Cor. 1. *Any two similar prisms are to each other as the cubes of their homologous edges.*

For, since the prisms are similar, their bases are similar, and may, therefore, be divided into the same number of similar triangles, similarly situated (B. III. Th. XVII.); hence, each prism may be divided into the same number of similar triangular prisms. But these triangular prisms are to each other as the cubes of their homologous edges; hence, the polygonal prisms which are respectively the sum of these triangular prisms must be to each other as the cubes of their homologous edges.

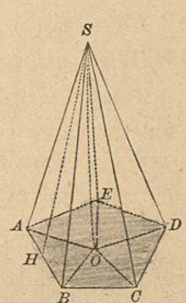
Cor. 2. Similar prisms are to each other as the cubes of their altitudes, or as the cubes of any other homologous lines.

THE PYRAMID.

THEOREM XII.

The convex surface of a right pyramid is equal to the perimeter of the base multiplied by one-half of the slant height.

Let $ABCDE—S$ be a right pyramid, and SH the slant height; then will the convex surface be equal to the perimeter $AB + BC + CD + DE + EA$ multiplied by $\frac{1}{2}$ of SH.



Draw SO perpendicular to the base; then, from the definition of a right pyramid, O is the centre of the base; consequently, the distances AO, BO, CO, etc. are all equal, and therefore the edges SA, SB, SC, etc., are all equal (B. V. Th. III.); and, since the sides AB, BC, etc., are all equal, the triangles SAB, SBC, etc. are all equal, and

their altitudes, which is the slant height of the pyramid, are equal.

Now, the area of each triangle is equal to its base multiplied by one-half of its altitude; hence, the sum of the areas of these triangles, which is the convex surface of the pyramid, equals the sum of their bases into one-half of the slant height SH; that is, the convex surface of the pyramid equals

$$(AB + BC + CD + DE + EA) \times \tfrac{1}{2} SH.$$

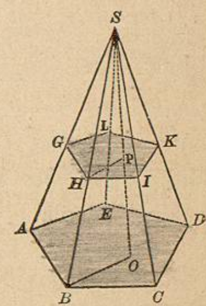
Therefore, etc.

THEOREM XIII.

If a pyramid be cut by a plane parallel to the base;

1. *The edges and altitude will be divided proportionally.*
2. *The section will be a polygon similar to the base.*

Let the pyramid $S—ABCDE$ be cut by a plane $GHIKL$ parallel to the base; then will the edges SA, SB, SC, etc., with the altitude SO, be divided proportionally, and the section $GHIKL$ will be similar to the base.



First. Since the planes $ABCDE$ and $GHIKL$ are parallel, the intersections AB and GH are parallel (B.V. Th. VIII.); for the same reason, BC is parallel to HI, and BO to HP. Hence, we have (B. III. Th. IX. C. 1),

$$SA : SG :: SB : SH;$$

and also, $$SB : SH :: SC : SI;$$

and also, $$SB : SH :: SO : SP.$$

Hence, the edges and altitude are divided proportionally.

Second. Since GH is parallel to AB, and HI to BC, the angle GHI is equal to ABC (B. V. Th. X.); and, for the same reason, each angle of the polygon $GHIKL$ is equal to the corresponding angle of the base; hence, the two polygons are mutually equiangular.

Again, since GH is parallel to AB, we have,

$$GH : AB :: SH : SB;$$

and, since HI is parallel to BC, we have,

$$HI : BC :: SH : SB.$$

Hence, from equal ratios, we have,

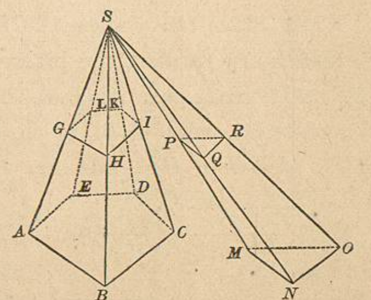
$$GH : AB :: HI : BC.$$

In the same manner, it may be shown that all the sides of the two polygons are proportional; hence, the section $GHIKL$ is similar to the base $ABCDE$ (B. III. D. 6).

THEOREM XIV.

If two pyramids have the same altitude, and their bases in the same plane, the sections made by a plane parallel to their bases are to each other as their bases.

Let $S—ABCDE$ and $S—MNO$ be two pyramids, having the same altitude, and their bases in the same plane; and let $GHIKL$ and PQR be sections made by a plane parallel to their bases; then will these sections be to each other as the bases.



For, the polygons $ABCDE$ and $GHIKL$, being similar, are to each other as the squares of their sides AB and GH (B. III. Th. XVIII.); but

$$AB : GH :: SA : SG.$$

Hence, $\quad ABCDE : GHIKL :: \overline{SA}^2 : \overline{SG}^2.$

For a similar reason,

$$MNO : PQR :: \overline{SM}^2 : \overline{SP}^2.$$

But (B. V. Th. XI.) we have,

$$SA : SG :: SM : SP:$$

Hence, $\quad ABCDE : GHIKL :: MNO : PQR.$

Therefore, etc.

Cor. 1. If the bases are equal, any two sections parallel to the bases at equal distances from the vertices are equal.

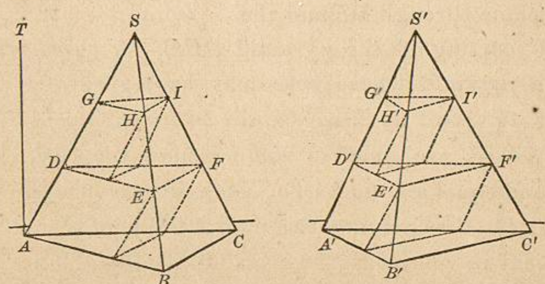
Cor. 2. Any two sections parallel to the base are proportional to the squares of their distances from the vertex.

THEOREM XV.

Two triangular pyramids having equivalent bases and equal altitudes are equal in volume.

Let $S-ABC$ and $S'-A'B'C'$ be two triangular pyramids having equivalent bases ABC and $A'B'C'$ and equal altitudes AT; then will these pyramids be equal in volume.

For, place the bases of the pyramids in the same plane,

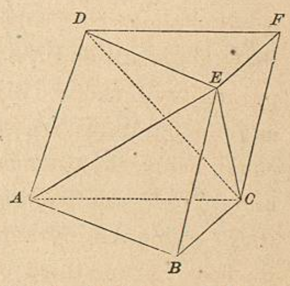


divide the altitude, AT, into any number of equal parts, and through the points of division pass planes parallel to the plane of their bases, forming the sections DEF, $D'E'F'$, etc., and construct prisms in the two pyramids with these sections as upper bases. Now, the corresponding sections DEF, $D'E'F'$, etc., are equivalent (Th. XIV.); hence, the corresponding prisms, having equivalent bases and equal altitudes, are equivalent (Th. X. C. 1).

Now, this is true whatever the equal number of inscribed prisms; hence, it is true when the number in each prism becomes indefinitely or infinitely great, in which case they will coincide respectively with the two pyramids; therefore, the pyramids are equal in volume. Therefore, etc.

THEOREM XVI.

A triangular prism may be divided into three equal triangular pyramids.



Let $ABC—F$ be a triangular prism; then may it be divided into three equal triangular pyramids.

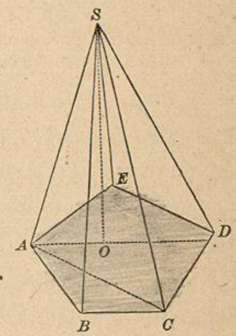
Pass a plane through the edge AC and the point E, cutting off the pyramid $ABC—E$; pass another plane through DE and the point C, cutting off the pyramid $DEF—C$; there will remain a pyramid whose base may be regarded as ACD, having its vertex at E. Now, the two pyramids $ABC—E$ and $DEF—C$ are equal in volume, since they have equal bases and equal altitudes (Th. XV.). Regarding the pyramid $DEF—C$ as having the base DCF and vertex at E, it is equal in volume to the pyramid $ACD—E$, since their bases are equal, being halves of the parallelogram $ACFD$, and their altitudes are equal, since their bases are in the same plane and vertices at the same point. Hence, the three pyramids into which the prism is divided are all equal in volume. Therefore, etc.

Cor. 1. A triangular pyramid is one-third of a prism having an equal base and an equal altitude.

Cor. 2. The volume of a triangular pyramid is one-third of the product of its base and altitude.

THEOREM XVII.

The volume of a pyramid is equal to one-third of the product of its base and altitude.



Let $S—ABCDE$ be a pyramid, and SO the altitude; then will its volume be equal to $ABCDE \times \frac{1}{3} SO$.

Draw the diagonals AC and AD, and pass the planes SAC and SAD through these diagonals and the vertex S; the pyramid will then be divided into triangular pyramids, whose altitudes are equal, being the altitude of the pyramid. Now, the volume of each of these triangular pyramids is equal to its base by one-third of the altitude (Th. XVI. C. 2); hence, the volume of the pyramid $S—ABCDE$, which is the sum of these triangular pyramids, is equal to the sum of their bases into one-third of the altitude; that is, base $ABCDE \times \frac{1}{3} SO$. Therefore, etc.

Cor. 1. The volume of a pyramid is one-third of the volume of a prism having an equal base and an equal altitude.

Cor. 2. Pyramids are to each other as the products of their bases and altitudes.

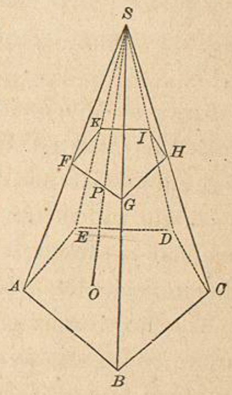
Cor. 3. Pyramids having equal bases are to each other as their altitudes; pyramids having equal altitudes are to each other as their bases.

Scholium. The volume of any polyedron may be found by dividing it into triangular pyramids, by passing planes through its vertices.

THEOREM XVIII.

Similar pyramids are to each other as the cubes of their homologous edges.

Let $S—ABCDE$ and $S—FGHIK$ be two similar pyramids; then will they be to each other as the cubes of any two homologous sides AB and FG.



For, since the pyramids are similar, they may be so placed that their homologous angles at the vertex will coincide. Then, since the faces SAB and SFG are similar, AB is parallel to FG; and since SBC and SGH are similar, BC is parallel to GH; hence, the planes of the bases are parallel (B. V. Th. X.).

Draw SO perpendicular to the base $ABCDE$; it will also be perpendicular to the base $FGHIK$ at some point, P; then (Th. XIII.),

$$SO : SP :: SB : SG :: AB : FG;$$

and, consequently,

$$\tfrac{1}{3}\, SO : \tfrac{1}{3}\, SP :: AB : FG.$$

But, the bases of the pyramids being similar, we have (B. III. Th. XVIII.),

$$base\ ABCDE : base\ FGHIK :: \overline{AB}^2 : \overline{FG}^2.$$

Multiplying these two proportions, term by term, we have,

$$base\ ABCDE \times \tfrac{1}{3}\, SO : base\ FGHIK \times \tfrac{1}{3}\, SP :: \overline{AB}^3 : \overline{FG}^3.$$

But, $base\ ABCDE \times \frac{1}{3}\, SO$ is equal to the volume of the pyramid $S—ABCDE$, and $base\ FGHIK \times \frac{1}{3}\, SP$ is equal to the volume of the pyramid $S—FGHIK$; hence, the two pyramids are to each other as the cubes of the homologous edges AB and FG. Therefore, etc.

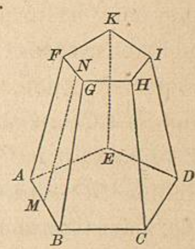
Cor. Similar pyramids are to each other as the cubes of their altitudes, or as the cubes of any two homologous lines.

FRUSTUM OF A PYRAMID.

THEOREM XIX.

The convex surface of a frustum of a right pyramid is equal to one-half of the sum of the perimeters of the upper and lower bases, multiplied by the slant height.

Let $ABCDE—K$ be the frustum of a right pyramid, and NM its slant height; then will its convex surface be equal to one-half of the sum of the perimeters of its two bases, multiplied by NM.



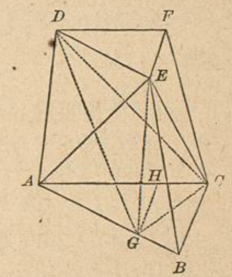
The faces forming the convex surface are equal trapezoids; for the faces of the pyramid of which this frustum is a part are equal, and the faces of the pyramid cut off are equal; hence, the figures which remain are equal, and their upper and lower bases being parallel, they are equal trapezoids, and have a common altitude NM, the slant height of the frustum.

Now, the area of each trapezoid, as $ABGF$, is equal to $\frac{1}{2}(AB + FG) \times NM$ (B. III. Th. IV.); hence, the area of the convex surface, which is the sum of all the trapezoids, is equal to one-half the sum of the perimeters of the upper and lower bases multiplied by the slant height. Therefore, etc.

THEOREM XX.

The volume of a frustum of a triangular pyramid is equal to the sum of the volumes of three pyramids, whose common altitude is the altitude of the frustum, and whose bases are the lower base of the frustum, the upper base of the frustum, and a mean proportional between the two bases.

Let $ABC—F$ be the frustum of a triangular pyramid. Through the points A, E, C, pass a plane cutting off the pyramid $E—ABC$. This pyramid has the altitude of the frustum, and for its base the lower base of the frustum. Through the points D, E, C, pass a plane cutting off the pyramid $C—DEF$. This pyramid has the altitude of the frustum, and for its base the upper base of the frustum. The remaining part of the frustum is a pyramid whose base is ACD, with its vertex at E.



Now, draw EG parallel to DA; draw also GD; then the pyramid $E - ACD$ is equal to the pyramid $G - ACD$, since they have the same base and equal altitudes. But the pyramid $G—ACD$ may be regarded as having AGC for its base, and its vertex at D; it will then have the altitude of the frustum. We will now show that its base AGC is a mean proportional between the two bases of the frustum.

Draw GH parallel to BC; then the triangles AGH and DEF, being similar to ABC, are similar to each other, and, hence, equiangular; and since AG equals DE, the triangle AGH equals DEF (B. I. Th. VII.). Now, AGC is a mean proportional between AGH and ABC (B. III. Th. IX. C. 3); hence, the base of the third pyramid is a mean proportional between the upper and lower bases. Therefore, etc.

Cor. This proposition is true for the frustum of any pyramid. For, since any pyramid is equal to a triangular pyramid having an equal base and equal altitude, by cutting the pyramids with a plane parallel to the base, and removing the upper part, it may be shown that the frustum of any pyramid is equal to the frustum of a triangular pyramid having equal bases and the same altitude; hence, if the proposition is true for triangular frustums, it is true for all frustums.

REGULAR POLYEDRONS.

A REGULAR POLYEDRON is one whose faces are all equal and regular polygons.

There can be five, and only five, regular polyedrons, namely:

1. The TETRAEDRON, or *regular pyramid*, a polyedron bounded by *four equal equilateral triangles.*
2. The HEXAEDRON, or *cube*, a polyedron bounded by *six equal squares.*
3. The OCTAEDRON, a polyedron bounded by *eight equal equilateral triangles.*
4. The DODECAEDRON, a polyedron bounded by *twelve equal regular pentagons.*
5. The ICOSAEDRON, a polyedron bounded by *twenty equal equilateral triangles.*

1st. In the tetraedron the polyedral angle is formed of *three* equilateral triangles; in the octaedron, of *four* such triangles; in the icosaedron, of *five* triangles. The combination of *six* such angles (each angle being $\frac{2}{3}$ of a right angle) gives four right angles, or a *plane*, and hence no polyedral angle; and the combination of more than six will not form a convex angle; hence only three regular polyedrons can be formed of triangles.

2d. In the hexaedron the polyedral angle is formed of *three* squares. The combination of four squares gives a plane, and a greater number would not give a convex angle; hence, but one regular polyedron can be formed of squares.

3d. In the dodecaedron the polyedral angle is formed of *three* regular pentagons. The combination of more than three such angles (each angle being $\frac{6}{5}$ of a right angle) exceeds four right angles, and will not give a convex angle; hence, but one regular polyedron can be formed of pentagons.

4th. Three or more angles of a regular hexagon (each angle being $\frac{4}{3}$ of a right angle) exceeds a right angle, and cannot form a convex polyedral angle; and the same is true of the heptagon, octagon, etc.

Therefore, only the five regular polyedrons named above are possible.

PRACTICAL EXAMPLES.

1. Required the convex surface of a right prism whose altitude is 14 inches and perimeter of the base 16 inches. *Ans.* 224 square inches.

2. Required the contents of a prism the area of whose base is 24 square feet and altitude 7 feet. *Ans.* 168 cubic feet.

3. Required the convex surface of a right pentangular pyramid whose slant height is 18 inches and each side of the base 6 inches. *Ans.* 270 square inches.

4. Required the volume of the frustum of a square pyramid, the sides of whose bases are 8 and 6 inches, and whose altitude is 12 inches. *Ans.* 592 cubic inches.

5. Required the entire surface of a cube whose sides are each 11 inches. *Ans.* 726 square inches.

6. A man wishes to make a cubical cistern whose contents are 373248 cubic inches; how many feet of inch boards will line it? *Ans.* 180 square feet.

7. What is the side of a cube which contains as much as a volume 20 feet 6 inches long, 10 feet 8 inches wide, and 6 feet 9 inches high? *Ans.* 11.4 feet.

8. What is the depth of a cubical cistern which shall contain 1600 gallons, each 231 cubic inches of water? *Ans.* 5.98 feet.

9. Required the dimensions of a cube whose surface shall be numerically equal to its contents. *Ans.* 6 units.

10. There are two similar prisms whose lengths are as 7 to 28 respectively; required the relation of their contents. *Ans.* 1 : 64.

11. Required the contents of a pyramid whose altitude is 20 inches and whose base is a regular hexagon, each side being 6 inches. *Ans.* 623.5386 cubic inches.

12. If we pass a plane parallel to the base of the pyramid of the 11th problem, half-way between its vertex and base, required the convex surface and contents of the frustum. *Ans. Vol.* = 545.596 cubic inches.

13. A farmer wishes to know what must be the depth of a cubical box which shall contain 100 bushels of grain, each bushel 2150.42 cubic inches. *Ans.* 4.9 feet.

THEOREMS FOR ORIGINAL THOUGHT.

1. Parallelopipedons having equal bases and equal altitudes are equal in volume.

2. The diagonals of a rectangular parallelopipedon are equal.

3. If a plane be passed through the opposite edges of a rectangular parallelopipedon, the triangular prisms formed are equal.

4. Two prisms having the same base are to each other as their altitudes.

5. Two similar pyramids are equal when the base and lateral edge of the one equal the base and lateral edge of the other.

6. The surfaces of similar polyedrons are to each other as the squares of their homologous edges