PROBLEMS.

The following problems are easily solved from the principles already presented.

1. To erect a perpendicular to a given plane at a given point of the plane. (See Prop. III.)

2. To construct a plane parallel to a given plane.

3. To construct a plane perpendicular to a given plane intersecting it in a given straight line.

4. To draw a line from a given point of a plane making any given angle with the plane.

5. To draw a plane intersecting a given plane and making any given angle with it.

BOOK VI.

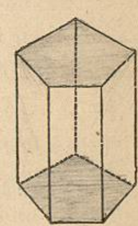
POLYEDRONS.

DEFINITIONS.

1. A POLYEDRON is a volume bounded by polygons.

The bounding polygons are called the *faces* of the polyedron; the lines in which the faces meet are called *edges;* and the points in which the edges meet are called *vertices* of the polyedron.

2. A PRISM is a polyedron, two of whose faces are equal polygons, having their homologous sides parallel; the other faces are parallelograms.



The equal polygons are called *bases* of the prism; one, the *upper base;* the other, the *lower base.* The parallelograms constitute the *lateral* or *convex surface* of the prism; the intersections of the lateral faces are called *lateral edges.*

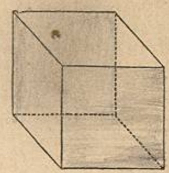
3. The ALTITUDE of a prism is the perpendicular distance between its bases.

4. A RIGHT PRISM is one whose lateral edges are perpendicular to the bases. In a right prism, each lateral edge is equal to the altitude.

5. An OBLIQUE PRISM is one whose lateral edges are oblique to the bases. Each lateral edge is, consequently, greater than the altitude.

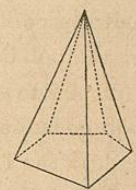
6. A prism is named from its bases. A *triangular prism* is one whose bases are triangles; a *quadrangular prism* is one whose bases are *quadrilaterals;* and so on.

7. A PARALLELOPIPEDON is a prism whose bases are parallelograms.



A RECTANGULAR PARALLELOPIPEDON is a *right* parallelopipedon with rectangular bases. A *cube* is a rectangular parallelopipedon, all of whose faces are equal squares.

8. A PYRAMID is a polyedron bounded by a polygon, and by triangles meeting at a common point.



The polygon is called the *base* of the pyramid; the triangles, its *lateral* or *convex surface;* and the point where the triangles meet, its *vertex.*

9. Pyramids are named from their bases; thus, we have *triangular*, *quadrangular*, *pentangular*, etc. pyramids, as the bases are triangles, quadrilaterals, pentagons, etc.

10. The ALTITUDE of a pyramid is the perpendicular distance from the vertex to the plane of the base.

11. A RIGHT PYRAMID is one whose base is a regular polygon, and in which a perpendicular from the vertex to the base passes through the centre of the base. This perpendicular is called the axis of the pyramid.

12. The SLANT HEIGHT of a right pyramid is the perpendicular distance from its vertex to any side of the base.

13. A FRUSTUM OF A PYRAMID is the part of a pyramid included between its base and a plane cutting the pyramid parallel to the base.

14. The ALTITUDE of a frustum of a pyramid is the perpendicular distance between its bases.

15. The SLANT HEIGHT of a frustum of a right pyramid is that portion of the slant height of the pyramid included between the bases of the frustum.

16. SIMILAR POLYEDRONS are those which are bounded by the same number of similar polygons, similarly placed. Parts which are similarly placed are *homologous*, whether faces, angles, or edges.

17. The DIAGONAL of a polyedron is a line joining the vertices of any two polyedral angles not in the same face.

18. The VOLUME of a polyedron is its numerical value, expressing how many times it contains some other polyedron as a unit.

19. A RIGHT SECTION of a prism is a section perpendicular to its lateral edges. An *oblique* section is one oblique to its lateral edges.

20. A TRUNCATED PRISM is a portion of the prism included between either base and an oblique section of the prism.

ANALYSIS.—This book treats of prisms, pyramids, and frustums. The object is to find the surface and volume of these polyedrons, and the relation of those which are similar. Their surface is readily determined by finding the area of the polygons which form their faces. In finding their volumes, we begin with the rectangular parallelopipedon, assuming for a *unit of measure* a cube whose edge is a unit of measure of the edges of the parallelopipedon. From the volume of a rectangular parallelopipedon we pass to that of any parallelopipedon, thence to the volume of a triangular prism, and from this to that of any prism. The division of a triangular prism into three equal parts gives the volume of a triangular pyramid, from which we pass to the volume of any pyramid, and also of any frustum.

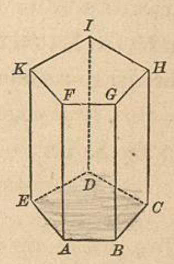
THE PRISM.

THEOREM I.

The convex surface of a right prism is equal to the perimeter of the base multiplied by the altitude.

Let $ABCDE—K$ be a right prism; then will its convex surface be equal to

$(AB + BC + CD + DE + EA) \times AF.$



For, the convex surface of the prism is equal to the sum of all the rectangles AG, BH, CI, etc. Now, the altitude of each of these rectangles is equal to the altitude of the prism, and the area of each rectangle is equal to its base multiplied by its altitude; hence, the convex surface, which is the sum of the areas of these rectangles, is equal to

$$(AB + BC + CD + DE + EA) \times AF;$$

or, the perimeter of the base multiplied by the altitude. Therefore, etc.

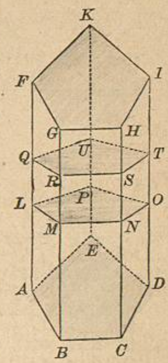
Cor. If two right prisms have the same altitude, their convex surfaces are to each other as the perimeters of their bases.

THEOREM II.

If a prism be cut by parallel planes, the sections formed will be equal polygons.

Let the prism $ABCDE—K$ be cut by the parallel planes

LO and QT; then will the sections LO and QT be equal polygons.



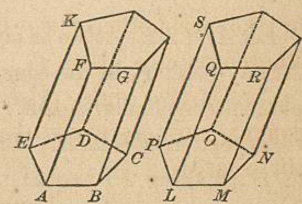
For, LM and QR are parallel, being the intersections of the two parallel planes with $ABGF$ (B. V. Th. VIII.); these lines LM and QR are also equal, since they are parallels included between the two parallels AF and BG (B. I. Th. XV. C. 2.). For a like reason, MN is equal and parallel to RS, NO to ST, OP to TU, etc.; hence, the angle LMN is equal to the angle QRS, MNO to RST, etc. (B. V. Th. X.); therefore, the sections are equal polygons.

Cor. Every section of a prism parallel to the base is equal to the base.

THEOREM III.

Two prisms are equal if three faces, including a triedral angle of one, are respectively equal to three faces similarly placed, including a triedral of the other.

Let the three faces of the triedral angles A and L of the prisms $ABCDE—F$ and $LMNOP—Q$ be equal and similarly placed; then will the prisms be equal.



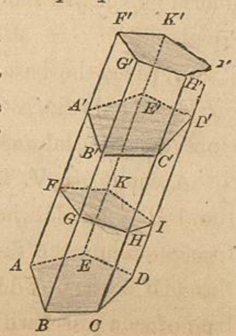
For, place the base $ABCDE$ on its equal $LMNOP$; then, since the triedral angles A and L are equal (B. V. Th. XIV.), they will coincide when applied to each other, the face AG will coincide with LR, and the face AK with LS; hence, the sides FG and FK of the upper base of one prism will coincide with the sides QR and QS of the upper base of the other prism; hence also the planes of their bases will coincide, and, since these bases are equal, they will coincide throughout; consequently, all the lateral faces of the two prisms will coincide, each to each, and the prisms will coincide throughout, and are therefore equal. Therefore, etc.

Cor. 1. *Two right prisms are equal if they have equal bases and equal altitudes.* For, if the faces are similarly placed, the prisms will coincide when applied to each other. If the faces are not similarly placed, by inverting one prism the faces will be similarly placed, and the prisms may be applied to each other and will coincide.

Cor. 2. *Two truncated prisms are equal if three faces including a triedral angle of one are respectively equal to three faces including a triedral angle of the other.* For, the above demonstration will apply whether the upper bases are parallel to the lower bases or are inclined to them, as they are in a truncated prism.

THEOREM IV.

An oblique prism is equivalent to a right prism whose base is equal to a right section of the oblique prism, and whose altitude is equal to a lateral edge of the oblique prism.



Let $ABCDE-A'$ be an oblique prism. At any point F, in the edge AA', pass a plane perpendicular to AA', forming the right section $FGHIK$. Produce AA' to F', making FF' equal to AA', and through F' pass a second plane perpendicular to the edge AA', intersecting all the faces of the prism produced, and forming another right section,

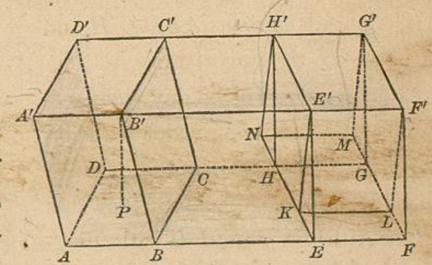
$F'G'H'I'K'$, parallel and equal to $FGHIK$. The prism $FGHIK-F'$ is a right prism whose base is the right section $FGHIK$, and whose altitude, FF', is equal to the lateral edge of the oblique prism.

Now, the truncated prism $ABCDE-F$ is equal to the truncated prism $A'B'C'D'E'-F'$ (Th. III. Cor. 2); if to each of these equal prisms we add the volume $FGHIK-A'$, we shall have the oblique prism $ABCDE-A'$, equivalent to the right prism $FGHIK-F'$. Therefore, etc.

THEOREM V.

Any parallelopipedon is equivalent to a rectangular parallelopipedon having the same altitude and an equivalent base.

Let $ABCD-D'$ be any oblique parallelopipedon whose base is $ABCD$ and altitude $B'P$. Produce the edges, AB, DC, $A'B'$, and $D'C'$; on AB produced take EF equal to AB, and through E and F pass planes perpendicular to the produced edges, forming the parallelopipedon $EFGH-H'$.



This parallelopipedon, regarding $EE'H'H$ as the base and EF the altitude, is a right prism, and is equivalent to the oblique parallelopipedon $ABCD-A'$ (Th. IV.).

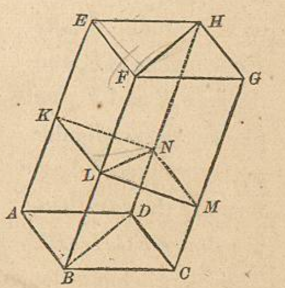
Again, from E' and H' let fall the perpendiculars $E'K$ and $H'N$ to EH produced, and from F' and G' let fall the perpendiculars $F'L$ and $G'M$ to FG produced, and draw LK and MN; then $KLMN-H'$ will be a rectangular parallelopipedon whose base is $KLMN$ and altitude $E'K$ equal to $B'P$.

Thus, the rectangular parallelopipedon $KLMN-H'$ is equivalent to the oblique parallelopipedon $EFF'E'-H$ (Th. IV.); but this parallelopipedon $EFF'E'-H$, which, with $EE'H'H$ as a base, is a right parallelopipedon, is equivalent to the parallelopipedon $ABCD-D'$ (Th. IV.); hence, $ABCD-D'$ is equivalent to the right parallelopipedon $KLMN-H'$. Also, the base KM is equal to the base $E'G'$, and hence to its equal EG, which is equivalent to the base AC; hence KM is equivalent to AC. Therefore, etc.

THEOREM VI.

The plane passed through two diagonally opposite edges of a parallelopipedon divides the parallelopipedon into two equivalent triangular prisms.

Let $ABCD-E$ be any parallelopipedon, and let a plane be passed through its opposite edges BF and DH; then will the triangular prisms $ABD-H$ and $BCD-H$ be equivalent.

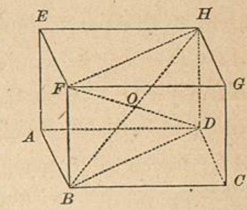


For, let $KLMN$ be any right section of the parallelopipedon made by a plane perpendicular to the edge AE; the intersection, LN, of this plane with the plane BH is the diagonal of the parallelogram $KLMN$, and divides the parallelogram into two equal triangles, LMN and KLN. The oblique prism $ABD-H$ is equivalent to a right prism whose base is the triangle KLN and whose altitude is AE (Th. IV.); and the oblique prism $BCD-H$ is equivalent to a right prism whose base is the triangle LMN and altitude AE; but the two right prisms are equal (Th. III. C. 1); hence, the two oblique prisms are equivalent to each other. Therefore, etc.

THEOREM VII.

The opposite faces of a parallelopipedon are equal and parallel.

Let $ABCD-H$ be a parallelopipedon; then will its opposite faces be equal and parallel.

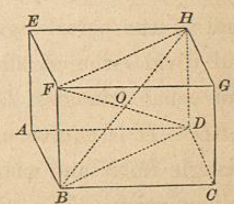


For, the bases are equal and parallel, by the definition of a parallelopipedon. Also, BC is equal and parallel to AD, since the figure $ABCD$ is a parallelogram, and, for a similar reason, BF and AE are equal and parallel; consequently, the angles EAD and FBC are equal, and their planes parallel (B. V. Th. X.), and, therefore, the parallelograms BG and AH are equal (B. I. Th. XV. C. 3). In a similar manner it may be shown that the faces AF and DG are equal and parallel. Therefore, etc.

Cor. 1. Any two opposite faces of a parallelopipedon may be taken as the bases.

Cor. 2. *The diagonals of a parallelopipedon bisect each other.*

Draw the diagonals FD and BH; draw also BD and FH; then, since BF and HD are equal and parallel, the figure $BDHF$ is a parallelogram; hence, the diagonals FD and BH bisect each other at O (B. I. Th. XVIII.). In the same manner it may be shown that either of these and any other diagonal bisect each other; hence, all the diagonals bisect each other.

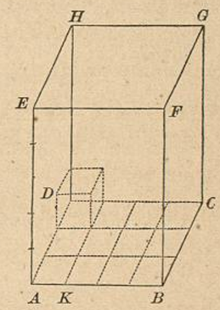


Cor. 3. In a rectangular parallelopipedon, the square of either diagonal equals the sum of the squares of the three edges which meet at the same vertex. Let the pupil show it; that $\overline{BH}^2 = \overline{BC}^2 + \overline{DC}^2 + \overline{DH}^2$.

THEOREM VIII.

The volume of a rectangular parallelopipedon is equal to the product of its base and altitude.

Let $ABCD—H$ be a rectangular parallelopipedon; then will its volume be equal to its base $ABCD$ multiplied by its altitude AE.



Suppose AK to be a common unit of measure of the three sides AB, AD, and AE, and suppose it to be contained 4 times in AB, 3 times in AD, and 5 times in AE; then divide AB into 4 equal parts, AD into 3, and AE into 5 equal parts, and pass planes through the points of division parallel to the faces of the parallelopipedon. The parallelopipedon will thus be divided into equal cubes, equal since their sides are equal and their angles are equal, all being right angles.

Now, the number of these little cubes upon the base is equal to the number of surface units in the base, and the whole number of cubes in the parallelopipedon is equal to the number upon the base multiplied by the number of layers, and the number of layers is the same as the number of units in the altitude; hence, the number of cubic units in the parallelopipedon is equal to the base multiplied by the altitude. Now, this is evidently true whatever be the size of the linear unit; hence, it is true when the linear unit is exceedingly small, and, consequently, when it is infinitely small, as it must be when the three sides are incommensurable. Therefore, the volume of a rectangular parallelopipedon is equal to the product of its base and altitude.

Cor. 1. It is evident that the number of cubic units upon the base is equal to the number of rows multiplied by the number in each row; that is, the length of the base multiplied by its breadth; hence, *the volume of a rectangular parallelopipedon equals the product of its length, breadth, and altitude, or the product of its three dimensions.*

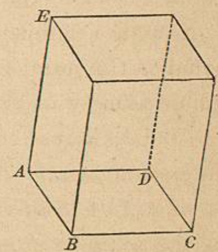
Cor. 2. Any two rectangular parallelopipedons are to each other as the products of their bases and altitudes, or as the products of their three dimensions.

Cor. 3. When their bases are equal, they are to each other as their altitudes; when their altitudes are equal, they are to each other as their bases.

THEOREM IX.

The volume of any parallelopipedon is equal to the product of its base and altitude.

Let $ABCD-E$ be any parallelopipedon whose base is $ABCD$ and altitude H; then will its volume be equal to the base $ABCD$ multiplied by the altitude H.

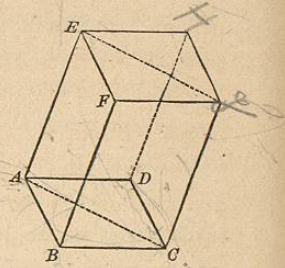


For, the parallelopipedon $ABCD—E$ is equivalent to a rectangular parallelopipedon having the same altitude and an equivalent base (Th. V.); but the volume of such a rectangular parallelopipedon is equal to the product of its base and altitude (Th. VIII.); hence, the volume of the parallelopipedon $ABCD-E$ is equal to the product of its base and altitude. Therefore, etc.

THEOREM X.

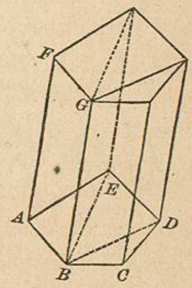
The volume of any prism is equal to the product of its base and altitude.

1st. Let $ABC-E$ be a triangular prism. This prism is half the parallelopipedon constructed on its edges AB, BC, and BF (Th. VI.). The volume of this parallelopipedon is equal to its base $ABCD$ multiplied by its altitude (Th. IX.); hence, the volume of the triangular prism $ABC-E$ is equal to its base ABC, the half of $ABCD$, multiplied by its altitude.



2d. Let $ABCDE-F$ be any prism. Divide it into triangular prisms by passing planes through any lateral edge BG; these prisms will have a common altitude, the altitude of the prism.

The volume of any triangular prism, $ABE-F$, is equal to the product of its base and altitude, as just shown; hence, the volume of the prism $ABCDE-F$, which is the sum of these triangular prisms, is equal to its base, which is the sum of the bases of the triangular prisms, multiplied by its altitude. Therefore, etc.



Cor. 1. Prisms having equivalent bases and equal altitudes are equivalent.

Cor. 2. Any two prisms are to each other as the products of their bases and altitudes.

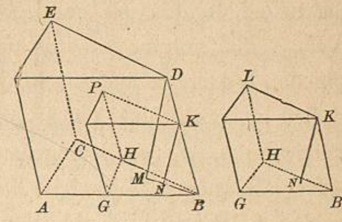
Cor. 3. Prisms having equal altitudes are to each other as their bases.

Cor. 4. Prisms having equal bases are to each other as their altitudes.

THEOREM XI.

Similar triangular prisms are to each other as the cubes of their homologous edges.

Let $ABC-E$ and $GBH-L$ be two similar triangular prisms; then will they be to each other as the cube of any two homologous edges AB and GB.



For, since the two prisms are similar, the faces containing the triedral angles B and B are respectively similar; therefore, the prism $GBH-L$ being applied to the prism $ABC-E$ will take the position $GBH-P$. From D draw DM perpendicular to the base, and from K draw KN perpendicular to the base; then the two triangles DMB and KNB must be similar, since they are mutually equiangular.

Now, since the bases are similar, we have (B. III. Th. XVI.),

$$base\ ABC : base\ GBH :: \overline{AB}^2 : \overline{GB}^2;$$

and, since the triangles DMB and KNB are similar, and also the parallelograms AD and GK, we have,

$$DM : KN :: DB : KB :: AB : GB.$$

Multiplying together the corresponding terms of the first and last couplets of these two proportions, we have,

$$base\ ABC \times DM : base\ GBH \times KN :: \overline{AB}^3 : \overline{GB}^3.$$

But $base\ ABC \times DM$ is the volume of the prism $ABC-E$, and $base\ GBH \times KN$ is the volume of the prism $GBH-L$; hence, the prisms are to each other as $\overline{AB}^3$ to $\overline{GB}^3$. Therefore, etc.