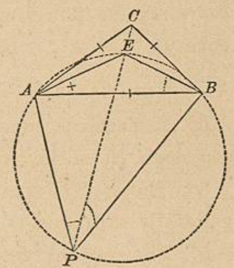
accessible objects when no point can be found from which both objects can be seen, two objects C and D are taken, 600 feet apart, from the former of which A can be seen, from the latter B. From C we measure the distance CF, not in the direction DC, equal to 600 feet, and from D a distance DE equal to 600 feet. We then measure the following angles:

$$CFA = 80° 16', BED = 86° 25',$$
$$ACF = 52° 24', BDE = 60° 24',$$
$$ACD = 56° 36', BDC = 150° 30'.$$

Required the distance AB. *Ans.* 1117.44 feet.

CASE IX.

62. *To find the distances from a given point to three objects whose distances from each other are known.*



METHOD.—The method is indicated in the problem and figure.

1. I wish to locate three buoys, A, B, and C, in a harbor, so that the distance between A and B is 800 yards, between A and C 600 yards, between B and C 400 yards, and from a fixed point on shore, the angle APC shall equal 33° 45′, and BPC 22° 30′; required the distances PA, PC, and PB.

Ans. $PA = 710.193$; $PC = 1042.522$; $PB = 934.291$.

NOTE.—This last problem is given by quite a number of authors, and seems to be general property.

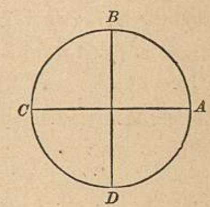
ANALYTICAL TRIGONOMETRY.

63. ANALYTICAL TRIGONOMETRY is that branch of Mathematics which treats of the properties and relations of trigonometrical functions.

64. Trigonometry, in its origin, was confined to triangles, the method of reasoning being geometrical. After the invention of *analysis*, mathematicians began to apply it to trigonometry, and, in course of time, developed the general properties of trigonometrical functions. This has enlarged the science and greatly increased its power as an instrument of investigation and discovery.

DEFINITIONS.

65. A circumference consists of four *quadrants*. AB is the *first quadrant*; BC is the *second quadrant*, etc.



66. The *origin* of arcs is at A, all arcs being generally supposed to begin at A.

67. The *extremity* of an arc is where it ends. An arc is said to be in that quadrant where its extremity is situated.

68. The sine, cosine, tangent, cotangent, etc. of an arc have already been defined, and need not be repeated here. The *versed sine* of an arc is the distance from the foot of the sine to the origin of the arc. The *co-versed sine* is the versed sine of the complement.

The sines, cosines, etc. are called the *circular functions* of the arcs.

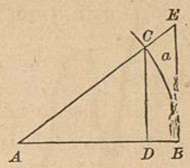
69. FUNDAMENTAL FORMULAS EXPRESSING THE RELATION BETWEEN THE CIRCULAR FUNCTIONS OF ANY ARC.

1. Let a represent the measuring arc of any angle. Draw the lines represented in the figure. Then, from the definitions,

$AB = 1,$ $\quad BE = \tan a,$

$CD = \sin a,$ $\quad AE = \sec a,$

$AD = \cos a,$ $\quad DB = \text{ver} \sin a.$



In the right-angled triangle ADC, we have,

$\overline{CD}^2 + \overline{AD}^2 = AC^2$, or, by substitution,

$$\sin^2 a + \cos^2 a = 1. \qquad (1)$$

Hence, $\sin^2 a = 1 - \cos^2 a$; (2) $\cos^2 a = 1 - \sin^2 a.$ (3)

2. From the figure, we also have,

$$DB = AB - AD; \text{ that is,}$$

$$\text{ver} \sin a = 1 - \cos a. \qquad (4)$$

Since this is true for any value of a, it is true for $90° - a$;

hence, $\quad \text{ver} \sin (90° - a) = 1 - \cos (90° - a),$

or, $\quad \text{co-ver} \sin a = 1 - \sin a. \qquad (5)$

3. Again, the triangles ADC and ABE being similar,

$$EB : AB :: CD : AD,$$

or, $\quad \tan a : 1 :: \sin a : \cos a;$

hence, $\quad \tan a = \dfrac{\sin a}{\cos a}. \qquad (6)$

Substituting $90° - a$ for a, we have,

$$\tan (90° - a) = \frac{\sin (90° - a)}{\cos (90° - a)},$$

or, $\quad \cot a = \dfrac{\cos a}{\sin a}. \qquad (7)$

4. Again, multiplying equations (6) and (7), we have,

$$\tan a \cot a = 1; \qquad (8)$$

hence, $\quad \tan a = \dfrac{1}{\cot a}$ (9), and $\cot a = \dfrac{1}{\tan a}. \qquad (10)$

5. Again, from the same triangles, we have,

$$AE : AB :: AC : AD,$$

or, $\quad \sec a : 1 :: 1 : \cos a;$

hence, $\quad \sec a = \dfrac{1}{\cos a}. \qquad (11)$

Substituting $90° - a$ for a,

$$\sec (90° - a) = \frac{1}{\cos (90° - a)},$$

or, $\quad \text{cosec}\, a = \dfrac{1}{\sin a}. \qquad (12)$

6. Again, from the triangle ABE, we have,

$$\sec^2 a = 1 + \tan^2 a; \qquad (13)$$

hence, $\quad \text{cosec}^2 a = 1 + \cot^2 a. \qquad (14)$

70. These are the fundamental formulas of trigonometry, and should be committed to memory. We will collect them, forming the following table:

TABLE I.

1. $\text{Sin}^2 a + \cos^2 a$	=	1	9. Tan a	=	$\dfrac{1}{\cot a}$
2. $\text{Sin}^2 a$	=	$1 - \cos^2 a$			
3. $\text{Cos}^2 a$	=	$1 - \sin^2 a$	10. Cot a	=	$\dfrac{1}{\tan a}$
4. Ver sin a	=	$1 - \cos a$			
5. Co-ver sin a	=	$1 - \sin a$	11. Sec a	=	$\dfrac{1}{\cos a}$
6. Tan a	=	$\dfrac{\sin a}{\cos a}$	12. Co-sec a	=	$\dfrac{1}{\sin a}$
7. Cot a	=	$\dfrac{\cos a}{\sin a}$	13. $\text{Sec}^2 a$	=	$1 + \tan^2 a$
8. Tan a cot a	=	1	14. $\text{Co-sec}^2 a$	=	$1 + \cot^2 a$

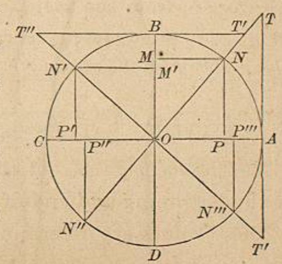
ALGEBRAIC SIGNS OF THE CIRCULAR FUNCTIONS.

71. In analytical trigonometry, we regard the *algebraic signs* of the functions as well as their numerical value. The sign of a function is determined by the following principles.

1. *All lines estimated upward from the horizontal diameter are* POSITIVE; *all lines estimated downward from it are* NEGATIVE.

2. *All lines estimated from the vertical diameter towards the right are* POSITIVE; *all lines estimated toward the left are* NEGATIVE.

Thus, the sines NP and $N'P'$ are positive, while $N''P''$ and $N'''P'''$ are negative; so also the cosines OP and OP''' are positive, while OP' and OP'' are negative.



72. The simplest way to determine the algebraic signs of the different functions is to derive those of the sine and cosine from the figure, and the others from the formulas.

1. The SINE is *positive* in the *first* and *second* quadrants, being measured above, and *negative* in the *third* and *fourth* quadrants.

2. The COSINE is *positive* in the *first* and *fourth* quadrants, and *negative* in the *second* and *third* quadrants.

3. The TANGENT is *positive* in the *first* and *third* quadrants, and *negative* in the *second* and *fourth.*

For, from formula (6),

$$\tan a = \frac{\sin a}{\cos a},$$

and this is positive when sine and cosine have like signs, and negative when they have unlike signs. In the first quadrant, both sine and cosine are plus, in the third both are minus, in the second and fourth one is plus and the other minus; hence, the tangent is positive in the first and third quadrants and negative in the second and fourth.

4. The COTANGENT is *positive* in the *first* and *third* quadrants, and *negative* in the *second* and *fourth;* as is readily shown from the formula,

$$\cot a = \frac{\cos a}{\sin a}.$$

5. The SECANT is *positive* in the *first* and *fourth* quadrants, and *negative* in the *second* and *third.* For, from formula (11),

$$\sec a = \frac{1}{\cos a};$$

hence, the secant has the same sign as the cosine.

6. The CO-SECANT is *positive* in the *first* and *second* quadrants, and *negative* in the *third* and *fourth,* as may be shown from For. (12).

NOTE.—Some of these may also be readily shown from the figure. In the secant, when the distance is estimated *toward* the extremity of the arc, it is *plus;* when *from* the extremity, *minus.*

LIMITING VALUES OF THE CIRCULAR FUNCTIONS.

73. The *limiting values* of the circular functions are their values at the beginning and end of the different quadrants.

These values are determined by the principle that *the value of a variable quantity up to the limit is its value at the limit.*

Beginning at the *origin*, we see that the value of sin 0 is 0, and the cos 0 is the radius, or 1. As the arc increases, the sine increases and the cosine decreases, until at 90° the sine is 1 and the cosine 0. As the arc increases from 90° to 180°, the sine decreases and cosine increases *numerically* (diminishes algebraically), until at 180° the sine is + 0 and cosine — 1. In the same way we see that sin 270° = — 1, and cos 270° = — 0; also, sin 360° = — 0, and cosine 360° = 1.

Now, since, by formula (6),

$$\tan a = \frac{\sin a}{\cos a},$$

substituting 0 for a, $$\tan 0 = \frac{\sin 0}{\cos 0} = \frac{0}{1} = 0;$$

and, also, $$\cot 0 = \frac{\cos 0}{\sin 0} = \frac{1}{0} = \infty.$$

74. By a similar examination of the limiting values of all the functions, we have the following table:

TABLE II.

Arc = 0	Arc = 90°	Arc = 180°	Arc = 270°	Arc = 360°
sin = 0	sin = 1	sin = 0	sin = — 1	sin = — 0
cos = 1	cos = 0	cos = — 1	cos = — 0	cos = 1
v-sin = 0	v-sin = 1	v-sin = 2	v-sin = 1	v-sin = 0
co-v-sin = 1	co-v-sin = 0	co-v-sin = 1	co-v-sin = 2	co-v-sin = 1
tan = 0	tan = ∞	tan = — 0	tan = ∞	tan = — 0
cot = ∞	cot = 0	cot = — ∞	cot = 0	cot = — ∞
sec = 1	sec = ∞	sec = — 1	sec = — ∞	sec = 1
cosec = ∞	cosec = 1	cosec = ∞	cosec = — 1	cosec = — ∞

FUNCTIONS OF THE SUM OR DIFFERENCE OF AN ARC AND ANY NUMBER OF QUADRANTS.

75. The trigonometrical function of any arc formed by adding an arc to or subtracting it from any number of quadrants, may be expressed in functions of the arc which is added to or subtracted from.

1. Let a represent any arc less than 90°; then, from the definitions, we have,

$$\sin(90° - a) = \cos a, \qquad \cot(90° - a) = \tan a,$$
$$\cos(90° - a) = \sin a, \qquad \sec(90° - a) = \operatorname{cosec} a,$$
$$\tan(90° - a) = \cot a, \qquad \operatorname{cosec}(90° - a) = \sec a.$$

2. Now, let a represent the arc BN', then will $ABN' = 90° + a$. From the figure, Art. 71, we see that

$$N'M' = \sin a, \qquad M'O = \cos a,$$
$$P'O = \cos(90° + a), \qquad N'P' = \sin(90° + a).$$

Hence, remembering that ABN', being in the second quadrant, its cosine is negative, we have,

$$\sin(90° + a) = \cos a, \text{ and } \cos(90° + a) = -\sin a.$$

Substituting these values in the formulas for tan, cot, etc. found in Table I., we have,

$$\tan(90° + a) = -\cot a, \qquad \sec(90° + a) = -\operatorname{cosec} a,$$
$$\cot(90° + a) = -\tan a, \qquad \operatorname{cosec}(90° + a) = \sec a.$$

3. Again, let a represent the arc CN', then will $ABN' = 180 - a$. From the figure, we have,

$$N'P' = \sin a, \qquad P'O = \cos a,$$
$$N'P' = \sin(180 - a), \qquad P'O = \cos(180 - a).$$

Hence, remembering that the cosine of ABN' ending in the second quadrant is negative, we have,

$$\sin(180° - a) = \sin a, \text{ and } \cos(180° - a) = -\cos a.$$

Substituting these values in the formulas for tan, cot, etc. in Table I., we have,

$$\tan(180° - a) = -\tan a, \qquad \sec(180° - a) = -\sec a,$$
$$\cot(180° - a) = -\cot a, \qquad \operatorname{cosec}(180° - a) = \operatorname{cosec} a.$$

From the above, we see that *the sine of an arc equals the sine of its supplement, and the cosine of an arc equals minus the cosine of its supplement*, etc.

76. In a similar manner, by deriving the values of the sines and cosines from the figure and making the substitutions in the proper formulas, we may obtain the functions of $180° + a$, $270° - a$, $270° + a$, and $360° - a$. All of these, with the above, are exhibited in the following table:

TABLE III.

Arc $= 90° + a$.				Arc $= 270° - a$.			
sin =	cos a,	cot =	$-\tan a$,	sin =	$-\cos a$,	cot =	$\tan a$,
cos =	$-\sin a$,	sec =	$-\operatorname{cosec} a$,	cos =	$-\sin a$,	sec =	$-\operatorname{cosec} a$,
tan =	$-\cot a$,	cosec =	$\sec a$.	tan =	$\cot a$,	cosec =	$-\sec a$.
Arc $= 180° - a$.				Arc $= 270° + a$.			
sin =	$\sin a$,	cot =	$-\cot a$,	sin =	$-\cos a$,	cot =	$-\tan a$,
cos =	$-\cos a$,	sec =	$-\sec a$,	cos =	$\sin a$,	sec =	$\operatorname{cosec} a$,
tan =	$-\tan a$,	cosec =	$\operatorname{cosec} a$.	tan =	$-\cot a$,	cosec =	$-\sec a$.
Arc $= 180° + a$.				Arc $= 360° - a$.			
sin =	$-\sin a$,	cot =	$\cot a$,	sin =	$-\sin a$,	cot =	$-\cot a$,
cos =	$-\cos a$,	sec =	$-\sec a$,	cos =	$\cos a$,	sec =	$\sec a$,
tan =	$\tan a$,	cosec =	$-\operatorname{cosec} a$.	tan =	$-\tan a$,	cosec =	$-\operatorname{cosec} a$.

77. This table can easily be committed to memory, by observing that when the arc is connected with 180° or 360°, the functions in both columns have the *same name;* but when connected with 90° or 270°, the functions in the two columns have *different names.*

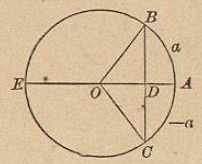
78. The principles of this table are of great value. By their means *the functions of any arc may be expressed in functions of an arc less than* 90°. Thus,

$$\sin 120° = \sin(90° + 30°) = \cos 30°,$$
$$\tan 243° = \tan(180° + 63°) = \tan 63°,$$
$$\cot 304° = \cot(270° + 34°) = -\tan 34°.$$

79. When the arc is greater than 360°, we may subtract 360° one or more times until we obtain an arc less than 360°; the *remainder* will have the same *origin* and *extremity:* hence, the circular function of the remainder will be the same as of the given arc, and this remainder being less than 360°, its functions can be expressed in functions of an arc less than 90°. Hence, *the functions of any arc can be expressed in functions of an arc less than* 90°.

CIRCULAR FUNCTIONS OF NEGATIVE ARCS.

80. Suppose AB to be any arc, and AC, estimated from the origin downward, be numerically equal to AB; then, if the arc AB be denoted by a, the arc AC will be denoted by $-a$; and CD will be the sine, and OD the cosine, of $-a$.



Now, since $BD = CD$ and OD is the cosine of both a and $-a$, we have,

$$\sin(-a) = -\sin a, \text{ and } \cos(-a) = \cos a.$$

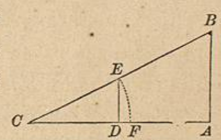
Substituting these in the formulas of Table I., we will have,

$$\begin{array}{llll} \text{ver sin} & (-a) = \text{ver sin } a, & \cot & (-a) = -\cot a, \\ \text{co-ver sin} & (-a) = 1 + \sin a, & \sec & (-a) = \sec a, \\ \tan & (-a) = -\tan a, & \text{co-sec} & (-a) = -\text{co-sec } a. \end{array}$$

81. From what has now been presented, we see that the circular functions of all arcs, whether positive or negative, may be expressed in functions of arcs less than 90°; hence, in the *tables of sines, cosines*, etc., we have only positive arcs and those less than 90°.

RELATION OF THE SIDES AND FUNCTIONS OF RIGHT-ANGLED TRIANGLES.

82. Let ACB be a right-angled triangle, the right angle being at A. Represent the angles by A, B, C, and their opposite sides by a, b, c. With a radius $CE = 1$, describe the arc EF, and draw the perpendicular ED; then $ED = \sin C$, and $CD = \cos C$.



Now, from the figure, we readily obtain,

$$1 : \sin C :: a : c,$$

and, also, $$1 : \cos C :: a : b;$$

hence, $$\sin C = \frac{c}{a} \quad (1), \quad \cos C = \frac{b}{a} \quad (2),$$

or, $$c = a \sin C \ (3), \ b = a \cos C \ (4).$$

Dividing (1) by (2) and then (2) by (1), we have,

$$\tan C = \frac{c}{b} \quad (5), \quad \cot C = \frac{b}{c} \quad (6),$$

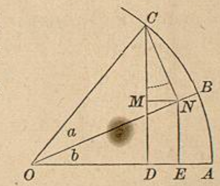
or, $$c = b \tan C \ (7), \text{ and } b = c \cot C \ (8).$$

83. These the pupil will commit to memory, and also translate into common language. The first, thus translated, is as follows:

1. *The sine of either acute angle of a right-angled triangle is equal to the opposite side divided by the hypothenuse.*

84. General formulas relating to the sum and difference of arcs, double arcs, etc.

1. Let AB and BC be two arcs having the common radius OA or $OC = 1$; denote AB by b and BC by a. From C draw CD perpendicular to OA, and CN perpendicular to OB; from N draw NE perpendicular to OA, and NM parallel to OA. Then,



$$CD = \sin(a + b), \ CN = \sin a, \ ON = \cos a.$$

Now, $$CD = CM + NE.$$

In the triangle OEN,

$$NE = ON \sin B = \cos a \sin b;$$

since CMN and NOE are similar, and the angle $MCN = NOE = b$,

$$CM = CN \cos b = \sin a \cos b.$$

Substituting these values in equation (1), we have,

$$\sin(a + b) = \sin a \cos b + \cos a \sin b. \quad \text{(A)}$$

This formula expresses the value of the sine of the sum of two arcs in terms of the sine and cosine of the single arcs. It is enunciated as follows:

The sine of the sum of two arcs or angles is equal to the sine of the first into the cosine of the second, plus the cosine of the first into the sine of the second.

2. If in formula (A) we substitute $-b$ for b, we have,

$$\sin(a - b) = \sin a \cos(-b) + \cos a \sin(-b);$$

but (Art. 80) $\cos(-b) = \cos b$, and $\sin(-b) = -\sin b$;

hence, $$\sin(a - b) = \sin a \cos b - \cos a \sin b. \quad \text{(B)}$$

3. If in formula (B) we substitute $90° - a$ for a, we have,

$$\sin(90° - a - b) = \sin(90° - a)\cos b - \cos(90° - a)\sin b;$$

but, $\sin(90° - a - b) = \sin(90° - (a+b)) = \cos(a+b)$,

and, $\sin(90° - a) = \cos a$, and $\cos(90 - a) = \sin a$;

hence, $\cos(a+b) = \cos a \cos b - \sin a \sin b$. (C)

4. Substituting $-b$ for b in formula (C), we have,

$$\cos(a-b) = \cos a \cos(-b) - \sin a \sin(-b),$$

or, $\cos(a-b) = \cos a \cos b + \sin a \sin b$. (D)

5. From Table I., For. (6), and formulas (A) and (C), we have,

$$\tan(a+b) = \frac{\sin(a+b)}{\cos(a+b)} = \frac{\sin a \cos b + \cos a \sin b}{\cos a \cos b - \sin a \sin b}.$$

Dividing both terms of the last member by $\cos a \cos b$, we have,

$$\tan(a+b) = \frac{\dfrac{\sin a \cos b}{\cos a \cos b} + \dfrac{\cos a \sin b}{\cos a \cos b}}{1 - \dfrac{\sin a \sin b}{\cos a \cos b}}.$$

Cancelling common factors, and reducing, we have,

$$\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a \tan b}. \quad \text{(E)}$$

6. Substituting $-b$ for b in formula (E), and reducing, we have,

$$\tan(a-b) = \frac{\tan a - \tan b}{1 + \tan a \tan b}. \quad \text{(F)}$$

7. Dividing formula (C) by (A), and reducing as in (5), we have,

$$\cot(a+b) = \frac{\cot a \cot b - 1}{\cot b + \cot a}. \quad \text{(G)}$$

8. Substituting $-b$ for b in formula (G), and reducing, we have,

$$\cot(a-b) = \frac{\cot a \cot b + 1}{\cot b - \cot a}. \quad \text{(H)}$$

85. FORMULAS FOR DOUBLE AND HALF ARCS.

1. Making $a = b$ in formulas (A), (C), (E), and (G), we have,

$$\sin 2a = 2 \sin a \cos a; \quad \text{(A}')$$

$$\cos 2a = \cos^2 a - \sin^2 a, \quad \text{(C}')$$

$$\tan 2a = \frac{2 \tan a}{1 - \tan^2 a}, \quad \text{(E}')$$

$$\cot 2a = \frac{\cot^2 a - 1}{2 \cot a}. \quad \text{(G}')$$

2. If now in (C′) we put $1 - \sin^2 a$ for $\cos^2 a$, and then $1 - \cos^2 a$ for $\sin^2 a$, we have,

$$\cos 2a = 1 - 2\sin^2 a, \quad (1)$$

$$\cos 2a = 2\cos^2 a - 1, \quad (2)$$

from which we have,

$$\sin a = \sqrt{\frac{1 - \cos 2a}{2}}, \quad \text{(A}'')$$

$$\cos a = \sqrt{\frac{1 + \cos 2a}{2}}. \quad \text{(C}'')$$

Dividing (A″) by (C″) and then (C″) by (A″), multiplying numerator and denominator by the denominator, and reducing,

$$\tan a = \frac{\sin 2a}{1 + \cos 2a}, \quad \text{(E}'')$$

$$\cot a = \frac{\sin 2a}{1 - \cos 2a}. \quad \text{(G}'')$$

3. Now, substituting $\frac{1}{2}a$ for a in (A″), (C″), (E″), and (G″),

$$\sin \tfrac{1}{2}a = \sqrt{\frac{1 - \cos a}{2}}, \quad \text{(A}_1)$$

$$\cos \tfrac{1}{2}a = \sqrt{\frac{1 + \cos a}{2}}, \quad \text{(C}_1)$$

$$\tan \tfrac{1}{2}a = \frac{\sin a}{1 + \cos a}, \quad \text{(E}_1)$$

$$\cot \tfrac{1}{2}a = \frac{\sin a}{1 - \cos a}. \quad \text{(G}_1)$$

Taking the reciprocals of (E_1) and (G_1), we have,

$$\cot \tfrac{1}{2}a = \frac{1 + \cos a}{\sin a}, \quad \text{(E}_{11})$$

$$\tan \tfrac{1}{2}a = \frac{1 - \cos a}{\sin a}. \quad \text{(G}_{11})$$

86. ADDITIONAL FORMULAS.

1. Adding and subtracting formulas (A) and (B), and doing the same with (C) and (D), we have,

$$\sin(a+b)+\sin(a-b)=2\sin a\cos b, \quad (1)$$
$$\sin(a+b)-\sin(a-b)=2\cos a\sin b, \quad (2)$$
$$\cos(a+b)+\cos(a-b)=2\cos a\cos b, \quad (3)$$
$$\cos(a-b)-\cos(a+b)=2\sin a\sin b. \quad (4)$$

2. Now, making

$$a+b=p \text{ and } a-b=q,$$

whence, $a=\frac{1}{2}(p+q)$ and $b=\frac{1}{2}(p-q)$;

and substituting these in the above, and we have,

$$\sin p+\sin q=2\sin\tfrac{1}{2}(p+q)\cos\tfrac{1}{2}(p-q), \quad (K)$$
$$\sin p-\sin q=2\cos\tfrac{1}{2}(p+q)\sin\tfrac{1}{2}(p-q), \quad (L)$$
$$\cos p+\cos q=2\cos\tfrac{1}{2}(p+q)\cos\tfrac{1}{2}(p-q), \quad (M)$$
$$\cos q-\cos p=2\sin\tfrac{1}{2}(p+q)\sin\tfrac{1}{2}(p-q). \quad (N)$$

3. Now, dividing (K) by (L),

$$\frac{\sin p+\sin q}{\sin p-\sin q}=\frac{\sin\frac{1}{2}(p+q)\cos\frac{1}{2}(p-q)}{\cos\frac{1}{2}(p+q)\sin\frac{1}{2}(p-q)}=\frac{\tan\frac{1}{2}(p+q)}{\tan\frac{1}{2}(p-q)}. \quad (P)$$

In a similar manner, we obtain

$$\frac{\sin p+\sin q}{\cos p+\cos q}=\frac{2\sin\frac{1}{2}(p+q)\cos\frac{1}{2}(p-q)}{2\cos\frac{1}{2}(p+q)\cos\frac{1}{2}(p-q)}=\tan\tfrac{1}{2}(p+q), \quad (Q)$$

$$\frac{\sin p-\sin q}{\cos p+\cos q}=\frac{2\sin\frac{1}{2}(p-q)\cos\frac{1}{2}(p+q)}{2\cos\frac{1}{2}(p+q)\cos\frac{1}{2}(p-q)}=\tan\tfrac{1}{2}(p-q), \quad (R)$$

$$\frac{\sin p+\sin q}{\sin(p+q)}=\frac{2\sin\frac{1}{2}(p+q)\cos\frac{1}{2}(p-q)}{2\sin\frac{1}{2}(p+q)\cos\frac{1}{2}(p+q)}=\frac{\cos\frac{1}{2}(p-q)}{\cos\frac{1}{2}(p+q)}, \quad (S)$$

$$\frac{\sin p-\sin q}{\sin(p+q)}=\frac{2\sin\frac{1}{2}(p-q)\cos\frac{1}{2}(p+q)}{2\sin\frac{1}{2}(p+q)\cos\frac{1}{2}(p+q)}=\frac{\sin\frac{1}{2}(p-q)}{\sin\frac{1}{2}(p+q)}, \quad (T)$$

$$\frac{\sin(p-q)}{\sin p-\sin q}=\frac{2\sin\frac{1}{2}(p-q)\cos\frac{1}{2}(p-q)}{2\sin\frac{1}{2}(p-q)\cos\frac{1}{2}(p+q)}=\frac{\cos\frac{1}{2}(p-q)}{\cos\frac{1}{2}(p+q)}. \quad (U)$$

These formulas may be enunciated in propositions; thus formula (P) gives,

The sum of the sines of two arcs is to the difference of their sines as the tangent of one-half of the sum of the arcs is to the tangent of one-half of their difference.

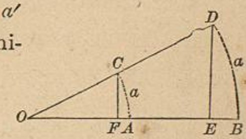
Comparing (S) and (U), we have,

$$\frac{\sin(p-q)}{\sin p-\sin q}=\frac{\sin p+\sin q}{\sin(p+q)}.$$

Hence, *the sine of the difference of two arcs is to the difference of their sines as the sum of the sines is to the sine of the sum.*

INTRODUCTION OF THE RADIUS.

87. In the preceding formulas, the radius, being unity, does not appear in any of the terms. When the radius is other than a unit, it should appear in these formulas, and we will now show how it may be introduced.



Let a be an arc whose radius is 1, and a' be an arc whose radius is R; then, by similar triangles,

$$\sin a : \sin a' :: 1 : R;$$

hence, $\sin a' = R \times \sin a$; $\sin a = \frac{\sin a'}{R}$;

and the same may be shown for the other circular functions.

Therefore, *any circular function whose radius is R is equal to the circular function whose radius is 1, multiplied by R.*

Also, *any circular function whose radius is 1 is equal to the circular function whose radius is R, divided by R.*

Now, if we substitute these in any of the formulas, we will find that R will be introduced in such a manner as to make the formulas *homogeneous*. Thus, For. 6, Tab. I., gives,

$$\frac{\tan a'}{R}=\frac{\frac{\sin a'}{R}}{\frac{\cos a'}{R}}; \text{ or, } \tan a'=\frac{R\sin a'}{\cos a'}.$$

Here, $\tan a'$ is a *line*, and $R\sin a' \div \cos a'$ is a *surface* divided by a *line*, which is also a *line;* hence, the formula is homogeneous. And since the same is generally true, therefore, we can introduce the radius in any formula by multiplying or dividing by R, so as to make the formula homogeneous.

CALCULATION OF A TABLE OF NATURAL SINES.

88. The circumference of a circle whose diameter is 1 is 3.14159 . . . . ; hence, when the *radius* is 1, the *semi-circumference* is 3.14159 . . . . . ; and if we divide this by 10800, the number of minutes in 180°, the quotient, .000290888 . . . . ., will be the length of an arc of *one minute.* Now, this arc is so small that it does not differ materially from its *sine;* hence, we may assume .000290888 . . . . . as the *sine of one minute.*

We then find the cosine of 1′ by For. 3, Table I. Thus,

$$\cos 1' = \sqrt{1 - \sin^2 1'} = .999999957 \ldots\ldots \quad (1)$$

To find the sine of other arcs, we take the formula under Art. 86, putting it in the form,

$$\sin(a + b) = 2 \sin a \cos b - \sin(a - b).$$

Now, make $b = 1'$, and then in succession, a equal to 1′, 2′, 3′, etc., and we have,

$$\sin 2' = 2 \sin 1' \cos 1' - \sin 0 \; = .0005817764 \ldots\ldots$$

$$\sin 3' = 2 \sin 2' \cos 1' - \sin 1' = .0008726646 \ldots\ldots$$

$$\sin 4' = \text{etc.}$$

We may thus obtain the sines of any number of degrees and minutes up to 45°, the corresponding cosines being obtained from equation (1).

Then, since the sine of an arc equals the cosine of its complement, etc., the sines and cosines of arcs between 45° and 90° are immediately derived from those between 0° and 45°.

The tangents are found by dividing the sines by the cosines; the cotangents are found by dividing the cosines by the sines, or by dividing 1 by the tangents.

CALCULATION OF A TABLE OF LOGARITHMIC SINES.

89. A table of logarithmic sines is computed from a table of natural sines. The process is as follows:

For the logarithmic sine, *take the logarithm of the natural sine, and add* 10.

For, let sin a represent the natural sine, and let Sin a represent the sine to a radius of 10,000,000,000; then, Art. 87,

$$\text{Sin } a = \sin a \times R;$$

taking logarithms, we have,

$$\log \text{Sin } a = \log \sin a + \log R.$$

But $\log R = \log 10{,}000{,}000{,}000 = 10.$

Hence, $\log \text{Sin } a = \log \sin a + 10.$

In the same manner, we find the log cosine; and in a similar manner, from the formulas of Table I., we can find all the other logarithmic circular functions.

THEOREMS AND PROBLEMS.

We now present a few exercises for original thought. The first and third are derived from a diagram; the 5th by For. 2. Art. 84; several which follow, by substituting values from Table I., obtaining an equation involving but one unknown quantity, which can then readily be found; the others, by judicious substitutions and reductions.

1. Prove that $\sin 60° = \frac{1}{2}\sqrt{3}$, and $\cos 60° = \frac{1}{2}$.
2. Prove that $\sin 30° = \frac{1}{2}$, and $\cos 30° = \frac{1}{2}\sqrt{3}$.
3. Prove that sin and cos of 45° equal $\frac{1}{2}\sqrt{2}$.
4. Prove that $\tan 45° = 1$, and $\sec 45° = \sqrt{2}$.
5. Prove $\sin 15°$, or $\sin(60° - 45°) = \dfrac{\sqrt{3} - 1}{2\sqrt{2}}$, and $\cos 15° = \dfrac{\sqrt{3} + 1}{2\sqrt{2}}$.
6. Prove $\tan 15° = 2 - \sqrt{3}$, and $\cot 15° = 2 + \sqrt{3}$.
7. If $\sin a \cos a = \frac{1}{4}\sqrt{3}$ find sin a and cos a.
   Ans. $\sin a = \frac{1}{2}\sqrt{3}$; $\cos a = \frac{1}{2}$.
8. If $3 \sin a + 5\sqrt{3} \times \cos a = 9$, find sin a. *Ans.* $\sin a = \frac{1}{2}$ or $\frac{1}{7}$.
9. If $\sin a(\sin a - \cos a) = \frac{4}{25}$, find sin a. *Ans.* $\sin a = \frac{4}{5}$.
10. If $\tan a = \frac{4}{3}$, find sin a and cos a. *Ans.* $\sin a = \frac{4}{5}$; $\cos a = \frac{3}{5}$.
11. If $\tan a + \cot a = 2$, find tan a. *Ans.* $\tan a = 1$.
12. Prove that $\tan^2 a - \sin^2 a = \tan^2 a \sin^2 a$.
13. Prove that $\sec^2 a \operatorname{cosec}^2 a = \sec^2 a + \operatorname{cosec}^2 a$.
14. Prove that $\sin(30° + a) + \sin(30° - a) = \cos a$.
15. Prove that $\cos(60° + a) + \cos(60° - a) = \cos a$.
16. If $a + b + c = 180°$, prove that
    $$\tan a + \tan b + \tan c = \tan a \tan b \tan c.$$
17. If $a + b + c = 90°$, prove that
    $$\cot a + \cot b + \cot c = \cot a \cot b \cot c.$$

SUGGESTION.—In 16th, $\tan(a + b) = \tan(180° - c)$, develop and simplify; and similarly in 17th.