paring the geometrical and arithmetical series, 1, 10, 100, etc., and 0, 1, 2, etc., and finding geometrical and arithmetical means; the arithmetical mean being the logarithm of the corresponding geometrical mean. This method was exceedingly laborious, involving so many multiplications and extractions of roots.

The method now generally used is that of series, by which the computations are much more easily made. The following formula is derived by algebraic reasoning.

$$\log\,(1+x) = A\left(\frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \text{etc.}\right)$$

In this the quantity A is called the *modulus*, which in the Napierian system is *unity*. The series, when A is *one*, put in a more convenient form, becomes,

$$\log.\,(z+1) - \log.\,z = 2\left(\frac{1}{2z+1} + \frac{1}{3(2z+1)^3} + \frac{1}{5(2z+1)^5} + \text{etc.}\right)$$

From which, knowing the logarithm of any number, we readily find the logarithm of the next larger number. The pupil will be interested in finding logarithms by this formula. Begin with 2, in which $z = 1$.

The logarithm found will be the Napierian logarithm, and this multiplied by 0.434294 will give the common logarithm.

PLANE TRIGONOMETRY.

DEFINITIONS AND PRIMARY PRINCIPLES.

1. PLANE TRIGONOMETRY is the science which treats of the solution of plane triangles.

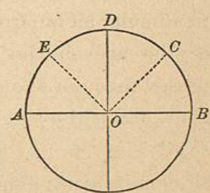
2. The SOLUTION of a triangle is the operation of finding the unknown parts when a sufficient number of the known parts are given.

3. In every triangle there are six parts; *three sides* and *three angles*. These parts are so related that when three of the parts are given, one being a side, the other parts may be found.

4. An angle is measured, as we have previously seen, by the arc included between its sides, the centre of the circumference being at the vertex of the angle.

5. For measuring angles, as has already been explained, the circumference is divided into 360 equal parts, called degrees, each degree into 60 equal parts, called minutes, etc.

6. A QUADRANT is one-fourth of the circumference of a circle; hence, if two lines be drawn through the centre of a circle at right angles to each other, they will divide the circumference into four quadrants. Each quadrant contains 90°.



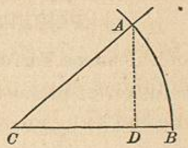
7. The COMPLEMENT of an arc is 90° minus the arc; thus, DC is the complement of BC; also, the angle DOC is the complement of BOC.

8. The SUPPLEMENT of an arc is 180° minus the arc; thus,

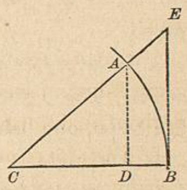
AE is the supplement of the arc *BDE;* also, the angle *AOE* is the supplement of the angle *BOE.*

9. In Trigonometry, instead of comparing the angles of tri-angles, or the arcs which measure them, we compare certain lines, called *functions* of the arcs. A function of a quantity is something depending upon the quantity for its value. These functions are the *sine, cosine, tangent, cotangent, secant,* and *cosecant.*

10. Thus, instead of reasoning with the angle *ACB*, or the arc *AB,* which measures it, we draw the perpendicular *AD*, and use the lines *AD* and *CD.* The line *AD* is called the *sine* of the arc or angle; the line *CD* is called the *cosine* of the arc or angle.



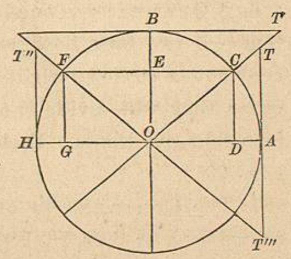
11. If we draw *BE* perpendicular to *CB,* meeting *CA* produced in *E,* the line *BE* is called the *tangent* of the angle, and the line *CE* is called the *secant.*



12. In comparing the sides and angles, these lines, we say, are used instead of the angles or the arcs. The necessity for such lines is evident, since we could not compare the sides, which are *straight lines,* with the *angles,* or the *curve lines,* which measure them.

We will now represent these lines in the first and second quadrants.

13. The Sine of an arc is the perpendicular let fall from one extremity of the arc on the diameter which passes through the other extremity. Thus, *CD* is the sine of the arc *AC.*



14. The Cosine of an arc is the sine of its complement; or it is the distance between the foot of the sine and the centre of the circle; thus, *CE* or *OD* is the cosine of the arc *AC.*

15. The Tangent of an arc is a line which is perpendicular to the radius at one extremity of the arc, and limited by a line passing through the centre of the circle and the other extremity; thus, *AT* is the tangent of *AC.*

16. The Cotangent of an arc is equal to the tangent of the complement of the arc; thus, *BT'* is the cotangent of *AC.*

17. The Secant of an arc is a line drawn from the centre of the circle through one extremity of the arc, and limited by a tangent at the other extremity; thus, *OT* is the secant of *AC.*

18. The Cosecant of an arc is the secant of the complement of the arc; thus, *OT'* is the cosecant of *AC.*

19. The sine, cosine, tangent, cotangent, etc. of an arc are indicated as follows:

sin *AC;*	tan *AC;*	sec *AC;*
cos *AC;*	cot *AC;*	cosec *AC.*

20. Principles.—From the definitions now given, we can readily derive the following simple principles.

1. *The sine of an arc equals the sine of its supplement, and also the cosine of an arc equals the cosine of its supplement.*

Dem.—Take the arc *ABF;* its sine is *FG,* its supplement is *FH,* and the sine of its supplement is *FG.* Hence, its sine equals the sine of its supplement. Its cosine is *GO,* which is also the cosine of *FH.* Hence, etc.

2. *The tangent and cotangent of an arc are respectively equal to the tangent and cotangent of the supplement of the arc.*

Dem.—The tangent of the arc *ABF* is *AT'''*, and the tangent of its supplement *FH* is *HT''*, and, by similar triangles, it may be shown that *AT'''* equals *HT'';* therefore, etc.

3. *The secant and cosecant of an arc are respectively equal to the secant and cosecant of the supplement of the arc.*

This may be demonstrated in a manner quite similar to those above. Let the pupil be required to show it.

4. If *a* equals any arc or angle, then we shall have, from the definitions,

$$\sin a = \cos\ (90° - a)$$
$$\tan a = \cot\ (90° - a)$$
$$\sec a = \text{cosec}\ (90° - a)$$

NATURAL SINES, COSINES, ETC.

21. The length of these trigonometrical lines may be expressed in numbers, differing, of course, as the radius of the circle is larger or smaller. If the radius is regarded as *unity*, or 1, we have what are called *natural sines, cosines*, etc. The method of calculating these sines, cosines, etc. will be explained hereafter.

The operation of multiplying and dividing by these natural sines being long and tedious, it has been found more convenient to use *logarithmic sines*, which we will now explain.

TABLE OF LOGARITHMIC SINES.

22. A LOGARITHMIC SINE, COSINE, TANGENT, or COTANGENT is the logarithm of the sine, cosine, tangent, or cotangent of an arc of a circle whose radius is 10,000,000,000.

23. A TABLE OF LOGARITHMIC SINES is a table containing the logarithmic sine, cosine, tangent, and cotangent of arcs.

24. The table of logarithmic sines may be calculated from a table of natural sines, as will be explained hereafter. In the table, the degrees are given at the top and bottom of the page, and the minutes at the sides, in the column headed M.

25. The column headed D contains the increase or decrease for 1 second. This is found by subtracting the logarithmic sine, etc. of an arc from that next exceeding it by 1 minute, and dividing the difference by 60.

26. *To find the logarithmic sines, cosines, etc. of arcs or angles.*

1. *When the arc is expressed in degrees, or in degrees and minutes.* If the angle is less than 45°, look for the degrees at the *top* of the page, and for the minutes in the *left*-hand column; then, opposite to the minutes, on the same horizontal line, in the column headed *Sine*, will be found the logarithmic sine; in that headed *Cosine* will be found the logarithmic cosine, etc. Thus,

log sin 23° 35′	9.602150
log tan 23° 35′	9.640027

If the angle exceeds 45°, look for the degrees at the bottom of the page, and for the minutes in the *right*-hand column; then, opposite to the minutes, in the same horizontal line, in the column marked at the bottom *Sine*, will be found the logarithmic sine, etc. Thus,

log cos 65° 24′	9.619386
log tan 65° 24′	10.339290

2. *When the arc contains seconds.*—Find the logarithmic sine, etc. as before; then multiply the corresponding number found in column D by the number of seconds, and *add* the product to the preceding logarithm for the sines or tangents, and *subtract* it for cosines or cotangents.

We subtract for cosine and cotangent, because the greater the arc the less the cosine or cotangent. In multiplying the tabular difference by the number of seconds, we observe the same rule for the decimal point as in logarithms. If the arc is greater than 90°, we find the sine, cosine, etc. of its supplement.

EXAMPLES.

1. Find the logarithmic sine of 36° 24′ 42″.

SOLUTION.

log sin 36° 24′,		9.773361
Tabular difference,	2.85	
No. of seconds,	42	
Product,	119.70 to be added,	120
log sin 36° 24′ 42″,		9.773481

2. Find the logarithmic cosine of 64° 30′ 30″.

SOLUTION.

log cos 64° 30′,			9.633984
Tabular difference,	4.41		
No. of seconds,	30		
Product,	132.30	to be subtracted,	132
log cos 64° 30′ 30″,			9.633852

3. Find the logarithmic tangent of 120° 15′ 24″.

SOLUTION.

	180° 00′ 00″		
The given arc,	120 15 24		
Supplement,	59 44 36		
log tan 59° 44′,			10.233905
Tabular difference,	4.84		
No. of seconds,	36	to be added,	174.24
log tan 120° 15′ 24″,			10.234079

4. Find the logarithmic sine of 40° 40′ 40″. *Ans.* 9.814117.
5. Find the logarithmic cosine of 140° 30′ 20″. *Ans.* 9.887441.
6. Find the logarithmic tangent of 85° 25′ 45″. *Ans.* 11.097200.
7. Find the logarithmic cotangent of 144° 44′ 28″. *Ans.* 10.150603.

27. *To find the arc corresponding to any logarithmic sine, cosine, tangent, or cotangent.*

1. Look in the proper column of the table for the given logarithm; if found there, and the name of the function be at the *head* of the column, take the degrees at the *top*, and the minutes on the *left;* but if the name of the function is at the *foot* of the column, take the degrees at the *bottom*, and the minutes on the *right.*

2. If the given logarithm is not exactly given in the table, then take the next less logarithm, subtract it from the given logarithm, and divide the remainder by the corresponding tabular difference; the quotient will be seconds, which must be *added* to the degrees and minutes corresponding to the logarithm taken from the table, for *sines* and *tangents*, and *subtracted* for *cosines* and *cotangents.*

EXAMPLES.

1. Find the arc whose logarithmic sine is 9.617033.

SOLUTION.

Given log sine,		9.617033
Next less in table,		9.616894
Tabular difference,	4.63)	139.00(30, to be added.

Hence, the arc or angle is 24° 27′ 30″.

2. Find the arc whose logarithmic cosine is 9.704682.

SOLUTION.

Given log cosine,		9.704682
Next less in table,		9.704610
Tabular difference,	3.58)	72.00(20, to be subtracted.

Hence, the arc or angle is 59° 33′ 40″.

3. Find the arc whose logarithmic sine is 9.438672. *Ans.* 15° 56′ 14″.
4. Find the arc whose logarithmic cosine is 9.634520. *Ans.* 64° 27′ 47″.
5. Find the arc whose logarithmic tangent is 10.753246. *Ans.* 79° 59′ 24″.
6. Find the arc whose logarithmic cotangent is 11.449852. *Ans.* 2° 1′ 40″.

28. Having learned how to find logarithmic sines, cosines, etc., we will next demonstrate some theorems for the solution of triangles.

THE THEOREMS OF TRIGONOMETRY.

29. The Theorems of Trigonometry express the relation between the sides and trigonometrical functions of the angles of triangles.

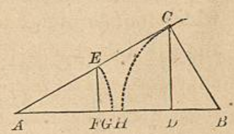
30. We give five theorems, the first three relating to triangles in general, the others to right-angled triangles.

THEOREM I.

31. *In any plane triangle, the sides are proportional to the sines of the opposite angles.*

Let ABC be a plane triangle; then will

$$CB : CA :: \sin A : \sin B.$$



For, with A as a centre, and a radius AE equal to BC, describe the arc EG, and draw the perpendicular EF. With B as a centre, and the equal radius BC, describe the arc CH, and draw the perpendicular CD; then will CD be the sine of the angle B, and EF be the sine of the angle A, to the same radius. Now, by similar triangles (B. III. Th. X.),

$$AE : AC :: EF : CD.$$

But AE equals CB, EF is sin A, and CD is sin B.

Hence, $$BC : AC :: \sin A : \sin B.$$

In a similar manner, it may be shown that

$$AC : AB :: \sin B :: \sin C.$$

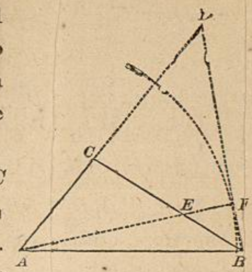
Therefore, etc.

THEOREM II.

32. *In any plane triangle, the sum of any two sides is to their difference as the tangent of half the sum of the opposite angles is to the tangent of half their difference.*

Let ABC be any plane angle; then will

$$BC + AC : BC - AC :: \tan \tfrac{1}{2}(A + B) : \tan \tfrac{1}{2}(A - B).$$



For, produce AC to D, making CD equal to CB, and draw BD; take CE equal to AC, draw AE, and produce it to F; then AD is the *sum* and BE the *difference* of the two sides AC and BC.

The sum of the angles CAE and AEC equals the sum of CAB and CBA, both sums being equal to 180° minus ACB (B. I. Th. XIII.); but the angle CAE equals AEC (B. I. Th. X.); hence, CAE or CAF is the *half sum* of CAB and CBA; also, BAF is the *half difference* of the angles CAB and ABC, since it equals the *half sum* CAE, subtracted from the greater angle CAB.*

The angle CDF equals CBD, since CB equals CD; also, CAE, which equals AEC, is equal to the vertical angle FEB; hence, the third angles of the triangles, AFD and EFB, are equal, and, therefore, AF is perpendicular to BD; consequently, if then we regard AF as the radius, FD will be the tangent of DAF, and FB will be the tangent of FAB. Now, by similar triangles,

$$AD : EB :: FD : FB; \text{ or,}$$

$$CB + AC : CB - AC :: \tan \tfrac{1}{2}(A + B) ; \tan \tfrac{1}{2}(A - B).$$

THEOREM III.

33. *In any plane triangle, if a line is drawn from the vertical angle perpendicular to the base, then the whole base will be to the sum of the other two sides as the difference of those sides is to the difference of the segments of the base.*

Let ABC be a triangle, and CD perpendicular to the base; then will

$$AB : AC + BC :: AC - BC : AD - DB.$$

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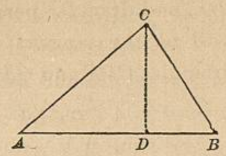
* This principle is thus proven:—Let a and b be any two quantities; then the half sum is $\frac{a+b}{2}$, and the half difference is $\frac{a-b}{2}$; and $a - \frac{a+b}{2} = \frac{a-b}{2}$; that is, *the greater minus the half sum equals the half difference.*



For, from Th. VI. Book III.,

$$AC^2 = \overline{AD}^2 + \overline{DC}^2,$$

and $$\overline{BC}^2 = \overline{BD}^2 + \overline{DC}^2.$$

Subtracting, $\overline{AC}^2 - \overline{BC}^2 = \overline{AD}^2 - \overline{BD}^2$.

Hence (B. III. Th. V. C. 2),

$$(AC + BC) \times (AC - BC) = (AD + BD) \times (AD - BD);$$

therefore, $AD + DB : AC + BC :: AC - BC : AD - DB$.

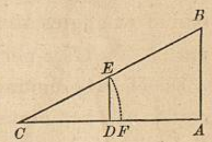
Therefore, etc.

THEOREM IV.

34. *In any right-angled plane triangle, radius is to the sine of either angle as the hypothenuse is to the side opposite.*

Let CAB be a triangle right-angled at A, and denote the radius by R: then will

$$R : \sin C :: CB : AB.$$



For, from the point C as a centre and any radius, as CE, describe the arc EF, and draw ED perpendicular to CA; then will ED be the sine of the angle C. The two triangles CED and CAB are similar; hence, we have (B. III. Th. X.),

$$CE : ED :: CB : BA,$$

or, $$R : \sin C :: CB : BA.$$

Therefore, etc.

Cor. It may also be shown that *radius is to the cosine of either acute angle as the hypothenuse is to the side adjacent.*

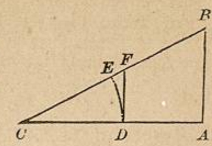
THEOREM V.

35. *In any right-angled plane triangle, radius is to the tangent of either acute angle as the side adjacent is to the side opposite.*

Let CAB be a triangle right-angled at A; then will

$$R : \tan C :: CA : AB.$$

For, with C as a centre and any radius CD, describe the arc



DE, and draw DF perpendicular to CA; FD will be the tangent of the angle C. The triangles CDF and CAB are similar; hence,

$$CD : DF :: CA : AB,$$

or, $$R : \tan C :: CA : AB.$$

Therefore, etc.

Cor. It may also be shown that *radius is to cotangent of either angle, as side opposite is to side adjacent.*

SOLUTION OF TRIANGLES.

36. The Solution of a Triangle is the process of finding the unknown parts when a sufficient number of the parts are given.

37. There are *six parts* in a plane triangle, and three of these—one of the three being a side—must be given to find the other parts.

38. If the angles alone were given, it is clear that the sides could not be determined, since there could be an indefinite number of triangles having their angles respectively equal.

39. There are four cases, as follows:

1. When two angles and a side are given.
2. When two sides and an angle are given.
3. When two sides and the included angle are given.
4. When the three sides are given.

CASE I.

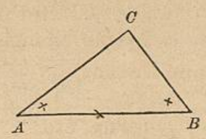
40. *Given two angles and one side, to find the remaining parts.*

Method.—We subtract the sum of the given angles from 180° to find the third angle, and then find the sides by Theorem I.

EXAMPLES.

1. In a triangle ABC, there are given the angle $A = 32° 24'$, the angle $B = 40° 32'$, and the side $AB = 240$; required the other parts.

SOLUTION.—Let ABC represent the triangle; then the sum of A and $B = 72° 56'$, and $C = 180° - 72° 56' = 107° 04'$. Then, to find AC, we have,



$$AC : AB :: \sin B : \sin C.$$

Hence, $AC = AB \times \sin B \div \sin C.$

From which AC is readily found by multiplying 240 by the natural sine of B, and dividing by the natural sine of C. It is simpler, however, to use logarithms. To find AC, we add the log of AB and log sin B, and subtract log sin C, or *add* the *arith. comp.* of log sin C.

a. c. log sin C	(107° 04′),	0.019558
log sin B	(40° 32′),	9.812840
log AB	(240),	2.380211
log AC,		2.212609 ∴ $AC = 163.158$

To find the side BC, we have,

$$BC : AB :: \sin A : \sin C:$$

or, by logarithms,

a. c. log sin C	(107° 04′),	0.019558
log sin A	(32° 24′),	9.729024
log AB	(240),	2.380211
log BC,		2.128793 ∴ $BC = 134.522$

2. In the triangle ABC, there are given the angle $A = 27° 40'$, the angle $C = 65° 45'$, and the side $AB = 625$, to find the other parts. *Ans.* $B = 86° 35'$; $BC = 318.29$; $AC = 684.266$.

CASE II.

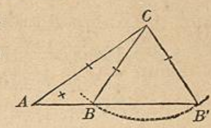
41. *Given two sides and an angle opposite one of them, to find the remaining parts.*

METHOD.—One of the required angles is found by Theorem I. The third angle is found by subtracting the sum of the two from 180°; the third side is found by Case I.

EXAMPLES.

1. In the triangle ABC, there are given $AC = 200$, $CB = 150$, and the angle $A = 44° 26'$, to find the other parts.

SOLUTION.—Let ABC be a triangle in which $A = 44° 26'$, $AC = 200$, and $BC = 150$; then, to find the angle B, we have,



$$\sin B : \sin A :: AC : BC,$$

or,

BC	(150)	a. c.	7.823009
: AC	(200)		2.301030
:: sin A	(44° 26′)		9.845147
: sin B	( )		9.969186

∴ $B = 68° 40' 16''$, or, $111° 19' 44''$

In this problem, if the side BC, opposite the given angle A, is shorter than the other given side AC, the solution will be *ambiguous;* for *two triangles*, ACB and ACB', may be formed, each of which will satisfy the conditions of the problem. Hence, the angle B found above may be either ABC or B'. But these, it will be seen, are supplements of each other; hence, in finding the angle corresponding to sin B, we take the angle or its supplement.

In practice, there is often some circumstance to determine whether the angle is acute or obtuse. *If the angle given is obtuse*, the other angles must be acute, and there will be but one solution. *If the side BC is equal to or greater than AC*, there will be but one triangle.

In the given diagram above, the angle $ABC = 111° 19' 44''$, and $AB'C = 68° 40' 16''$; hence, the angle $ACB = 24° 13' 16''$, and the angle $ACB' = 113° 6' 16''$.

To find the side AB, we have,

$$AB : CB :: \sin ACB : \sin A;$$

from which, by logarithms, we find $AB = 88.085$.

To find the side AB', we have,

$$AB' : CB' :: \sin ACB' : \sin A;$$

from which, by logarithms, we find $AB' = 197.484$.

2. In a triangle ABC there are given AB 45.96, BC 62.50, and the angle A 79° 21′; find the remaining parts.

Ans. $C = 46° 16′ 38″$; $B = 54° 22′ 22″$; $AC = 51.69$.

(There is no ambiguity, since the side BC is greater than AC.)

3. In a triangle ABC there are given $BC = 15.71$, $AC = 21.12$, and the angle $A = 27° 50′$; find the other parts.

Ans. $C = 113° 17′ 13″$; $B = 38° 52′ 47″$; $AB = 30.906$.
or, $C = 11° 2′ 47″$; $B = 141° 7′ 13″$; $AB = 6.447$.

CASE III.

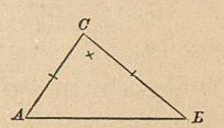
42. *Given two sides and the included angle, to find the remaining parts.*

Method.—We find the sum of the two angles by subtracting the given angle from 180°, and divide this by 2 for the *half sum.* We then find the *half difference,* by Theorem II. Having found the half sum and half difference of the two angles, we find the greater angle by adding the half difference to the half sum; and the less by subtracting the half difference from the half sum. The third side is found by Theorem I.

EXAMPLES.

1. In the triangle ABC, let $BC = 680$, $AC = 460$, and the included angle 84°; required the other parts.

Solution.—Let ABC represent the triangle, $AC = 460$, $BC = 680$, and the angle $C = 84°$. Then, $AC + BC = 460 + 680 = 1140$; $BC - AC = 680 - 460 = 220$. $A + B = 180° - 84° = 96°$; hence, *half sum* $= 48°$. The half difference we find by the following proportion.



$BC + AC$	1140	ar. co.	6.943095
$: BC - AC$	220	. .	2.342423
$:: \tan \frac{1}{2}(A + B)$	48°	. .	10.045563
$: \tan \frac{1}{2}(A - B)$	12° 5′ 49″		9.331081
Hence, $A =$	60° 5′ 49″;		and $B = 35° 54′ 11″$.

The other side, found by Theorem I., equals 783.733.

2. Given two sides of a plane triangle 240 and 360, and the included angle 68° 36′; required the other parts.

Ans. 72° 02′ 26; 39° 21′ 34″; 352.349.

CASE IV.

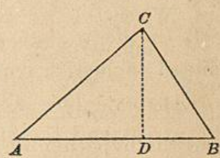
43. *Given the three sides of a plane triangle, to find the angles.*

Method.—Let fall a perpendicular upon the greater side from the angle opposite, dividing the triangle into two right-angled triangles. Find the difference of the segments of the base by Theorem III.; half this difference added to half the base gives the greater segment, and subtracted from half the base gives the less. We will then have two sides and the right angle of two right-angled triangles, from which we can find the acute angles by Theorem I.

EXAMPLES.

1. In a triangle ABC, given $AB = 60$, $AC = 50$, and $BC = 40$, to find the angles.

Solution.—Let ABC represent the triangle; then $AB = 60$, $AC = 50$, $BC = 40$; then, by Th. III.,



$AB : AC + BC :: AC - BC : AD - BD$,
or, $60 : 90 :: 10 : AD - BD$.

hence, $AD - BD = 90 \times 10 \div 60 = 15$;
then, $AD = \frac{1}{2}(60 + 15) = 37.5$
and $BD = \frac{1}{2}(60 - 15) = 22.5$

Then, in the triangle ACD, to find the angle ACD,

a. c.	AC	(50)	8.301030
	$: AD$	(37.5)	1.574031
	$:: \sin D$	(90°)	10.000000
	$: \sin ACD$	48° 35′ 25″	9.875061

Then, in the triangle BCD, to find the angle BCD,

a. c.	BC	(40)	8.397940
	: BD	(22.5)	1.352183
	: : sin D	(90°)	10.000000
	: sin BCD	34° 13′ 44″	9.750123

Hence, $A = 90° - 48° 35' 25'' = 41° 24' 35''$,

and $B = 90° - 34° 13' 44'' = 55° 46' 16''$,

and $C = 48° 35' 25'' + 34° 13' 44'' = 82° 49' 09''$.

2. In a plane triangle the sides are 1005, 1210, and 1368; required the angles.

Ans. 45° 22′ 35″; 58° 58′ 18″; 75° 39′ 7″.

SOLUTION OF RIGHT-ANGLED TRIANGLES.

44. In the solution of right-angled triangles we have the four following cases:

1. When the hypothenuse and an acute angle are given.
2. When the hypothenuse and a side are given.
3. When one side and the angles are given.
4. When the two sides about the right angle are given.

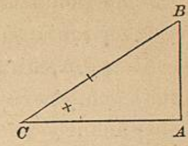
METHOD.—The first three cases are readily solved by Theorem ..; remembering that the sine of 90° is *radius*, the log. sin. being 10. The fourth case may be solved by Theorem V.; or we may find the hypothenuse by B. III. Th. VI., and then find the angles by Theorem I.

These four cases may also be solved by Theorems IV. and V.; but the method suggested above is preferred, since it is simpler and more easily remembered.

EXAMPLES.

1. In a right-angled triangle, given the hypothenuse 475 and the angle at the base 36° 34′; find the other parts.

SOLUTION.—Let CAB represent the triangle, BC being equal to 475 and the angle $C = 36° 34'$; then, to find AB, we have,



sin A	90°	a. c.	0.000000
: sin C	36° 34′		9.775070
: : CB	475		2.676694
: AB	282.985		2.451764

The angle $B = 90° - (36° 34') = 53° 26$; then, by a similar proportion, we can find the side $CA = 381.503$.

2. Given the hypothenuse 45.36 and the angle at the base 45° 36′; required the other parts. *Ans.*

3. Given the hypothenuse 396 and the base 218, to find the other parts. *Ans.* 330.59; 33° 24′ 05″; 56° 35′ 55″.

4. Given the two sides 58.75 and 74.58, to find the remaining parts. *Ans.* 94.94; 38° 13′ 45″; 51° 46′ 15″.

PRACTICAL APPLICATIONS.

HEIGHTS AND DISTANCES.

45. A HORIZONTAL PLANE is one which is parallel to the plane of the horizon.

46. A VERTICAL PLANE is one which is perpendicular to a horizontal plane.

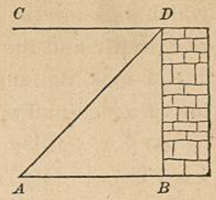
47. A HORIZONTAL LINE is any line in a horizontal plane. A *vertical line* is a line perpendicular to a horizontal plane.

48. A HORIZONTAL ANGLE is an angle in a horizontal plane.

49. A VERTICAL ANGLE is an angle in a vertical plane.

50. An ANGLE OF ELEVATION is a vertical angle having one

side horizontal, and the inclined side above the horizontal side; as *BAD*.



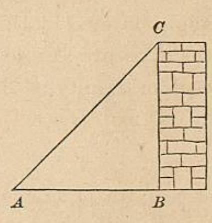
51. An ANGLE OF DEPRESSION is a vertical angle having one side horizontal, and the inclined side under the horizontal side; as *CDA*.

52. *Distances upon the ground* are usually measured by a chain, called *Gunter's Chain*. This chain is 4 rods or 66 feet long, and consists of 100 links. Sometimes a half chain is used, consisting of 50 links.

53. Angles are measured by various instruments. Horizontal angles are measured by an instrument called *The Compass*. Horizontal and vertical angles are both measured by the *Theodolite*, or, what is still better for general use, a *Transit-Theodolite*.

CASE I.

54. *To determine the height of a vertical object standing upon a horizontal plane.*



METHOD.—Measure from the foot of the object any convenient horizontal distance *AB;* at the point *A* take the angle of elevation *BAC;* then, in the triangle *ABC* we have a side and an acute angle; hence, we can readily find the altitude.

1. From the foot of a tower I measure a horizontal line 120 feet, and at its extremity find the angle of elevation to be 48° 36′; what was the height of the tower?

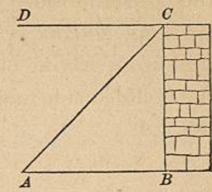
Ans. 136.113 feet.

CASE II.

55. *To find the distance of a vertical object whose height is known.*

METHOD.—Measure the angle of elevation to the top of the object, as before; we will then have a right-angled triangle in

which we know the perpendicular and an acute angle; hence, we can readily find the base.



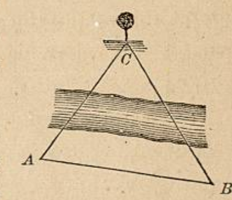
1. I took the angle of elevation to the top of a flag-staff whose height I knew to be 160 feet, and found it to be 20°; how far was I from the staff?

Ans. 439.60 feet.

CASE III.

56. *To find the distance of an inaccessible object.*

METHOD.—Measure a horizontal base-line *AB*, and then take the angles formed by this line and lines from the object to the extremities of this base-line, as *CAB* and *ABC;* the distance *AC* or *BC* can then be readily found.



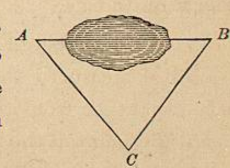
1. I am on one side of a river, and wish to know the distance to a tree on the other side. I measure 300 yards by the side of the river, and find that the two angles formed by this line and the lines from its extremities to the tree are 72° 40′ and 45° 36′; required the distance from each extremity of the base-line to the tree.

Ans. 243.362 yards; 325.15 yards.

CASE IV.

57. *To find the distance between two objects separated by an impassable barrier.*

METHOD.—Select any convenient station, as *C*, and measure the distance from it to each of the objects *A* and *B*, and the angle *C* included between these lines. We can then readily find the distance *AB*.



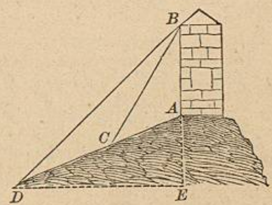
1. The distance between two trees cannot be directly measured: I therefore take a third position from

which each of the trees can be seen, and find the distances from it to the trees to be 300 and 250 yards, and the included angle 43° 16′; required the distance between the trees.

Ans. 208.02 yards.

CASE V.

58. *To find the height of a vertical object standing upon an inclined plane.*



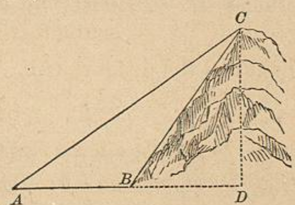
METHOD.—Measure any convenient distance AD on a line from the foot of the object, and at the point D measure the angles of elevation, EDA and EDB, to foot and top of the tower. By means of the two triangles DEA and DEB, we can find the height of AB.

1. Wishing to determine the height of a tower situated upon a hill, I measured a distance down the slope of the hill 400 feet, and found the angles of elevation to the foot of the tower 42° 28′, and to the top of the tower 68° 42′; required the height of the tower.

Ans. 486.747.

CASE VI.

59. *To find the height of an inaccessible object above a horizontal plane.*



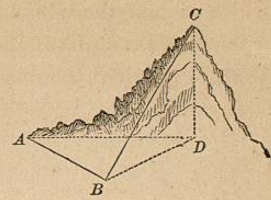
FIRST METHOD.—Measure any convenient horizontal line AB directly toward the object, and take the angles of elevation at A and B; we will then have conditions sufficient to find DC.

1. Wishing to find the altitude of a hill, I measured the angle of elevation at the bottom 60° 37′, and 460 feet from the foot in a right line of the top of the hill and the point at the foot, and in the same horizontal plane as the foot, I measured the angle of elevation 36° 52′; required the height of the hill.

Ans. 597.092.

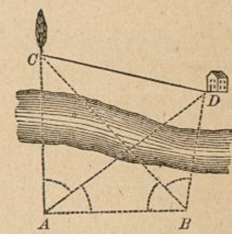
SECOND METHOD.—*If it is not convenient to measure a horizontal base-*

line towards the object, we measure any line AB, and also measure the horizontal angles BAD, ABD, and the angle of elevation DBC. Then, by means of the two triangles ABD and CBD, the height CD can be found.



CASE VII.

60. *To find the distance between two inaccessible objects when points can be found at which both objects can be seen.*



METHOD.—The method of measurement is indicated in the following problem. The method of solution we prefer leaving to the ingenuity of the pupil, that he may learn to think for himself.

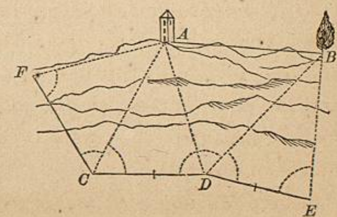
1. Wishing to know the horizontal distance between a tree and house on the opposite side of a river, I took the following measurements:

$$AB = 400; \quad CAD = 56° 30′,$$
$$BAD = 42° 24′; \; ABC = 44° 36′,$$
$$\text{and } DBC = 68° 50′.$$

Required the distance CD. *Ans.* 747.913.

CASE VIII.

61. *To find the distance between two inaccessible objects when no points can be found from which both objects can be seen.*



METHOD.—The method is indicated in the following problem and figure. This and the following case may be omitted with young pupils.

1. Wishing to know the horizontal distance between two in-

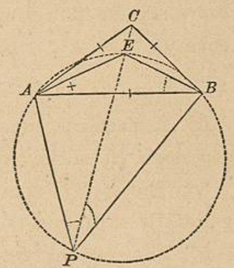
accessible objects when no point can be found from which both objects can be seen, two objects C and D are taken, 600 feet apart, from the former of which A can be seen, from the latter B. From C we measure the distance CF, not in the direction DC, equal to 600 feet, and from D a distance DE equal to 600 feet. We then measure the following angles:

$$CFA = 80° 16', BED = 86° 25',$$
$$ACF = 52° 24', BDE = 60° 24',$$
$$ACD = 56° 36', BDC = 150° 30'.$$

Required the distance AB. *Ans.* 1117.44 feet.

CASE IX.

62. *To find the distances from a given point to three objects whose distances from each other are known.*



Method.—The method is indicated in the problem and figure.

1. I wish to locate three buoys, A, B, and C, in a harbor, so that the distance between A and B is 800 yards, between A and C 600 yards, between B and C 400 yards, and from a fixed point on shore, the angle APC shall equal 33° 45′, and BPC 22° 30′; required the distances PA, PC, and PB.

Ans. $PA = 710.193$; $PC = 1042.522$; $PB = 934.291$.

Note.—This last problem is given by quite a number of authors, and seems to be general property.

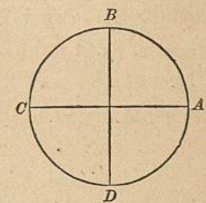
ANALYTICAL TRIGONOMETRY.

63. Analytical Trigonometry is that branch of Mathematics which treats of the properties and relations of trigonometrical functions.

64. Trigonometry, in its origin, was confined to triangles, the method of reasoning being geometrical. After the invention of *analysis*, mathematicians began to apply it to trigonometry, and, in course of time, developed the general properties of trigonometrical functions. This has enlarged the science and greatly increased its power as an instrument of investigation and discovery.

DEFINITIONS.

65. A circumference consists of four *quadrants*. AB is the *first quadrant*; BC is the *second quadrant*, etc.



66. The *origin* of arcs is at A, all arcs being generally supposed to begin at A.

67. The *extremity* of an arc is where it ends. An arc is said to be in that quadrant where its extremity is situated.

68. The sine, cosine, tangent, cotangent, etc. of an arc have already been defined, and need not be repeated here. The *versed sine* of an arc is the distance from the foot of the sine to the origin of the arc. The *co-versed sine* is the versed sine of the complement.

The sines, cosines, etc. are called the *circular functions* of the arcs.

69. Fundamental formulas expressing the relation between the circular functions of any arc.

1. Let a represent the measuring arc of any angle. Draw the lines represented in the figure. Then, from the definitions,