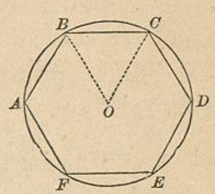
PROBLEM XVII.

To inscribe a regular hexagon in a circle.

Suppose the problem to be solved, and that $ABCDEF$ is a regular hexagon; draw the radii OB and OC. Now, the arc BC is one-sixth of a circumference, or 60°; hence, the angle BOC is 60°, and the other angles OBC and BCO equal 180° minus 60°, or 120°, and, OB being equal to OC, the angles OBC and BCO are equal; hence, each is equal to one-half of 120°, or 60°. Consequently, the triangle OBC is equiangular, and therefore equilateral; hence, the side BC is equal to the radius OB. Therefore, to inscribe a regular hexagon in a circle, we apply the radius six times as a chord to the circumference.



PROBLEMS FOR ORIGINAL THOUGHT.

1. Given the three sides of a triangle, to construct the triangle.
2. Given two sides of a triangle, and the angle opposite one of them, to construct the triangle.
3. To inscribe a circle in a given triangle.
4. To inscribe a circle in a square, and a square in a circle.
5. To find the side of a square which shall be equal to the sum of two given squares.
6. To find the side of a square which shall be equal to the difference between two given squares.
7. To construct a rectangle equal in area to a given triangle.
8. To find a fourth proportional to three given lines.
9. On a given line to construct a rectangle which shall be equal to a given rectangle.
10. To construct a square that shall be equal in area to a given parallelogram.

BOOK V.

PLANES AND THEIR ANGLES.

DEFINITIONS.

1. A PLANE is a surface such that a straight line connecting any two of its points will lie entirely in the surface.

2. A straight line is PERPENDICULAR TO A PLANE when it is perpendicular to every line of the plane passing through its foot. The *foot* is the point where the line meets the plane.

Reciprocally, the plane is also perpendicular to the line.

3. A straight line is PARALLEL TO A PLANE when it cannot meet the plane, however far both be produced.

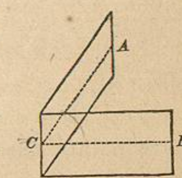
Reciprocally, the plane is also parallel to the line.

4. TWO PLANES ARE PARALLEL when they cannot meet, however far both be produced.

5. When two planes meet, they form a line, which is called their LINE OF INTERSECTION.

6. A DIEDRAL ANGLE is the divergence of two planes. The line in which the planes intersect is called the *edge of the angle;* the planes are called the *faces of the angle.*

A diedral angle is measured by the angle formed by two lines, one in each plane and perpendicular to the edge at the same point. Thus, the diedral angle in the margin is measured by the angle ACB.



7. A Polyedral Angle is the divergence of three or more planes proceeding from a common point.

The common point is called the *vertex* of the angle; the planes are its *faces*; the intersection of the planes, its *edges*.

8. A Triedral Angle is a polyedral angle of three faces.

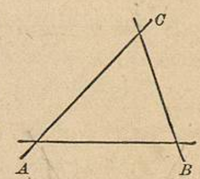
9. Two planes are **Perpendicular to each other** when their diedral angle is a right angle.

Analysis.—This Book treats of planes, the lines and angles formed by their intersection. It is not so valuable in itself as the other Books of Geometry, and much less interesting. Its object is to prepare for the Book which immediately follows it.

THEOREM I.

Through three points not in the same straight line, one plane can be passed, and but one.

Let A, B, and C be the three points; then can one plane be passed through them.



For, join two of the points, as A and C, by the line AC. Pass a plane through AC, and turn it around AC until it contains the point B; it will then pass through the three points A, C, and B. If now the plane be turned about AC, it will no longer contain the point B; hence, only this one plane can be passed through the three points. Therefore, etc.

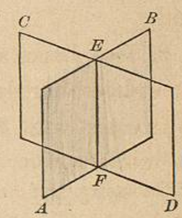
Cor. 1. Since only one plane can be passed through three points, three points are said to determine the position of a plane.

Cor. 2. Two lines which are parallel or which intersect determine the position of a plane.

THEOREM II.

If two planes cut one another, their common section is a straight line.

Let the two planes AB and CD cut one another in the points E and F; then will their common section be a straight line.



For, draw the line EF uniting the two common points E and F of the planes. Now, this line, having two points in the plane AB, will lie wholly in the plane AB (B. I. Def.), and, having two points in the plane CD, it will lie wholly in the plane CD; hence, the line EF is common to both planes, and must therefore be in their common intersection. Therefore, etc.

THEOREM III.

If from a point without a plane lines be drawn to the plane,

1. *The perpendicular is the shortest distance from the point to the plane;*

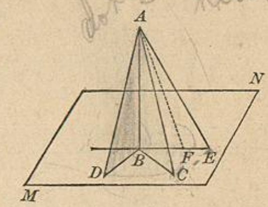
2. *Oblique lines which meet the plane at equal distances from the foot of the perpendicular are equal;*

3. *Of two oblique lines which meet the plane at unequal distances from the foot of the perpendicular, the one which meets it at the greater distance is the longer.*

Let A be a point without the plane MN; let AB be a perpendicular to the plane, and let AC, AD, and AE be oblique lines.

First. AB will be shorter than any oblique line AC. For, through B draw the line BC; then in the triangle ABC, AB is less than AC (B. I. Th. XIV.).

Second. Let AC and AD meet the plane at equal distances from the point B; then AC will be equal to AD. For, draw BC and BD; then the right-angled triangles ABC and ABD will have BC equal to BD, and the side AB common; hence, the triangles are equal, and AC equals AD.



Third. Let AC and AE meet the plane so that the distance BE is greater than BC; then AE will be greater than AC. For, take BF equal to BC and draw AF; then $AE > AF$ (B. I. Th. XIV); but $AF = AC$; hence, $AE > AC$.

Cor. 1. Equal oblique lines drawn from a point to a plane meet the plane at equal distances from the foot of the perpendicular; and of two unequal oblique lines, the greater meets the plane at the greater distance from the foot of the perpendicular.

Cor. 2. The equal oblique lines meet the plane in the circumference of a circle whose centre is B; hence, to draw a perpendicular from a point A to a given plane MN, find any three points, C, D, and F, of the plane equally distant from A, then find the centre of the circumference passing through these points; then AB will be the perpendicular required.

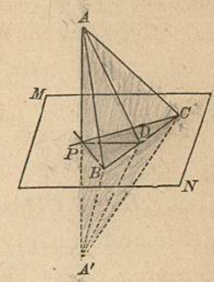
THEOREM IV.

If a straight line is perpendicular to two straight lines of a plane at the point of their intersection, it is perpendicular to the plane of those lines.

Let AP be perpendicular to PB and PC at the point P; then will it be perpendicular to MN, the plane of those lines.

For, let PD be any other straight line of the plane MN

drawn through P. Draw BC, cutting PB, PD, and PC in B, D, and C; produce AP making $PA' = AP$; and draw AB, AD, AC, $A'B$, $A'D$ and $A'C$. Then, since BP and CP are perpendicular to AA' at its middle point, AB equals $A'B$, and AC equals $A'C$, and the triangles ABC and $A'BC$ are equal (B. I. Th. IX.), and also AD equals $A'D$; whence PD is perpendicular to AA' (B. I. Th. XIV. C. 3). Hence, AP is perpendicular to any line passing through its foot; it is, therefore, perpendicular to the plane MN.

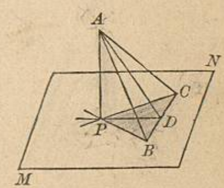


Cor. Only one perpendicular can be erected to a plane from a point of the plane.

THEOREM V.

If from the foot of a perpendicular to a plane a line is drawn at right angles to any line of the plane, and the point of intersection is joined with any point of the perpendicular, the last line will be perpendicular to the line of the plane.

Let AP be a perpendicular to the plane MN, P its foot, BC the given line, and A any point of AP; draw PD perpendicular to BC, and join the points A and D; then will AD be perpendicular to BC.



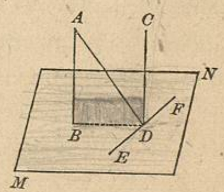
For, lay off BD equal to DC, and draw PB, PC, AB, and AC. Since PD is perpendicular to BC, and DB equals DC, PB equals PC (B. I. Th. XIV.); hence, in the triangles APB and APC, AB equals AC. Therefore the line AD, having two points, A and D, equally distant from B and C, is perpendicular to BC (B. I. Th. XIV. C. 3).

Cor. The line BC is perpendicular to the plane of the triangle APD, because it is perpendicular to AD and PD at the point D (Th. IV.).

THEOREM VI.

If one of two parallels is perpendicular to a plane, the other is also perpendicular to the plane.

Let AB and CD be two parallel lines, and let AB be perpendicular to the plane MN; then will CD also be perpendicular to MN.



For, pass a plane through the parallels cutting MN in BD; draw AD, and in the plane MN draw EF perpendicular to BD at the point D. Then, EF is perpendicular to the plane $ABDC$ (Th. V. C.); hence, the angle EDC is a right angle; but CDB is a right angle, since CD is parallel to AB (B. I. Th. III. C.); hence, CD is perpendicular to the two lines BD and EF at their point of intersection; it is, therefore, perpendicular to the plane MN (Th. IV.). Therefore, etc.

Cor. 1. Conversely.—*Two lines which are perpendicular to the same plane are parallel.* For, suppose the two lines AB and CD to be perpendicular to the plane MN; then, if they are not parallel, draw from the point D a line which is parallel to BA; this line will be perpendicular to MN (Th. VI.); we shall then have two perpendiculars to the plane MN from the same point, which is impossible (Th. IV. C.); therefore, AB and CD are parallel.

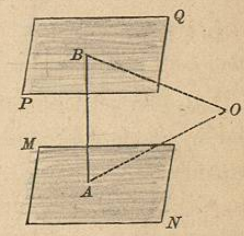
Cor. 2. *Two lines parallel to a third line are parallel to each other.* Let the two lines A and B be parallel to a third line ; pass a plane perpendicular to C, it will be perpendicular

to both A and B (Th. VI.); hence, A and B, being perpendicular to the same plane, are parallel (Th. VI. C. 1).

THEOREM VII.

If two planes are perpendicular to the same straight line, they are parallel.

Let the two planes MN and PQ be perpendicular to the straight line AB; then will they be parallel.

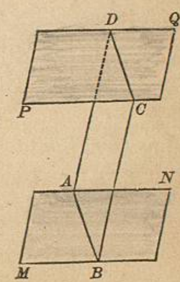


For, if they are not parallel, they will meet in some point O. From O draw the lines OA and OB; then, since OA lies in the plane MN, it will be perpendicular to AB at A (D. 2); and since OB lies in the plane PQ, it will be perpendicular to AB at B. Hence, we have two perpendiculars drawn from the same point to the same straight line, which is impossible (B. I. Th. XIV. C. 1); consequently, the planes cannot meet, and are, therefore, parallel.

THEOREM VIII.

If a plane meet two parallel planes, the lines of intersection are parallel.

Let the plane AC intersect the two parallel planes MN and PQ; then will AB and CD be parallel.



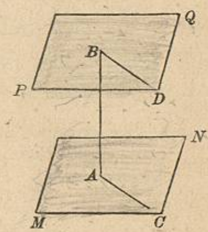
For, if the lines AB and CD are not parallel, since they lie in the same plane, they will meet if sufficiently produced, and, consequently, the planes MN and PQ will meet; but the planes cannot meet, since they are parallel; hence, the lines AB and CD cannot meet; they are, therefore, parallel.

Cor. Parallel lines included between parallel planes are equal. For, the opposite sides of the figure *AC* being parallel, it is a parallelogram, and hence *AD* equals *BC*.

THEOREM IX.

If a straight line is perpendicular to one of two parallel planes, it is perpendicular to the other also.

Let *MN* and *PQ* be two parallel planes, and let the line *AB* be perpendicular to *PQ;* then will it also be perpendicular to the plane *MN*.

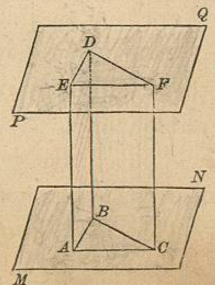


For, pass any plane through *AB;* the intersections *AC* and *BD* will be parallel (Th. VIII.); since *AB* is perpendicular to *PQ*, it will be perpendicular to *BD* (D. 2), and since *BD* and *AC* are parallel, it will be perpendicular to *AC* (B. I. Th. III. C.); hence, *BA*, being perpendicular to any line of the plane *MN* passing through its foot, is perpendicular to the plane *MN*. Therefore, etc.

THEOREM X.

If two angles not in the same plane have their sides parallel and lying in the same direction, the angles will be equal, and their planes parallel.

Let *BAC* and *DEF* be two angles not in the same plane, having their sides respectively parallel and lying in the same direction; then will these angles be equal and their planes parallel.



Take *ED* equal to *AB*, and *EF* equal to *AC*, and draw *BC*, *DF*, *AE*, *BD*, and *CF*.

First. The angles *BAC* and *DEF* will be equal.

For, since *AC* and *EF* are equal and parallel, the figure *ACFE* is a parallelogram (B. I. Th. XVII.), and *AE* and *CF* are equal and parallel. Since *AB* and *ED* are equal and parallel, *ABDE* is a parallelogram, and *AE* and *BD* are equal and parallel; hence, *BD* and *CF* are equal and parallel (Th. VI. C. 2), and, consequently, *DF* is equal and parallel to *BC*. Hence, the triangles *ABC* and *EDF* have their corresponding sides equal; they are, therefore, equal, and the angle *DEF* equals the angle *BAC*.

Second. The planes are parallel.

For, three lines which intersect determine the position of a plane; and since the three sides of the triangles are respectively parallel, their planes must be parallel.

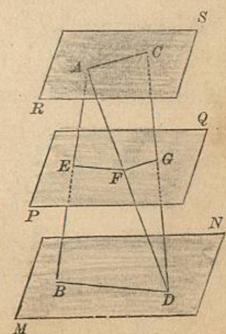
Cor. If three straight lines not in the same plane are equal and parallel, the triangles formed by joining the extremities of these lines will be equal, and their planes parallel. This is readily proved; let the pupil show it.

THEOREM XI.

If two straight lines are cut by three parallel planes, they will be divided proportionally.

Let the lines *AB* and *CD* be cut by the parallel planes *MN*, *PQ*, and *RS*, in the points *A*, *E*, *B*, and *C*, *G*, *D;* then will

$$AE : EB :: CG : GD.$$



For, draw the line *AD*, meeting the plane *PQ* in *F;* draw also *AC*, *EF*, *FG*, and *BD*. Now, since the planes *MN* and *PQ* are parallel, *EF* is parallel to *BD* (Th. VIII.); and since *PQ* and *RS* are parallel, *AC* is parallel to *FG*. Hence (B. III. Th. IX.), we have,

$AE : EB :: AF : FD$; and also,
$AF : FD :: CG : GD$.

Hence, from the principles of proportion, we have,

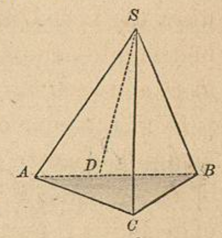
$$AE : EB :: CG : GD.$$

Therefore, etc.

THEOREM XII.

Either angle of the three plane angles which form a triedral angle, is less than the sum of the other two.

Let the triedral angle whose vertex is S be formed by the three plane angles ASC, ASB, and CSB; then will any one of these be less than the sum of the other two.



If the angle considered is less than either of the other two, it is evidently less than their sum. Suppose, however, the angle greater than either of the other two, and let ASB be that angle. In the plane ASB make the angle BSD equal to BSC, draw the line AB at pleasure, make SC equal to SD, and draw AC and BC.

In the two triangles BSC and BSD, BS is common, CS equals DS, and the angle BSC equals BSD by construction; hence, the triangles are equal, and BD equals BC. Now (B. I. A. 10, C.),

$$AD + DB < AC + BC.$$

And, taking away the equals DB and BC, we have,

$$AD < AC.$$

Hence (B. I. Th. VIII. C.), we have,

$$\text{angle } ASD < \text{angle } ASC;$$

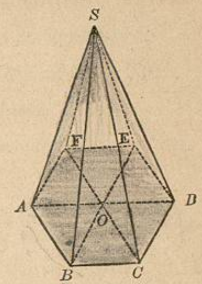
and, adding the equal angles DSB and CSB, we have,

$$\text{angle } ASB < \text{angle } ASC + \text{angle } CSB.$$

Therefore, etc.

THEOREM XIII.

The sum of the plane angles which form any polyedral angle, is less than four right angles.



Let S be the vertex of a polyedral angle formed by the plane angles ASB, BSC, CSD, etc.; then will the sum of these plane angles be less than four right angles.

For, pass a plane cutting the edges in the points A, B, C, D, E, and F, and the faces in the lines AB, BC, etc. From any point, O, in the polygon thus formed, draw the lines OA, OB, OC, etc. We then have two sets of triangles, one set having their vertices at S, the other at O, and both having the common bases AB, BC, etc.

Now, the sum of the angles of the upper set of triangles is equal to the sum of the angles of the lower set of triangles, since both sets consist of the same number of triangles. But the sum of the angles SBA and SBC is greater than ABC, or $ABO + OBC$ (Th. XII.); and also $SCB + SCD$ is greater than $OCB + OCD$; and so on with the other angles at D, E, etc. Hence, the sum of all the angles at the bases of the upper set of triangles is greater than the sum of all the angles at the bases of the lower set of triangles; therefore, the sum of the angles at S must be less than the sum of the angles at O. But the sum of the angles at O is equal to four right angles (B. I. Th. II. C. 2); hence, the sum of the angles at S is less than four right angles. Therefore, etc.

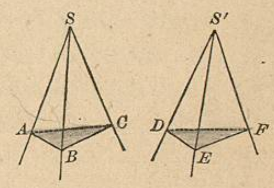
Scholium. This proposition supposes that the polyedral angle is convex; if it were not, the sum of the plane angles would be unlimited.

THEOREM XIV.

If the three face angles of a triedral angle are respectively equal, the triedral angles are either equal or symmetrical.

Let S and S' be the vertices of two triedral angles in which ASB equals $DS'E$, $BSC = ES'F$ and $ASC = DS'F$; then the diedral angles are respectively equal, and the triedral angles are either equal or symmetrical.

For, on SB take any point B, and in the faces ASB and BSC draw the lines BA and BC perpendicular to SB; then the angle ABC will measure the diedral angle of the faces ASB and BSC (Def. 6). On $S'E$ lay off $S'E$ equal to SB, and on the faces $DS'E$ and $ES'F$ draw DE and EF perpendicular to $S'E$; then the angle DEF will measure the diedral angle of the faces $DS'E$ and $ES'F$.



The right-angled triangles SBA and $S'ED$ are equal, since $SB = S'E$ and $ASB = DS'E$; hence, AB equals DE and AS equals DS'. In a similar way it may be shown that BC equals EF and CS equals FS'. Hence, the triangles ASC and $DS'F$ have their sides respectively equal; and ASC equals $DS'F$ by hypothesis; therefore AC equals DF. Hence, the triangles ABC and DEF have their sides respectively equal, and consequently their corresponding angles are equal. Hence the angle ABC, which measures the diedral angle of the planes ASB and BSC, is equal to the angle DEF, which measures the diedral angle of the planes $DS'E$ and $ES'F$, or the diedral angles are equal. In the same way it may be shown that the other diedral angles are respectively equal.

Now, if the face angles of these triedral angles are similarly placed, the triedral angles may be applied to each other and they will coincide; if, however, the face angles are not similarly placed, the triedral angles will not coincide, but are then said to be *symmetrical*. Therefore, etc.

Scholium. Polyedral angles are said to be *symmetrical* when, having the same number of face angles, these angles and the successive diedral angles are respectively arranged in a reverse order.

THEOREMS FOR ORIGINAL THOUGHT.

1. Prove that but one plane can be passed through a given point perpendicular to a given line.

2. If a line is perpendicular to a plane, every plane passed through the line is perpendicular to that plane.

3. If two planes are perpendicular to each other, a line drawn in one of them perpendicular to their intersection is perpendicular to the other.

4. If two planes which cut each other are both perpendicular to a third plane, their intersection is perpendicular to that plane.

5. Prove that through a given line of a given plane, only one plane perpendicular to the given plane can be passed.

6. Prove that through a line parallel to a given plane, only one plane perpendicular to the given plane can be passed.

7. If two planes which intersect contain two lines parallel to each other, the intersection of the planes will be parallel to the lines.

8. If a line is parallel to one plane and perpendicular to another, these two planes are perpendicular.

9. If two planes are parallel to a third, they are parallel to each other.

10. Only one plane can be drawn through a given point parallel to a given plane.

11. If two lines are parallel in space, and planes be passed through them perpendicular to a third plane, the two planes will be parallel.

PROBLEMS.

The following problems are easily solved from the principles already presented.

1. To erect a perpendicular to a given plane at a given point of the plane. (See Prop. III.)
2. To construct a plane parallel to a given plane.
3. To construct a plane perpendicular to a given plane intersecting it in a given straight line.
4. To draw a line from a given point of a plane making any given angle with the plane.
5. To draw a plane intersecting a given plane and making any given angle with it.

BOOK VI.

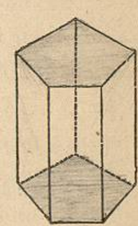
POLYEDRONS.

DEFINITIONS.

1. A POLYEDRON is a volume bounded by polygons.

The bounding polygons are called the *faces* of the polyedron; the lines in which the faces meet are called *edges;* and the points in which the edges meet are called *vertices* of the polyedron.

2. A PRISM is a polyedron, two of whose faces are equal polygons, having their homologous sides parallel; the other faces are parallelograms.



The equal polygons are called *bases* of the prism; one, the *upper base;* the other, the *lower base.* The parallelograms constitute the *lateral* or *convex surface* of the prism; the intersections of the lateral faces are called *lateral edges.*

3. The ALTITUDE of a prism is the perpendicular distance between its bases.

4. A RIGHT PRISM is one whose lateral edges are perpendicular to the bases. In a right prism, each lateral edge is equal to the altitude.

5. An OBLIQUE PRISM is one whose lateral edges are oblique to the bases. Each lateral edge is, consequently, greater than the altitude.

6. A prism is named from its bases. A *triangular prism* is one whose bases are triangles; a *quadrangular prism* is one whose bases are *quadrilaterals;* and so on.