

FIG. 4975.—Fruit of "split" Vanilla, one-fourth natural size.

20 to 25 cm. (8 to 10 in.) long and 5 to 8 mm. ($\frac{1}{4}$ to $\frac{1}{2}$ in.) thick, straight, except for the short-hooked base, flexible, longitudinally wrinkled, dark chocolate-brown, shin-

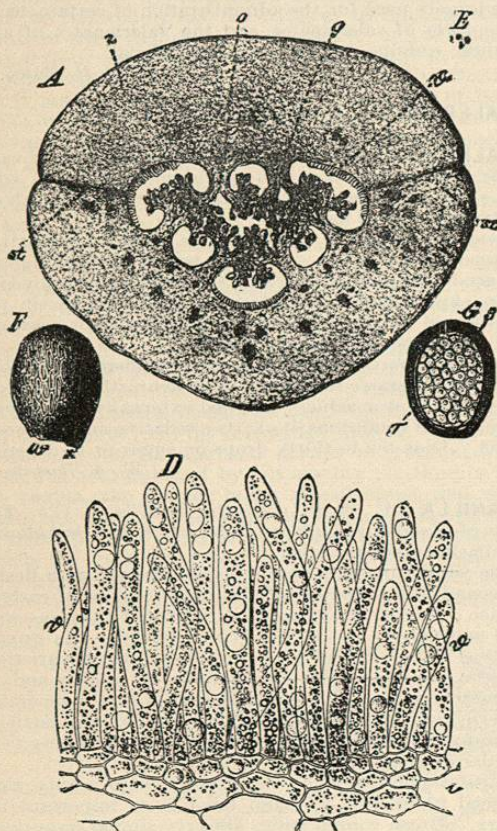


FIG. 4976.—A, Transverse section of Vanilla Pod, enlarged; B, seed, natural size; C, the same, magnified; D, section of same; E, the papillary gland hairs lining the cavity of the pod. (Berg.)

ends in drying, which renders them less salable; though some good judges believe that these "splits," as they are commercially termed, are not inferior. The details of the curing process vary greatly in different countries, and even among different operators, but its essential features depend upon the fact that the odorous principle, vanillin, does not exist in the natural fruit and must be developed in the curing process. This usually consists of a sweating or steaming operation in a closed space, followed by alternating exposures in closed containers, or between woollen blankets. The "beans" thus treated gradually shrink in thickness and become darker until they assume the characteristic appearance described below. They are assorted into lengths, the more long and slender ones being preferred and selling higher. They are tied tightly into bundles, wrapped in paraffin paper, and packed in tight tin cans. Because of the loss in weight from over-drying, there is a temptation to underdry them, and many bundles, owing to this mistake, are found mouldy within, on being opened. The finest product is thus described: Bean-shaped fruits, from

ing or covered with white crystals of vanillin, giving them a frosted appearance; one-celled, with three fleshy placenta and innumerable minute seeds; having a strong, pleasant, and characteristic fragrance, and a sweetish, aromatic, and slightly fruity taste.

CONSTITUENTS.—The important constituent of vanilla, and that which represents its odor and taste, is from 1.5 to 2.5 per cent. of *vanillin*, *vanillic aldehyde*, or *methyl-protocatechuic aldehyde*, ($C_8H_8O_2$), which occurs in colorless prisms having the characteristic odor and taste of vanilla in intense degree; they melt at 80° to 81° C., and are soluble in alcohol, ether, and chloroform, and less so in water. With the vanillin, occur a little volatile and about ten per cent. of fixed oil, sugar, and other unimportant constituents.

SUBSTITUTES AND ADULTERANTS.—On account of the high price of vanilla and vanillin, both have been enormously substituted and adulterated. The use of inferior fruits of other species of vanilla, short, broad, flat, and of a strong fruity odor, must be classed as substitution. These usually cost only from one-tenth to one-twelfth as much as the best. The use of tonka beans and other substances containing cumarin is even more extensive. Finally, various compounds of prunes, raisins, and other substances are sold as vanilla substitutes. Cumarin is itself substituted for vanillin. Artificial vanillin is largely prepared from coniferin and eugenol, and, although apparently identical chemically with the natural, it has not been found equal to it in useful properties.

USES.—Vanilla has no active properties as a medicine, though the official ten-per-cent. tincture is sometimes used as an antispasmodic in fluidrachm doses. As an adjunct for perfuming and flavoring, it has many uses in the pharmacy, as well as in confectionery and pastry.

Henry H. Rusby.

VANILLIN PARAPHENETIDIN, $C_6H_3.OH.OCH_3.CH.N.C_6H_4.OC_2H_5$ is formed by heating vanillin with paraphenetidin. It occurs in the form of crystals which are soluble in water, and in dose of 1 to 2 gm. (gr. xv.-xxx.) it has a hypnotic and antineuralgic action.

W. A. Bastedo.

VARIATION.—In its modern biological sense, *variation* is used to signify a difference or the presence of differences from some *type* taken as the standard of comparison. With a somewhat more limited meaning, the word *deviation* also is used as the name for a difference from the type of any organ or character. By *variability* is meant the extent of the deviations, together with the relative

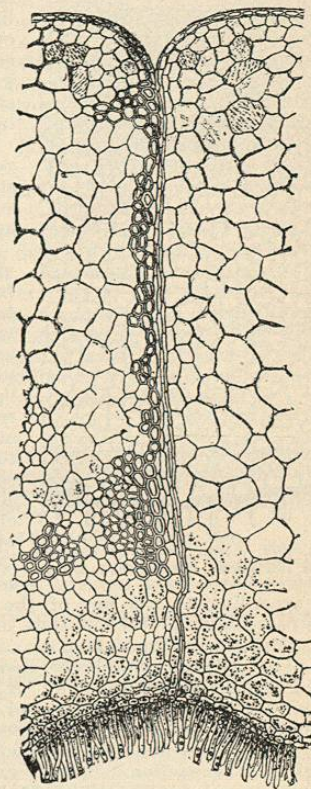


FIG. 4977.—Section of Vanilla through the Commissure where the Fruit splits. (Berg.)

number of individuals possessing each degree of variation.

What Variation is Not.—Much confusion in regard to variation has arisen from the word having been employed loosely with meanings not in accord with the present usage. Therefore it may be well at the outset to call attention to the more important variations in the use of the word.

In the first place, it is necessary to distinguish between variation and *modification*. Following the suggestion of C. L. Morgan (1897), which has been adopted largely by biologists, we may employ the word *modification* as a technical term, to signify a change in any characteristic of an individual organism in response to some condition of its environment or the process by which such a change is brought about. Modifications were spoken of formerly as *acquired characters*, but the term *acquired* has frequently led to confusion in the minds of persons not perfectly familiar with the facts. It seems best, therefore, to abandon that term. A modification is a change produced in an individual. A variation is any difference in the individual from a certain ideal condition taken as the type. Thus a modification may produce or modify a variation, but modification and variation are two entirely distinct and independent concepts. An electric discharge may produce sound, but sound and electricity are not the same thing, and sound may be produced by other causes. Likewise, modification may produce variation, but they are not the same thing and variation may be due to other causes.

Theoretically we may follow Weismann in making a distinction between blastogenic and somatogenic variations; *blastogenic* variations being those due to differences in the constitution of the germs from which the group of individuals under consideration have developed, while *somatogenic* variations are differences which have arisen in the bodies (*somata*) of the individuals as the result of modification following exposure to different environmental conditions. But as has been pointed out by Vernon (1903), in the practical study of variation by statistical methods, it is generally impossible to make this distinction. This is due, in the first place, to the fact that blastogenic and somatogenic variations may be of the same kind and degree. Thus deafness may result from a pathological lesion, as in scarlet fever, or it may be due to a congenital malformation, which, owing to observed parental correlations, we must suppose to be of blastogenic origin. But even in such an extreme variation as this Fay (1898) found it impossible to determine in many cases to which class a given individual should be assigned. In the second place, every individual and even parts of individuals, are subject to constantly fluctuating environmental conditions from the moment of conception until death. These changes of surrounding conditions are met by corresponding responses, functional or structural, or both, on the part of the organism, but these reactions are themselves probably subject to variations independent of the exciting stimuli, and who can tell, therefore, how much of the similarities and differences in any group of organisms is due to similarities and differences in the structure of their germs, and how much to the various conditions of the environment to which they have been subjected? The importance of modifications in the normal development of organisms has been discussed in another article (see *Differentiation*).

The clearest examples of purely blastogenic variation are to be found in such mammalia as give birth to a considerable number of offspring at one time. For example, in a litter of puppies or kittens, all sired by the same male, all developed at the same time in the same uterus, we have a group of organisms where the conditions of parentage and environment are as uniform as possible, and yet variations in color, form, and mental traits will be readily apparent. These differences appear to be clearly blastogenic in origin, but the very general presence of a "runt" in such a litter indicates that the conditions of nutrition at least have not been quite uniform. Variation, then, as we understand the

term at present, is not a broader term than modification, as Vernon (1903) would have it, but is an entirely distinct concept. Modification, on the other hand, may stand toward variation in the relation of cause to effect. Variations so effected are classed as somatogenic, as distinguished from blastogenic variations of purely germinal origin.

Moreover, our idea of variation does not include any change in the individual taking place during the course of its life, no matter whether that change be of blastogenic or somatogenic origin. Thus the changes which an individual insect undergoes in the course of its metamorphosis would not come within our definition of variation. But a group of insects, say the potato beetles in a certain field, would exhibit in each stage of their development variations from the type of that stage, and the degree of variability might differ in the successive stages.

Finally, the idea of variation does not necessarily include the parental relation. It is true that we may select one or the other parent, or both parents combined in an imaginary mid-parent (see *Heredity*) as the type, when dealing with the characters of offspring of identical or like parents. This is done in the study of heredity. But in the ordinary study of variation the investigator is usually ignorant of the parentage of his material. So a variation is not a difference from the parental type necessarily, but is a difference from any type that may be selected.

In fact, variation is a phenomenon by no means confined to organisms. It is shown as well by lifeless objects and events. Thus we may, if we choose, study the variability of the stones on the beach, of snow crystals, of the forms of clouds, of rainfall, of the height of the barometer, or of the price of wheat.

Aim and Method of Study.—Having now clearly in mind what is meant by variation, let us consider what it is that we may expect to gain by the study of this subject and the method to be pursued in such an investigation.

Every one who is at all familiar with modern theories of organic evolution knows that variation is of very great and very fundamental importance from the theoretical points of view. Variation is at the basis of every theory of evolution. Therefore, in order to understand evolution, we must know the facts of variation. (See *Evolution*.)

Variation is an important factor in the classification of animals and plants. Formerly the systematist simply took as the type of a species the individual that was first described. The modern methods enable him, when provided with sufficient material, to define the limits of a species with much greater accuracy and to select as the type a form that is truly representative.

To the physician the study of variation is of great importance. Every practitioner knows that the same disease may be manifested in diverse forms, and that patients vary in their reactions to treatment. A study of the types and variability of diseases and of the effects of treatment may be expected to add much toward a true appreciation of these subjects.

For the surgeon, variation is still more important. He must know what anomalies are likely to be met with and their relative frequency.

Variation has been studied extensively by teachers of gymnastics and the results have had practical application in the determination of the relative rank of the pupils and their degrees of improvement.

The method employed in the modern study of variation is statistical. This, as Pearson (1902, p. 320) has well said, is "because the whole problem of evolution is a problem in vital statistics—a problem of longevity, of fertility, of health, and of disease, and it is as impossible for the evolutionist to proceed without statistics as it would be for the registrar-general to discuss the national mortality without an enumeration of the population, a classification of deaths, and a knowledge of statistical theory." This is true, not only for the problem of evolution, but also for all problems for which a solu-

tion may be sought in the phenomena of organic variation.

In order that this method may be employed, the data must be expressed in quantitative terms. We may no longer speak of things as being longer or shorter, heavier or lighter, but we must give the results of observations in exact units of length, weight, and the like. Or, if this be impossible, the observations must be divided into groups, so that each group shall be as nearly as may be equally divergent from the one on either side of \bar{x} , in order that to each one may be assigned a serial number that can be used as a quantitative expression.

The statistical method for the study of variation was founded by Quetelet (1846), who showed that certain human characteristics follow in their relative frequency the mathematical law of the probability of error, which had been applied previously to astronomical and other physical observations. Darwin (1868) gathered a great many cases of variation, especially among domestic animals and plants, but he seems to have overlooked Quetelet, or, at any rate, he did not employ his method. The first attempt to study the variations of wild animals by exact quantitative measurements appears to have been made by J. A. Allen. But we are especially indebted to Galton (1889) for calling the attention of both sociologists and naturalists to the importance of the statistical method for the study of variation and heredity. He made considerable advances beyond Quetelet's use of the theory of probability, and his methods were quickly appreciated and used by persons interested in anthropometry. Weldon (1890) was the first to apply these methods in the study of variations of animals, and de Vries (1894) applied them in botany. He was quickly followed by Ludwig (1895), who had made previously extensive statistical investigations on plants (1883 *et seq.*).

While others had paved the way, naturalists are especially indebted to Pearson for his mathematical investigations beginning in 1893, which, it is safe to say, will be the foundation for all future work in statistical biology. Unfortunately, however, much of his writing is unavailable to any but those versed in the language of the higher mathematics. Dunker (1899) has done good service in giving an excellent elementary treatment of the subject as it stood at the end of the year 1898, and Davenport (1899) has furnished a useful manual of formulæ and tables. But both of these works are now (1903), in part at least, out of date, and a text-book that will enable the average naturalist or physician to understand and apply the principles laid down by Pearson is still very much to be desired.

Classification of Variations.—Before proceeding with the details of the statistical method it may be well to pause to consider the nature of the material to be studied.

In the first place it should be noted that every individual has many characteristics, and one of the limitations of the method is that it deals primarily, not with individuals as entire objects, but with single characters only. Part of this difficulty may be overcome in some cases by combining two or more characters in a ratio, usually called an *index*, as the *cephalic index*, the ratio of breadth to length of head, or the *strength-weight index*, which is the ratio of the sum of the strength tests to the weight. Such a ratio may then be treated like a single character.

Another difficulty that would be met with in an attempt to appreciate the variability of entire organisms arises from the fact that single characters in one individual differ greatly among themselves in the amount of their deviations from the values typical for the population as a whole. For example, it is very seldom that we meet with a man who has more or less than one nose, two eyes, a mouth, and the like; while, on the other hand, we find within certain limits all degrees of variability in stature, proportions of the face, color of the hair and eyes, and in various mental traits. So at the outset we may divide the characteristics of organisms into (1) those that are normally stable and (2) those that are normally variable. When a variation is observed in a character that is normally stable, it is so unusual an occur-

rence as to attract at once the attention of an observer familiar with the normal form. Gardeners call such variations "sports." Nectarines, our seedless fruits, double flowers, and many new varieties of flowers and fruits have arisen in the first place as sports. In animals polydactylism, albinism, and other monstrosities come within this class of unusual variations, which we may designate as *anomalies* in contradistinction to the *normal variations* of normally variable characters. Perhaps the normally variable characters may sometimes show variations that are so extraordinary as to be classed as anomalies, but it is hard to draw a line between these and the extremes of normal variation.

Bateson (1894) has divided variations into substantive, meristic, and homeotic. *Substantive* variations are differences in the substance of the organism, such as variations in size, weight, color, chemical composition, and the like. *Meristic* variations are differences in the number, division, and geometrical position of parts. *Homeosis*, or the occurrence of a homeotic variation, may be seen in a meristic series when a member of one series appears in the form characteristic of another series, as when in a flower one in the series of stamens appears in the form of a petal, or as when in a spinal column a vertebra of the lumbar series develops with the structure of a thoracic vertebra. In a case like the spinal column, however, where several series of structures are closely connected, it is not always possible to distinguish between homeosis and ordinary meristic variation.

It will be seen from this classification that variation opens a wide field for study. It would be useless to attempt to give illustrations of these classes in the present article, for every one will call to mind cases within his own experience, and to summarize the literature on the subject would require volumes. One wishing information in detail should consult Darwin's "Variation of Animals and Plants under Domestication" (1868), Bateson's "Materials for the Study of Variation" (1894), and especially Vernon's "Variation of Animals and Plants" (1903), which is an excellent summary of the results of recent investigations.

From another point of view variations may be classified according to their mode of occurrence into (1) variations between parts of an individual, (2) variations between individuals, and (3) variations between races or other groups of individuals.

In both animals and plants there may be produced in one individual a number of similar parts which show among themselves no differentiation connected in any marked degree with either their time of development, their relative position, or their relation to the environment (Pearson, 1902, p. 335). Such "undifferentiated-like organs" Pearson calls *homotypes*, and we may speak of the variation of such parts within a single individual as *homotypic variation*. Examples are variations in the number of veins in the leaves of a beech tree, the number of seeds in the pods of a pea vine, and the like.

If, instead of comparing similar characters of the same individual, we take like characters of different individuals, we may find what is called *individual variation*.* For example, we might count the number of ray flowers on the highest head of each one of a thousand plants in a field of daisies, or we might measure the stature of each man in a regiment of soldiers. Such observations would give us the data for the study of individual variations in these groups.

Finally, if a study of the data of individual variation shows that the observations tend to group themselves, not about a single type, but around two or more types (as determined by appropriate methods to be mentioned later), then each type may be taken as representation of a race or of a species, and we are confronted by the phenomena of *racial variation*. Under this heading would be included differences of type in the same locality as

* Pearson uses the term "individual variation" for what we have called homotypic variation and his racial variation is our individual variation. But our use of the latter term is more in accord with current American usage.

well as differences in different localities. In the first case the material would be said to be *heterogeneous*; in the second case, while it might be *homogeneous*, that is, variable about a single type, in each separate locality, it would likewise be heterogeneous as a whole. The heterogeneity of mixed material in the same locality is difficult to determine, but the presence of racial variability associated with difference of dwelling-place is a familiar phenomenon, especially to American ornithologists. Witness the races of Bob Whites, song sparrows, flichers, Western blue jays, etc. It is important to note the distinction between what we have called individual and racial variation, because a failure to do this leads to confused ideas as to the method of evolution.

Variation and Chance.—All human knowledge is an estimate of probabilities. But in the statistical study of variation the mathematical theory of probability, or doctrine of chance, is of especial importance, for it forms the foundation upon which the method is erected.

Most persons think of chance as something quite independent of law. They imagine an event occurring by chance as one quite impossible of prediction, and the Darwinian theory has been criticised because it supposes race progress to depend upon the survival of "mere chance variations." On the contrary, while it is perfectly true that we are frequently unable to predict single events, it may be shown, nevertheless, that chance follows very definite laws, which may be expressed by exact mathematical formulæ.

In order that the discussion of statistical methods may be clear, it is necessary to call to mind at this point those principles of probability that especially concern our subject.

In the first place, "the theory of chance consists in reducing all the events of the same kind to a certain number of cases equally possible, that is to say, to such as we may be equally undecided about in regard to their existence, and in determining the number of cases favorable to the event whose probability is sought. The ratio of this number to that of all the cases possible is the measure of this probability, which is thus simply a fraction whose numerator is the number of favorable cases, and whose denominator is the number of all the cases possible" (Laplace, 1902, p. 6).

This definition of probability, which is the first principle of Laplace (*loc. cit.*, p. 11), may be illustrated by the tossing of a coin. Suppose I drop a silver quarter upon the table: it may come to rest either with the goddess of liberty or with the eagle on the upper side, "heads or tails." We have, then, two equally possible events resulting from the tossing of the coin. Suppose A bets that the throw will be heads, and B takes him up. There are two possible events, of which one is favorable to A, so the probability of his winning is expressed by the ratio one-half, but the other one of the two possible events is favorable to B, and the probability of his winning is also expressed by the ratio one-half. The probabilities for and against heads being the same, the bets on the game should be even. All the equally possible events that may produce or prevent a result are called the *chances* of that result (Quetelet, p. 10). The chances may be favorable or unfavorable; if all favorable, the result is a certainty. It will be noticed that the probabilities for and against the event added together equal unity, which is the mathematical expression for certainty. Thus in mathematics the various degrees of probability are expressed by fractions which may have any value between 0 and 1. It should be noted also that the probability of a given result remains the same no matter how many possible events may affect it, provided that the ratio of favorable events to total events remains the same. Suppose, for example, that A and B play with dice instead of a coin. A cube has six faces, any one of which may rest uppermost. Let A bet that the upper face will show more than three spots, he wins if it shows four, five, or six; that is, there are three favorable out of six possible chances, and the probability of his winning is $\frac{3}{6} = \frac{1}{2}$, as before. But suppose that he wagers on the throw being an ace, then he will lose if any one of the other five sides

turns up. His probability of success is then $\frac{1}{6}$, and the probability of the stakes going to B is $\frac{5}{6}$. In making the bets, therefore, B should give odds to A of five to one. What this means is, that, while we cannot predict the result of a single throw of coin or dice (for if we could the probability would be certainty), this calculation enables us to foresee that in the game of heads and tails or in throwing dice for more than three spots, the number of successes will in the long run equal the failures, or in the throwing of aces will be in the proportion of one to five, provided always that the coin or dice are perfectly symmetrical in form and weight.

The closeness of observation to theory is illustrated by an experiment made by Quetelet (*loc. cit.*, p. 93). He placed in an urn 40 white and 40 black balls, and then drew one ball at a time, which was replaced after its color had been recorded. He made in all 4,096 drawings, and according to the theory of probability the result should have been 2,048 white balls and the same number of black. The actual results were 2,066 white and 2,030 black balls, an error of only $\frac{18}{1000}$.

The theory of probability teaches us how to judge of future events by past experience; it is thus common sense reduced to mathematical expression. Suppose, as does Quetelet (*loc. cit.*, p. 14), that we wish to determine the probability of drawing a white ball from an urn. We upset the urn and count the balls. If we find two white balls, three red, and four black, we answer that the probability of drawing a white ball at the first draw, the balls having been replaced, will be $\frac{2}{9}$.

But let it be impossible to upset or to see into the urn, and let only one ball be drawn at a time, which must be replaced before drawing another. Having no previous knowledge of the contents of the urn, we may suppose it to contain any number of balls of any color mixed in any proportion. The probability of drawing a white ball at the first draw may have any value from 1 to 0. We may suppose this value to be $\frac{1}{3}$. We draw a ball which is white, and it is then replaced. Now we know that the urn contains one white ball, and we may guess as before that the rest of the balls are in equal proportions of white and colored. There would then be on this hypothesis two chances of drawing a white ball to a total of three chances, and the probability would be $\frac{1+1}{1+2} = \frac{2}{3}$.

A second draw is made with the same result. Now we know that the urn contains two white balls, and the probability that the third ball will be white becomes $\frac{2+1}{2+2} = \frac{3}{4}$. Similarly after ten white balls have been drawn in succession, the probability that the next ball will be white is $\frac{10+1}{10+2} = \frac{11}{12}$, and after ninety-eight

repetitions of this result it is increased to $\frac{98+1}{98+2} = \frac{99}{100}$

(Quetelet, *loc. cit.*, p. 367). This is but a rough illustration of a principle that mathematicians arrive at by a somewhat different line of reasoning involving the processes of the integral calculus. The rule is: "That an event having occurred successively any number of times, the probability that it will happen again the next time is equal to this number increased by unity and divided by the same number increased by two units" (Laplace, p. 19).

Thus from the analogy of the urn we see how we may judge of the probability of future events in nature. If, for example, I examine one hundred specimens of a certain kind of animal and find that each one has six legs, the probability that the next one examined will have six legs is $\frac{100}{100+1}$, a value so near unity that I may safely infer that the number of legs in this species is a normally stable character. Putting the extent of historical time at 5,000 years, or 1,826,213 days, during which time the sun is known to have risen regularly during every twenty-four hours, the probability, aside from all other considerations, that it will rise to-morrow is $\frac{1,826,214}{1,826,215}$

or the chances that it will not rise are about one in two million. Thus we see how the factor of doubt in regard to any deduction decreases with the increase in the number of observations upon which the deduction is based. Whatever may be the true value of this unknown factor we may reduce its relative value to any extent we please by continuing the observations to any desired limit. An explorer who upon entering an unknown island should meet two men, both black, would still have considerable doubt as to the color of the rest of the inhabitants. But after he had seen a thousand of the natives and found them all black, he would be justified in thinking that to be the color of the whole population.

Another principle similar to the preceding may be illustrated by returning again to the urn. Suppose that the result of the first nine consecutive drawings was five white and four black balls, the probability of drawing a white ball next time would be expressed by $\frac{5+1}{5+4+2} = \frac{6}{11}$; and if the result of ninety-nine drawings was fifty-five white and forty-four black balls the probability that the hundredth ball will be white is $\frac{55+1}{55+44+2} = \frac{56}{101}$. Thus in

such cases in which observations show two possible results, the probability that one of these results will occur again is expressed by a fraction whose numerator is the number of times that result has been observed plus one, and whose denominator is the total number of observations plus two (Quetelet, p. 27). Here, again, we see the factor of doubt decreasing in relative value with the increase of the number of observations, so that, if after a large number of drawings we still get the same relative numbers, we may conclude that the urn contains only black and white balls, and that the proportion of white to black is five to four. Hence with a sufficient number of observations the factor of doubt practically vanishes and we may consider the chances favorable to each of the results to be in the same proportion to the total number of chances that the number of times each result has been observed is to the total number of observations (Quetelet, p. 29). When there are several possible results the calculation is rather complicated, but this rule practically holds good.

This principle is applied in calculating the proportion of the sexes. Thus Quetelet, taking the data from all the rural districts of Belgium during the year 1841, found that there had been born 53,437 male children and 49,788 females, giving a proportion of 1,072 males to 1,000 females. But taking the data for nine years, 1834 to 1842, he found the proportion to be 1,063 to 1,000, which is the same as the result obtained later by Oesterlen from the records of 59,350,000 births gathered from several countries of Europe. (See article *Sex*.)

One of the most important of the principles of probability is that, "If the events are independent of one another, the probability of their combined existence is the product of their respective probabilities." (Third principle of Laplace, *l.c.*, p. 12.) For example, if A toss a coin, the probability of heads is one-half; likewise the probability of B's coin giving heads is one-half. If A and B toss their coins at the same time, there are four possible combinations, but in only one of these do both get heads, thus:

(1)	(2)	(3)	(4)
A—heads	heads	tails	tails
B—heads	tails	heads	tails

So the probability that both A and B will throw heads (1) at the same time is $\frac{1}{4}$, which is equal to the product of the probabilities that A and B will throw heads, respectively, multiplied together, $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$. For the same reason the probability that both will throw tails (4) at one time is $\frac{1}{4}$, and $\frac{1}{4}$ is likewise the probability that A will throw heads at the same time that B throws tails (2), and that B throws tails when A throws heads (3). But combinations (2) and (3) give like results, namely, heads and tails. So there are two chances of heads and

tails against two that the throws will be either both heads or both tails, and the probability is $\frac{1}{2}$. Thus heads and tails thrown with two coins is a complex combination, and its probability is the sum of the probabilities of the two simple combinations of which it is composed, $\frac{1}{4} + \frac{1}{4} = \frac{1}{2}$. It may be noticed that these values, $\frac{1}{4}$, $\frac{1}{2}$, $\frac{1}{4}$, are the terms of the expanded binomial $(\frac{1}{2} + \frac{1}{2})^2$.

If C comes into the game the number of possible combinations is increased as follows:

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
A—h	h	h	t	t	h	t	t
B—h	h	t	h	t	t	h	t
C—h	t	h	h	h	t	t	t

The number of possible combinations is now $8 = (2)^3$, and of these only one favors the throwing of heads by all three players, the probability of that result being, therefore, $\frac{1}{8} = (\frac{1}{2})^3$. But there are three combinations, giving two heads to one tail. The probability of each of these is $\frac{1}{8}$ and the sum of the probabilities is $\frac{3}{8} = 3 \times \frac{1}{8} \times (\frac{1}{2})^2$. The probability of two tails and one head is also $\frac{3}{8}$, and of all tails is $\frac{1}{8}$. Again we have a series, $\frac{1}{8}$, $\frac{3}{8}$, $\frac{3}{8}$, $\frac{1}{8}$, which forms the terms of an expanded binomial, $(\frac{1}{2} + \frac{1}{2})^3$.

Now putting these figures into more general terms, we may represent the probability that a single throw will be heads by p , and that it will be tails by q . Let n equal the number of coins used and N the number of throws of the n coins together. Then the frequency of each combination of heads and tails will be represented by the terms of the binomial $N(p+q)^n$ when expanded. But in order that all the terms may be whole numbers, N must at least be equal to 2^n (greater when p and q are not equal), and the number of terms will be $n+1$.

With the view of illustrating the truth of this formula Weldon (1898) made four series of 4,096 throws of 12 dice ($4,096 = 2^{12}$), and counted the number in each throw of dice that showed more than 3 spots. The results are as follows:

Number of dice with more than three spots.	Terms of $4096 (\frac{1}{2} + \frac{1}{2})^{12}$.	OBSERVED FREQUENCIES.			
		i.	ii.	iii.	iv.
12.....	1	0	1	0	1
11.....	12	11	13	8	14
10.....	66	71	86	61	66
9.....	220	257	246	241	241
8.....	495	536	540	513	586
7.....	792	847	836	856	861
6.....	924	948	913	948	896
5.....	792	731	750	802	728
4.....	495	430	446	420	474
3.....	220	198	198	182	204
2.....	66	60	55	51	67
1.....	12	7	12	13	6
0.....	1	0	0	1	0

The closeness of observation to theory is more clearly apparent in the diagram (Fig. 4978). The solid line is a polygon of thirteen points, the abscissa and ordinate of each point being proportional, respectively, to a number from the first column and the corresponding number on the same line in the second column of the table. The other polygons are formed in the same way, but with ordinates taken from the columns of observed frequencies. The first polygon is a graphic representation of the result expected on the theory of probability, supposing the dice to be perfectly symmetrical.

It will be noticed that the observed polygons lean slightly toward the side of the greater number of spots. This is readily explained when it is noticed that the dice are slightly excavated at each spot, and the side with six spots is opposite the ace. There is therefore a slight eccentricity of the centre of gravity, which tends to cause the greater number of spots to turn upward. These facts serve as a simple illustration of the way in which the study of probabilities may lead to the discovery of previously unknown causes of observed facts. Thus we see that in this case, in order to make our formula exact-

ly represent the facts, p should be made slightly greater than q . This is illustrated further in another series of 4,096 throws of 12 dice, in which Weldon counted the number

When the chances for and against the event to be observed are even, when $p = q$, the curve will be symmetrical about the middle point, which is the highest. When the chances are uneven, when p is greater or

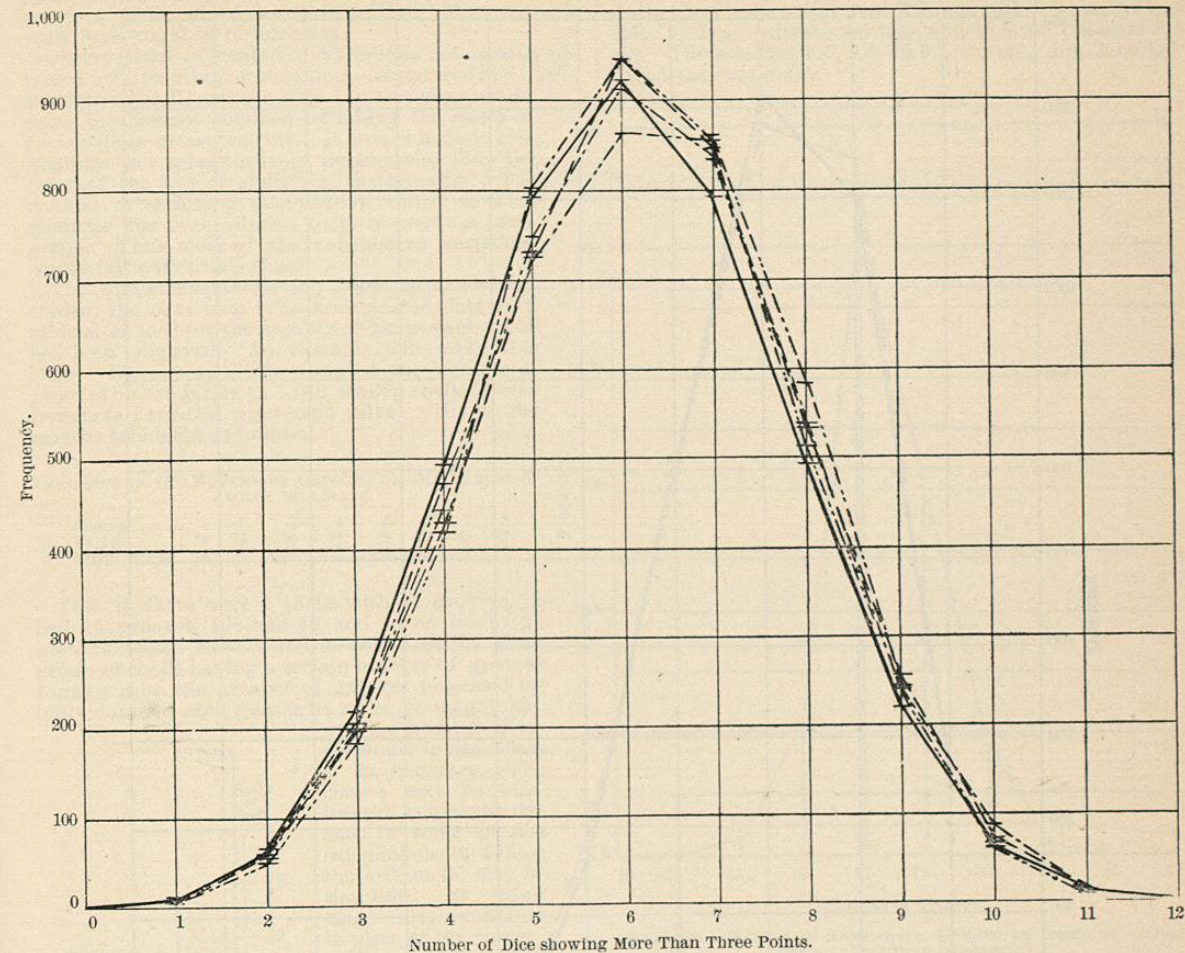


Fig. 4978.—Binomial Curve of Probability, terms of $4,096 (\frac{1}{2} + \frac{1}{2})^{12}$, compared with the frequencies of the numbers of dice showing more than three points, in each of four series of trials with twelve dice, the number of throws in each series being $2^{12} = 4,096$. Binomial curve; —, observations series I — — —, II — — —, III — — —, IV — — —.

of sixes. Each die having six faces, the probability that one of them will turn up is $\frac{1}{6}$, and that it will not is $\frac{5}{6}$. So in this case $p = \frac{1}{6}$ and $q = \frac{5}{6}$. Thus our formula becomes $4,096 (\frac{1}{6} + \frac{5}{6})^{12}$.

The calculated and observed results are as follows:

Number of sixes.	Terms of $4096 (\frac{1}{6} + \frac{5}{6})^{12}$.	Observed frequencies.
8.....	0.58	1
7.....	4.96	7
6.....	27.18	24
5.....	116.43	115
4.....	363.84	380
3.....	808.53	796
2.....	1,211.44	1,181
1.....	1,102.56	1,145
0.....	459.52	447

The graphic representation of these results in Fig. 4979 is strikingly different from the curves in Fig. 4978. The latter were symmetrical or nearly so, while in the former both curves are distinctly asymmetrical, or *skew*. Thus we have two types of curves, or polygons, of frequency.

less than q , the curve will be skew, and the *skewness* will be greater the greater the difference between p and q .

When we abandon games of chance and seek to apply these formulae to the study of physical and biological phenomena, the problem is reversed. The measurements made and their "observed frequencies," whose sum = N , are our facts, and p , q , and n become unknown quantities. For example, a man's stature depends upon n factors, such as food, climate, ancestry, etc., and p and q are the probabilities for or against an increase or diminution of stature as the result of each of them. The problem of finding a mathematical curve that will fit the facts now becomes too complex to be solved by the binomial theorem and requires formulae derived by means of the calculus, but which may be used, nevertheless, with the aid of proper tables, by persons not familiar with that important branch of mathematics. When variations are distributed symmetrically on both sides of the type, the curve may be found by the "method of least squares," which was employed originally to determine the range of error in astronomical observations. But when the distribution is skew, Pearson (1902) has shown that it is