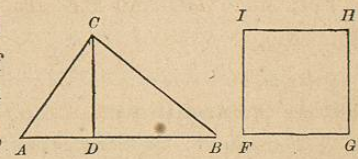


PROBLEM XVI.

To construct a square equal to a given triangle.

Let ABC be the given triangle, AB its base, and CD its altitude.

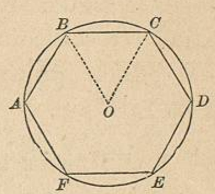
Find a mean proportional between CD and one-half of AB (Prob. XV.). Let FG be that mean proportional, and on it, as a side, construct the square $FGHI$; this will be the square required. For, by the construction, we have $\overline{FG}^2 = \frac{1}{2} AB \times CD$, which equals the area of ABC.



PROBLEM XVII.

To inscribe a regular hexagon in a circle.

Suppose the problem to be solved, and that $ABCDEF$ is a regular hexagon; draw the radii OB and OC. Now, the arc BC is one-sixth of a circumference, or 60°; hence, the angle BOC is 60°, and the other angles OBC and BCO equal 180° minus 60°, or 120°, and, OB being equal to OC, the angles OBC and BCO are equal; hence, each is equal to one-half of 120°, or 60°. Consequently, the triangle OBC is equiangular, and therefore equilateral; hence, the side BC is equal to the radius OB. Therefore, to inscribe a regular hexagon in a circle, we apply the radius six times as a chord to the circumference.



PROBLEMS FOR ORIGINAL THOUGHT.

1. Given the three sides of a triangle, to construct the triangle.
2. Given two sides of a triangle, and the angle opposite one of them, to construct the triangle.
3. To inscribe a circle in a given triangle.
4. To inscribe a circle in a square, and a square in a circle.
5. To find the side of a square which shall be equal to the sum of two given squares.
6. To find the side of a square which shall be equal to the difference between two given squares.
7. To construct a rectangle equal in area to a given triangle.
8. To find a fourth proportional to three given lines.
9. On a given line to construct a rectangle which shall be equal to a given rectangle.
10. To construct a square that shall be equal in area to a given parallelogram.

BOOK V.

PLANES AND THEIR ANGLES.

DEFINITIONS.

1. A PLANE is a surface such that a straight line connecting any two of its points will lie entirely in the surface.

2. A straight line is PERPENDICULAR TO A PLANE when it is perpendicular to every line of the plane passing through its foot. The *foot* is the point where the line meets the plane.

Reciprocally, the plane is also perpendicular to the line.

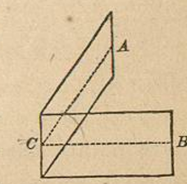
3. A straight line is PARALLEL TO A PLANE when it cannot meet the plane, however far both be produced.

Reciprocally, the plane is also parallel to the line.

4. Two PLANES ARE PARALLEL when they cannot meet, however far both be produced.

5. When two planes meet, they form a line, which is called their LINE OF INTERSECTION.

6. A DIEDRAL ANGLE is the divergence of two planes. The line in which the planes intersect is called the *edge of the angle;* the planes are called the *faces of the angle.*



A diedral angle is measured by the angle formed by two lines, one in each plane and perpendicular to the edge at the same point. Thus, the diedral angle in the margin is measured by the angle ACB.