ELEMENTS OF TRIGONOMETRY.

INTRODUCTION.

LOGARITHMS.

1. LOGARITHMS are a species of numbers used to abbreviate Multiplication, Division, Involution, and Evolution.

2. The *logarithm of a number* is the exponent denoting the power to which a fixed number must be raised in order to produce the first number.

3. This *fixed number* is called the *base* of the system. The base of the common system is 10.

4. Raising 10 to different powers, we have,

$10^0 = 1$; hence, 0 is the log of 1;
$10^1 = 10$ " 1 " 10;
$10^2 = 100$ " 2 " 100;
$10^3 = 1000$ " 3 " 1000;
etc.

5. From this we have the following principles:

PRIN. 1. *The logarithm of a number between* 1 *and* 10 *is between* 0 *and* 1, *and is, therefore, a decimal.*

PRIN. 2. *The logarithm of a number between* 10 *and* 100 *is between* 1 *and* 2, *and is, therefore,* 1 *and a decimal.* Thus, it has been found that the log. of 76 is 1.880814.

PRIN. 3. *The logarithm of a number between* 100 *and* 1000 *is between* 2 *and* 3, *and is, therefore,* 2 *and a decimal.* Thus, the log. of 458 is 2.660865.

6. When the logarithm consists of an integer and a decimal,

the integer is called the *characteristic*, and the decimal part the *mantissa*. Thus, in 2.660865 the 2 is the characteristic, and .660865 is the mantissa.

PROPERTIES OF LOGARITHMS.

PRIN. 1.—*The characteristic is always one less than the number of integral places in the number.*

For, from Art. 4, we see that the log. of 100 is 2, the log. of 1000 is 3, and of any number between 100 and 1000 it is 2 and a decimal; hence, the characteristic is one less than the number of integral places.

PRIN. 2.—*The logarithm of the base is* 1, *and the logarithm of* 1 *is zero.*

For, since $10^1 = 10$, the log. of 10 is 1; and since $10^0 = 1$, the logarithm of 1 is 0.

PRIN. 3.—*The characteristic of the logarithm of a decimal is negative, and is numerically one greater than the number of ciphers between the decimal point and the first significant figure.*

For, if we raise the base, 10, to powers which give decimals, we will have,

$$\begin{array}{llll} 10^0 = 1 & ; \text{ hence, } & \log 1 & = 0; \\ 10^{-1} = .1 & \text{"} & \log .1 & = -1; \\ 10^{-2} = .01 & \text{"} & \log .01 & = -2; \\ 10^{-3} = .001 & \text{"} & \log .001 & = -3; \\ \text{etc.} & & & \text{etc.} \end{array}$$

which proves the principle. Thus, the log. of .458 is $\bar{1}.660865$.

PRIN. 4.—*The logarithm of the product of two numbers is equal to the sum of the logarithms of those numbers.*

For, let M and N be any two numbers, and m and n their logarithms; then we shall have, according to the definition,

$$10^m = M, \qquad 10^n = N.$$

Multiplying these equations, member by member, we have,

$$10^{m+n} = M \times N.$$

Hence, $\log (M \times N) = m + n$; or, $= \log M + \log N$.

PRIN. 5.—*The logarithm of the quotient of two numbers equals the difference of the logarithms of those numbers.*

For, from the definition, we have,

$$10^m = M, \qquad 10^n = N.$$

Dividing the first by the second, we have,

$$10^{m-n} = \frac{M}{N}.$$

Hence, $\log \left(\frac{M}{N}\right) = m - n$, or, $= \log M - \log N$.

PRIN. 6.—*The logarithm of any power of a number is equal to the logarithm of the number multiplied by the exponent of the power.*

For, since

$$10^m = M,$$

if we raise both members to the nth power, we have,

$$10^{mn} = M^n.$$

Hence, $\log M^n = mn$, or, $= \log M \times n$.

PRIN. 7.—*The logarithm of the root of any number is equal to the logarithm of the number divided by the index of the root.*

For, since

$$10^m = M,$$

if we take the nth root of both members, we have,

$$10^{\frac{m}{n}} = \sqrt[n]{M}.$$

Hence, $\log \sqrt[n]{M} = \frac{m}{n}$, or, $\log M \div n$.

PRIN. 8.—*The logarithm of the product of any number multiplied by* 10 *is equal to the logarithm of the number increased by* 1.

Suppose $\log M = m$; then, by Prin. 4,

$\log (M \times 10) = \log M + \log 10$. But $\log 10 = 1$;

Hence, $\log (M \times 10) = m + 1$.

Thus, $\log (76 \times 10) = 1.880814 + 1$; or, $\log 760 = 2.880814$.

PRIN. 9.—*The logarithm of the quotient of any number divided by* 10 *is equal to the logarithm of the number diminished by* 1.

Suppose $\log M = m$; then, by Prin. 5,

$$\log (M \div 10) = \log M - \log 10; \text{ from which}$$

$$\log (M \div 10) = m - 1.$$

Thus, $\log (458 \div 10) = 2.660865 - 1$;

or, $\log 45.8 = 1.660865$.

7. The following examples will illustrate Principles 1, 3, 8, and 9.

log 234	is	2.369216,
log 23.4	"	1.369216,
log 2.34	"	0.369216,
log .234	"	$\bar{1}$.369216,
log .0234	"	$\bar{2}$.369216.

From this, we see that when we change the place of the decimal point we change the characteristic, but do not change the decimal part of the logarithm.

The minus sign is written over the characteristic, showing that it only is negative.

TABLE OF LOGARITHMS.

8. A Table of Logarithms is a table by means of which we can find the logarithms of numbers, or the numbers corresponding to given logarithms.

9. In the annexed table the entire logarithms of the numbers up to 100 are given. For numbers greater than 100 the mantissa alone is given; the characteristic being found by Prin. 1.

10. The numbers are placed in the column on the left, headed N; their logarithms are opposite, on the same line. The first two figures of the mantissa are found in the first column of logarithms.

11. The column headed D shows the average differences of the ten logarithms in the same horizontal line. This difference is found by subtracting the logarithm in column 4 from that in column 5, and is very nearly the mean or average difference.

TO FIND THE LOGARITHM OF ANY NUMBER.

12. *To find the logarithm of a number of* ONE *or* TWO *figures.*

Look on the first page of the table, in the column headed N, and opposite the given number will be found its logarithm. Thus,

the logarithm of 25 is 1.397940,
" " 87 is 1.939519.

13. *To find the logarithm of a number of* THREE *figures.*

Look in the table for the given number; opposite this, in column headed 0, will be found the decimal part of the logarithm, to which we prefix the characteristic 2, Prin. 1. Thus,

the logarithm of 325 is 2.511883,
" " 876 is 2.942504.

14. *To find the logarithm of a number of* FOUR *figures.*

Find the three left-hand figures in the column headed N, and opposite to these, in the column headed by the fourth figure, will be found four figures of the logarithm, to which two figures from the column headed 0 are to be prefixed. The characteristic is 3, Prin. 1. Thus,

the logarithm of 3456 is 3.538574,
" " 7438 is 3.871456.

15. In some of the columns, *small dots* are found in the place of figures; these dots mean zeros, and should be written zeros. If the four figures of the logarithm fall where zeros occur, or if, in passing back from the four figures found to the zero column, any of these *dots are passed over*, the two figures to be prefixed must be taken from the line just below. Thus,

the logarithm of 1738 is 3.240050,
" " 2638 is 3.421275.

16. *To find the logarithm of a number of* MORE THAN FOUR *figures.*

Place a decimal point after the fourth figure from the left hand, thus changing the number into an integer and a decimal. Find the mantissa of the entire part by the method just given. Then from the column headed D take the corresponding *tabular difference*, multiply it by the decimal part, and add the product to the mantissa already found; the result will be the mantissa of the given number. The characteristic is determined by Prin. 1.

If the decimal part of the product exceeds .5, we add 1 to the entire part; if less than .5, it is omitted.

EXAMPLES.

1. Find the logarithm of 234567.

SOLUTION.—The characteristic is 5, Prin. 1. Placing a decimal point after the fourth figure from the left, we have 2345.67. The decimal part of the logarithm of 2345 is .370143; the number in column D is 185; and $185 \times .67 = 123.95$, and since .95 exceeds .5, we have 124, which, added to .370143, gives .370267; hence, log $234567 = 5.370267$.

2. Find the logarithm of 4567. *Ans.* 3.659631.
3. Find the logarithm of 3586. *Ans.* 3.554610.
4. Find the logarithm of 11806. *Ans.* 4.072102.
5. Find the logarithm of .4729. *Ans.* $\bar{1}$.674769.
6. Find the logarithm of 29.337. *Ans.* 1.467416.

17. *To find the number corresponding to a given logarithm.*

1. Find the *two left-hand* figures of the *mantissa* in the column headed 0, and the other four, if possible, in the same or some other column, on the same line; then, in column N, opposite to these latter figures, will be found the *three left-hand* figures, and at the top of the page the other figure of the required number.

2. *When the exact mantissa is not given in the table*, take out the four figures corresponding to the *next less mantissa* in the table; subtract this mantissa from the given one; divide the remainder, with ciphers annexed, by the number in column D, and annex the quotient to the four figures already found.

3. Make the number thus obtained correspond with the characteristic of the given logarithm, by pointing off decimals or annexing ciphers.

EXAMPLES.

1. Find the number whose logarithm is 5.370267.

SOLUTION.—The mantissa of the given logarithm is . . .370267
The mantissa of the next less logarithm of the table is . .370143
and its corresponding number is 2345.
Their difference is 124
The tabular difference is 185
The quotient is . . . 185)124.00(.67
Hence, the required number is . 234567.

NOTE.—If the characteristic had been 2, the number would have been 234.567; if it had been 7, the number would have been 23456700; if it had been $\bar{2}$, the number would have been .0234567, etc.

2. Find the number whose logarithm is 3.659631. *Ans.* 4567.
3. Find the number whose logarithm is 2.554610. *Ans.* 358.6.
4. Find the number whose logarithm is 1.072102. *Ans.* 11.806.
5. Find the number whose logarithm is $\bar{2}$.674769. *Ans.* .04729.
6. Find the number whose logarithm is $\bar{3}$.065463. *Ans.* .0011627.

MULTIPLICATION BY LOGARITHMS.

18. From Prin. 4, for the multiplication of numbers by means of logarithms, we have the following

RULE.—*Find the logarithms of the factors, take their sum, and find the number corresponding to the result; this number will be the required product.*

NOTE.—The term *sum* is used in its algebraic sense. Hence, when any of the characteristics are negative,—the mantissa is always positive,—we take the difference between the sums of the positive and negative characteristics, and prefix to it the sign of the greater. If any thing is to be carried from the addition of the mantissas, it must be added to a positive characteristic, or subtracted from a negative one.

EXAMPLES.

1. Multiply 35.16 by 8.15.

SOLUTION. log 35.16 = 1.546049
log 8.15 = 0.911158

2.457207
457125

Product, 286.554

152)82.00(.54

2. Find the product of .7856, 31.42. *Ans.* 24.6835.
3. Find the product of 31.42, 56.13, and 516.78. *Ans.* 911393.7.
4. Find the product of 31.462, .05673, and .006785. *Ans.* 01211168.
5. Product of .06517, 2.16725, .000317, and 42.1234 *Ans.* .001886.
6. Product of 2.3456, .00314, 123.789, .00078, and 67.105. *Ans.* .04772076.

DIVISION BY LOGARITHMS.

19. From Prin. 5, to divide by means of logarithms, we have the following

RULE.—*Find the logarithms of the dividend and divisor, subtract the latter from the former, and find the number corresponding to the result: this number will be the required quotient.*

NOTE.—The term *subtract* is here used in its algebraic sense; hence, we must subtract according to the principles of algebra.

EXAMPLES.

1. Divide 783.5 by 6.25.

SOLUTION. log 783.5 = 2.894039
log 6.25 = 0.795880

2.098159
.097951

Quotient, 125.36

346)208(6

2. Divide 272.636 by 6.37. *Ans.* 42.8.
3. Divide 50.38218 by 67.8. *Ans.* .7431.
4. Divide 155 by .0625. *Ans.* 2480.

ARITHMETICAL COMPLEMENT.

20. The operation of division when combined with multiplication is somewhat simplified by using the principle of the *arithmetical complement.*

21. The ARITHMETICAL COMPLEMENT of a logarithm is the result arising from subtracting the logarithm from 10. Thus, the arithmetical complement of the logarithm 5.623427 is 10 — 5.623427, or 4.376573.

22. The arithmetical complement may be written directly from the table, *by subtracting each figure of the logarithm from* 9, *except the right-hand figure, which must be taken from* 10. This is the same as subtracting the logarithm from 10.

23. We will now prove that *the difference between two logarithms is equal to the first logarithm, plus the arithmetical complement of the second, minus* 10.

Let $a =$ the first logarithm,
$b =$ the second logarithm,
and $c = 10 - b =$ arith. comp. of b.
The difference is $a - b$.
But, $-b = c - 10$.
Hence, $a - b = a + c - 10$,
which proves the principle.

24. Hence, to divide by means of the arithmetical complement, we have the following

RULE.—*Add the arithmetical complement of the logarithm of the divisor to the logarithm of the dividend, subtract* 10, *and find the number corresponding to the difference, this number will be the required quotient.*

EXAMPLES.

1. Divide 856.3 by 45.32.

SOLUTION. log 856.3 . .		2.932626
(a. c.) log 45.32 . .		8.343710
Quotient,	18.8945	1.276336

2. Divide 0.3156 by 78.35.

log 0.3156 . . .		$\bar{1}$.499137
(a. c.) log 78.35 . . .		8.105961
Quotient,	.004028	$\bar{3}$.605098

3. Divide 3.7521 by 18.346. *Ans.* .204519.
4. Divide 483.72 by .30751. *Ans.* 1573.02.
5. Multiply 32.16 by 7.856, and divide the product by 45.327. *Ans.* 5.574.
6. Divide the product of 31.57 and 123.4 by the product of 316.2 and .0316. *Ans.* 389.8884.
7. Find by logarithms the first term of the proportion, $x : 73.15 :: 48.16 : 3167$. *Ans.* 1.11237.

INVOLUTION BY LOGARITHMS.

25. From Prin. 6, to raise a number to any power, we have the following

RULE.—*Find the logarithm of the number, multiply it by the exponent of the power, and find the number corresponding to the result.*

EXAMPLES.

1. Find the 4th power of 45.

SOLUTION.

	log 45 =	1.653213
		4
Power,	4100625	6.612852

2. Find the cube of 0.65. *Ans.* 0.2746.
3. Find the 6th power of 1.037. *Ans.* 1.243.
4. Find the 7th power of .4797. *Ans.* 0.005846.

EVOLUTION BY LOGARITHMS.

26. From Prin. 7, to extract any root of a number, we have the following

RULE.—I. *Find the logarithm of the number, divide it by the index of the root, and find the number corresponding to the result.*

II. *If the characteristic is negative and not divisible by the index of the root, add to it the smallest negative number that will make it divisible, prefixing the same number with a plus sign to the mantissa.*

EXAMPLES.

1. Find the square root of 576.

SOLUTION. $\log 576 = 2.760422$

$2.760422 \div 2 = 1.380211$

Hence, the root is 24.

2. Find the fourth root of .325.

SOLUTION. $\log .325 = \bar{1}.511883 = \bar{4} + 3.511883.$

Then $(\bar{4} + 3.511883) \div 4 = \bar{1}.877971$

Hence, the quotient is, .75504.

3. Find the fifth root of .0625. *Ans.* .574348.
4. Find the cube root of 7. *Ans.* 1.9129.
5. Find the fifth root of 5. *Ans.* 1.3797.
6. Find the tenth root of 8764.5. *Ans.* 2.479.

CALCULATION OF LOGARITHMS.

The pupil will by this time naturally inquire how these logarithms are calculated. This we have not room to explain here; in fact, an explanation of the modern methods would be almost too difficult for the majority of pupils who study this book. Only a general idea can here be given.

In computing logarithms, it is only necessary to calculate the logarithms of prime numbers, since the logarithms of composite numbers may be obtained by adding the logarithms of their prime factors.

The logarithms of the prime numbers were first computed by comparing the geometrical and arithmetical series, 1, 10, 100, etc., and 0, 1, 2, etc., and finding geometrical and arithmetical means; the arithmetical mean being the logarithm of the corresponding geometrical mean. This method was exceedingly laborious, involving so many multiplications and extractions of roots.

The method now generally used is that of series, by which the computations are much more easily made. The following formula is derived by algebraic reasoning.

$$\log\,(1+x) = A\left(\frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \text{etc.}\right)$$

In this the quantity A is called the *modulus*, which in the Napierian system is *unity*. The series, when A is *one*, put in a more convenient form, becomes,

$$\log.\,(z+1) - \log.\,z = 2\left(\frac{1}{2z+1} + \frac{1}{3(2z+1)^3} + \frac{1}{5(2z+1)^5} + \text{etc.}\right)$$

From which, knowing the logarithm of any number, we readily find the logarithm of the next larger number. The pupil will be interested in finding logarithms by this formula. Begin with 2, in which $z = 1$.

The logarithm found will be the Napierian logarithm, and this multiplied by 0.434294 will give the common logarithm.

PLANE TRIGONOMETRY.

DEFINITIONS AND PRIMARY PRINCIPLES.

1. PLANE TRIGONOMETRY is the science which treats of the solution of plane triangles.

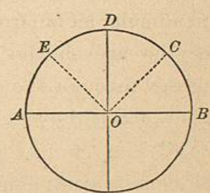
2. The SOLUTION of a triangle is the operation of finding the unknown parts when a sufficient number of the known parts are given.

3. In every triangle there are six parts; *three sides* and *three angles*. These parts are so related that when three of the parts are given, one being a side, the other parts may be found.

4. An angle is measured, as we have previously seen, by the arc included between its sides, the centre of the circumference being at the vertex of the angle.

5. For measuring angles, as has already been explained, the circumference is divided into 360 equal parts, called degrees, each degree into 60 equal parts, called minutes, etc.

6. A QUADRANT is one-fourth of the circumference of a circle; hence, if two lines be drawn through the centre of a circle at right angles to each other, they will divide the circumference into four quadrants. Each quadrant contains 90°.



7. The COMPLEMENT of an arc is 90° minus the arc; thus, DC is the complement of BC; also, the angle DOC is the complement of BOC.

8. The SUPPLEMENT of an arc is 180° minus the arc; thus,