better to use the "method of moments," borrowed from the science of mechanics.

Collection and Tabulation of the Data.—The object of the statistical method for the study of variation is to represent in intelligible form the facts as they occur in nature. It is of the greatest importance, therefore, that the facts should not be obscured by any personal bias in the collection of the data. The data must be collected methodically at random.

Sometimes, however, the forces of nature themselves may tend to falsify the results and must be guarded against. For example, if we wish to determine the variability of shells due to the factors affecting growth, we must be sure that the proportions of the shell have not been affected by the erosive action of chemical substances or of parasitic algæ.

Finally, the data may not give true results if the observations are not sufficiently numerous. We must have a fair sample of the population. It has been seen in the case of the dice, that with 12 independent factors we cannot expect to get all the possible combinations in less than 4,096 observations. In practice it is found that usually 1,000 observations give sufficiently close results; 2,000 would be better, but sometimes 500 will do.

For statistical purposes we may divide normal variations into two classes: (1) *discrete* variations, in which the differences between individuals are sharply defined; and (2) *conjacent* (L. *conjacere,* to lie together) variations,

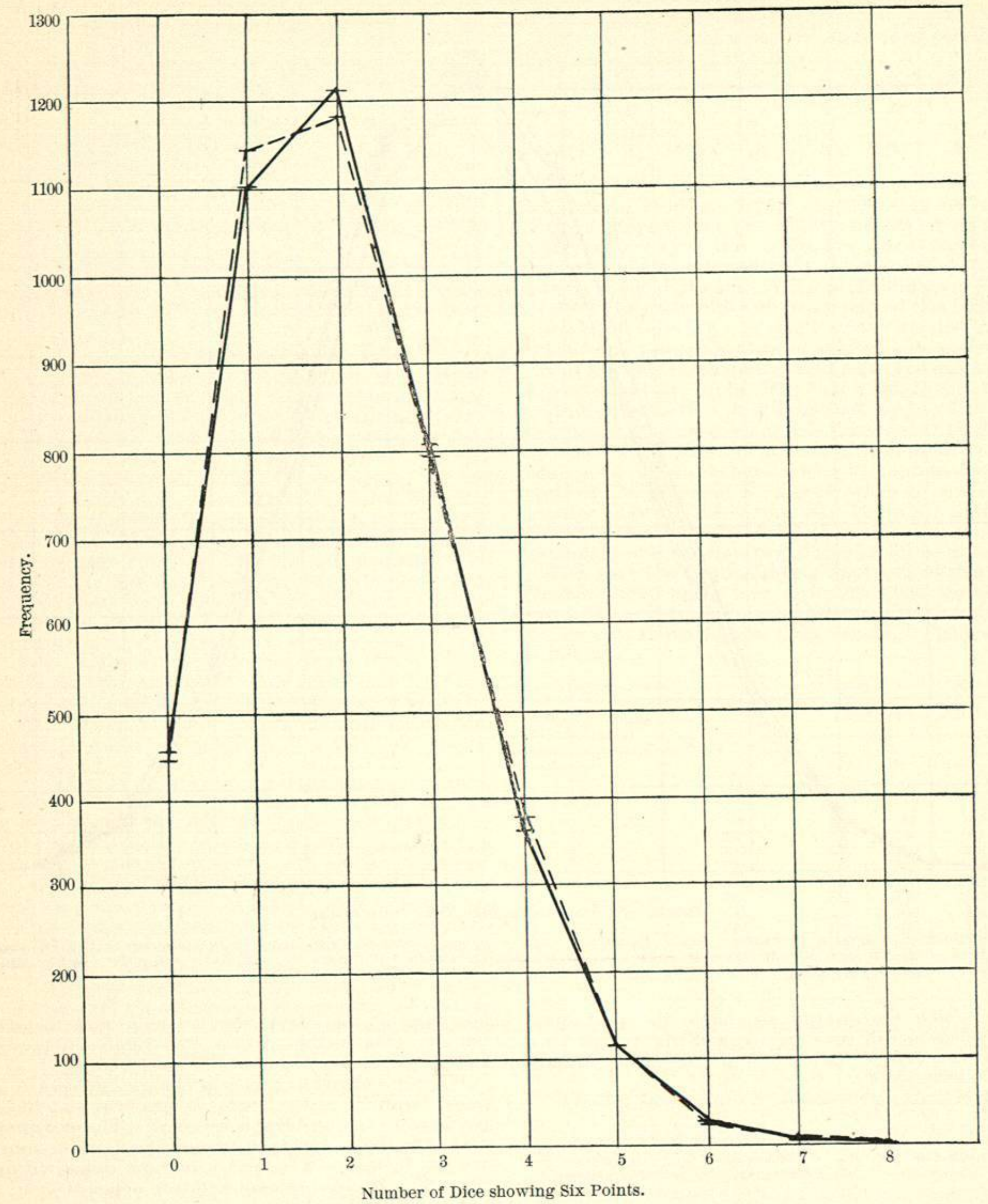

FIG. 4979.—Binominal Curve of Probability, terms of $4{,}096(\frac{1}{6}+\frac{5}{6})^{12}$, compared with the frequencies of the numbers of sixes thrown in a series of trials with twelve dice, the number of throws being $2^{12} = 4{,}096$. Binominal ——, dice — — —.

in which the smallness of the differences that may be observed is limited only by the precision of our instruments.

Discrete variations are usually observed by counting, as the number of ribs on a shell, the number of ray flowers on a daisy, or of veins in a leaf. Most meristic variations would be in this class.

Observations of conjacent variations are usually obtained by measuring or weighing—as stature (Fig. 4980), strength of pull, and the like. In this class would come the indexes obtained by taking the ratios of two or more measurements. It would include also qualities not quantitatively measurable, like the color of the eye or ability in mathematics. The method of assigning quantitative values to such qualities has been alluded to in a previous paragraph. Thus most of the substantive variations would fall within this class.

The measurements having been made and recorded, the next step is to arrange the data in a table so as to show the number of times each value has been observed. An example will make this clear. Davenport (1900) counted the number of grooves on the inside of 1,046 scallop shells (*Pecten irradians* Lamarck, right-hand valve). His results may be tabulated as follows:

VARIATION IN THE NUMBER OF GROOVES ON THE SHELLS OF PECTEN IRRADIANS.

Classes.......	1	2	3	4	5	6	7	8
Values.......	14	15	16	17	18	19	20	21
Frequencies..	2	15	108	515	308	90	7	1

That is, there were 2 shells with 14 grooves, 15 had 15 grooves, 108 had 16, and so on up to 21 grooves, which were found in only 1 shell. Each group of shells having a certain number of grooves forms a *class,* the number of grooves possessed by individuals of each class is its *value,* or magnitude, and the number of individuals in each class is its *frequency.* The classes may be numbered consecutively, and in some of the calculations to follow the arithmetic may be simplified by using these serial numbers in place of the values.

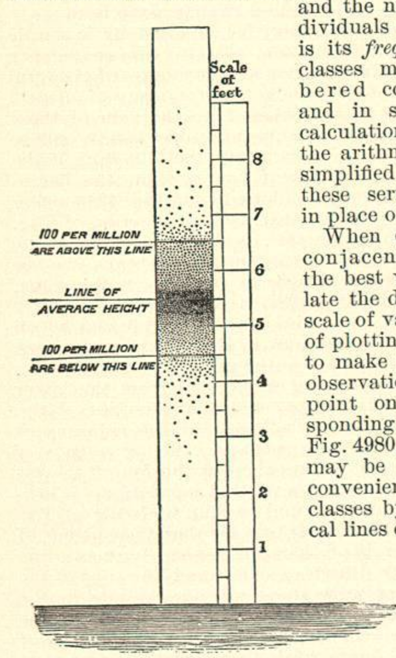

FIG. 4980.—An Example of Conjacent Variation. A supposed record of the statures of a large number of men. (From Galton.)

When dealing with conjacent variations the best way to tabulate the data is to lay off a scale of values upon a sheet of plotting paper, and then to make a mark for each observation opposite the point on the scale corresponding to its value, as in Fig. 4980. Then the marks may be divided into any convenient number of classes by a series of vertical lines drawn from points equal distances apart upon the scale. The number of marks between each pair of lines will be the frequency of the class. For example, De Vries gives the results of measuring the length of four hundred and forty-eight common red spotted beans (*Phaseolus vulgaris*), which may be tabulated in the following manner:

VARIATIONS IN LENGTHS OF BEANS OF PHASEOLUS VULGARIS.

Classes..........................	1	2	3	4	5	6	7	8	9
Mid-values, mm..............	8	9	10	11	12	13	14	15	16
Frequencies..................	1	2	23	108	167	106	33	7	1

In this case all specimens shorter than 8.5 mm. would go in class 1, those between 8.5 mm. and 9.5 mm. go in class 2, those between 9.5 mm. and 10.5 mm. in class 3, etc. These values, 8.5, 9.5, 10.5, . . . etc., mm. may be called the *class limits.*

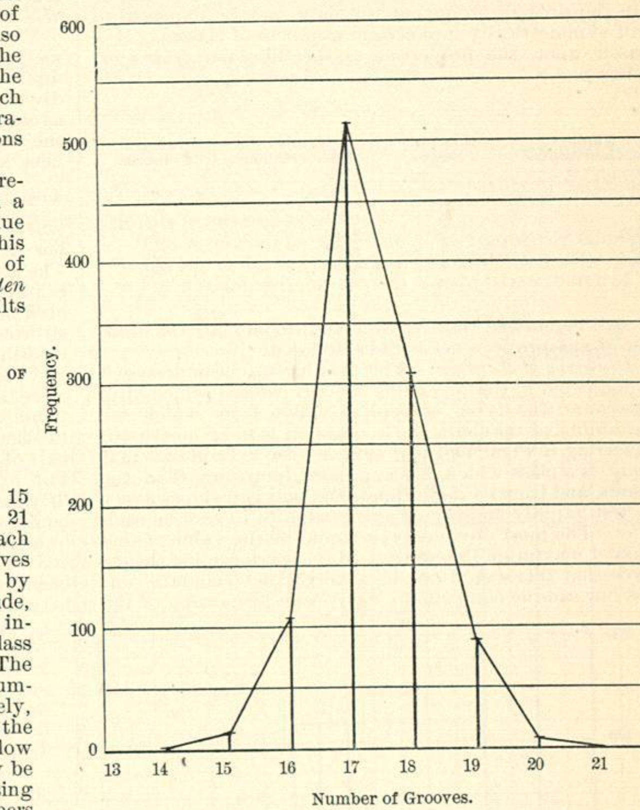

FIG. 4981.—Polygon of Frequency: Grooves on inside of scallop shells. (Data from Davenport.)

The results of these tabulations may be represented to the eye by polygons of frequency similar to those used to illustrate the results of throwing dice. Using plotting paper ruled in squares, we lay off a horizontal scale of values and a vertical scale of frequencies. Then the method of procedure differs according to whether we are dealing with discrete or with conjacent variations. In the case of discrete variations, the value of each class is taken as an abscissa from an imaginary zero point on the left and the corresponding ordinate is raised from zero at the base line to a point on the vertical scale corresponding to its frequency. Then connecting the tops of the ordinates by straight lines gives the polygon of frequency (Fig. 4981).

In the case of conjacent variations where there are many classes, the polygon may be constructed in a similar manner by taking the mid-values as abscissæ for the ordinates. But a more exact method, especially when there are few classes, is to construct the polygon of a series of rectangles, as in Fig. 4982, where the sides of the rectangles have for abscissæ the limiting values of the classes, and their heights are proportional to the corresponding frequencies.

A comparison of Figs. 4981 and 4982 with Figs. 4978 and 4979 show striking resemblances and suggests at once that these variations in shells and beans follow the same general laws of chance that are found in the throwing of dice. This inference is confirmed by the results of a large number of studies of organic variation.

In a case of discrete variation the division of the data into classes presents no difficulties, but with conjacent variations the division is purely arbitrary, and here the theory of probability comes in as an aid to one's judgment. The following table, extracted from one given by Quetelet (1846, p. 90) shows how many observations are required in order that the data may be expected to fall symmetrically into certain numbers of classes. It is based upon the properties of the binomial $N(p+q)^n$ when $p=q$.

Number of observations.	Number of classes.	Number of observations.	Number of classes.
4	3	256	9
8	4	512	10
16	5	1,024	11
32	6	2,048	12
64	7	4,096	13
128	8	8,192	14

It will be noticed that for each additional class the number of observations needs to be doubled.

Discovery of the Type.—The data having been arranged as indicated in the preceding section, we are prepared to determine the type, or typical value, from which the variability of the character in question is to be measured. Referring to Figs. 4981 and 4982, we see in each case that there is a class which has a greater frequency than the others, and from its midordinate the polygon slopes away at first rapidly and then more gradually to zero on each side. The most obvious type would be the value of this class of maximum frequency. But experience and theory show that this would not be a very exact standard, unless our sample comprised a very large proportion of the

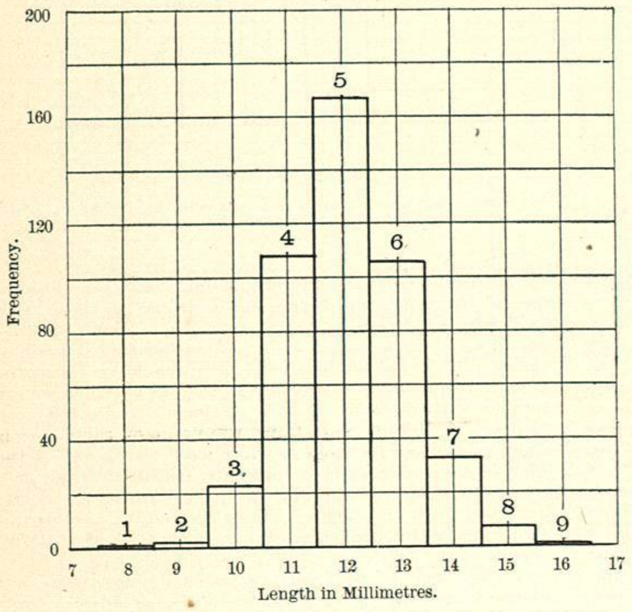

FIG. 4982.—Polygon of Frequency: Lengths of red-spotted beans. The serial numbers of the classes are placed above the corresponding rectangles. (Data from de Vries.)

total population. This is especially true of conjacent variations on account of the artificial character of the class boundaries. For example, in our case of the beans the class of maximum frequency has a medium length of 12 mm. But suppose the classes had been arranged so that the midordinates would fall at 8.5, 9.5, 10.5, etc., mm.; then the maximum frequency might have been found at either 11.5 or 12.5 mm. There is, however, a corresponding type called the *mode*, which is measured by the position of the ordinate of maximum frequency of the *theoretical curve* corresponding to the observed polygon of frequency (Pearson, 1896, p. 345). But the exact calculation of the mode, which is of considerable theoretical importance, involves a rather extended mathematical operation. A less exact, but very easy method will be described later.

The type usually employed is the average, or arithmetical mean, commonly called the *mean*. This may be discovered by adding all the observations together and dividing the sum by their number. But in order to avoid this long addition, the usual method is to multiply the value of each class by its frequency, add the products, and divide the sum by the number of observations. This operation is represented by the formula $M=\frac{\Sigma(Vz)}{N}$ where $M=$ mean, $V=$ values, $z=$ frequency, $N=$ number of observations, and Σ is the sign for summation. The same result may be obtained by taking the serial numbers of the classes for the values of V, and, having obtained the value of M in these units, it is easy to substitute the corresponding value in terms of the units of measurement employed in making the observations.

A third type, of considerable importance in certain cases, is called the *median*. This is the middle value, the value above which and below which fifty per cent. of the cases fall. It is easily found by simply counting off half of the observations arranged in order of magnitude. For example, if we have measured four hundred and fifty-one beans and arranged them in the order of their lengths, then the median will be the length of the two hundred and twenty-sixth bean, counting from either end of the series. If there are four hundred and fifty beans, an even number, the median will be intermediate between the length of the two hundred and twenty-fifth and the two hundred and twenty-sixth bean.

The same result may be obtained by a simple geometrical construction. Having for convenience reduced the frequencies to percentages of the total number of observations, the frequency of class 2 is added to that of class 1, to the sum of these class 3 is added, to this sum add class 4, and so on for all the classes, as in the following table. Thus we get a series of "sums from the beginning," which may be plotted as in Fig. 4983, using the method employed in the construction of Fig. 4982. Comparing the two figures, both made from De Vries's measurements of beans, we see that the first rectangle in the two is the same, the second in Fig. 4983 is composed of 1 and 2 in Fig. 4982, the third, of 1, 2, and 3, and so on until the ninth column in Fig. 4983 has the same area as the sum of all the nine rectangles in Fig. 4982. Now, drawing a diagonal from the lower left-hand to the upper right-hand corner of the upper rectangle in each one of the columns of Fig. 4983, we obtain what is called a *polygon of distribution*. It shows the distribution of the variations in our sample of the population. Turning the diagram around so that the scale of frequencies is horizontal, we see that the outline of this polygon is of the same general form as a line that might be drawn so as to touch the tops of the beans if they were stood up side by side in the order of their lengths. First there would be two or three very short beans, then a little row of beans 9.5–10.5 mm. high, then a much longer row, 10.5–11.5 mm. high, etc. Now, with the diagram in its first position, draw a horizontal line from the fifty-per-cent. point on the scale of frequencies to the middle rectangle, cutting the side *ac* at *d* and the diagonal, *cb* at *m*. Then draw a vertical line from *m*, and at its point of intersection with the scale of measurements read off the value of the median. To insure ac-

DATA USED IN CONSTRUCTION OF POLYGON OF DISTRIBUTION (FIG. 4983).

Class numbers.	Mid-values mm.	Frequency.	Per cent.	Sums from beginning per cent.
1	8	1	0.2	0.2
2	9	2	0.4	0.6
3	10	23	5.2	5.8
4	11	108	24.1	29.9
5	12	167	37.2	67.1
6	13	106	23.7	90.8
7	14	33	7.4	98.2
8	15	7	1.6	99.8
9	16	1	0.2	100.0
	....	448	100.0	

CALCULATION OF THE VARIABILITY IN LENGTH OF 448 BEANS, PHASEOLUS VULGARIS (DATA FROM DE VRIES).

Classes.	V.	z.	x.	xz.	x^2z.
1	8	1	−4	−4	16
2	9	2	−3	−6	18
3	10	23	−2	−46	92
4	11	108	−1	−108	108
5	12	167	0	0	0
6	13	106	1	106	106
7	14	33	2	66	132
8	15	7	3	21	63
9	16	1	4	4	16
				$\Sigma(xz)=361$	$\Sigma(x^2z)=551$

$N=448$

$M=12$ mm. $\text{A D}=\frac{\Sigma(xz)}{N}=0.8023$ mm.

$\sigma=\sqrt{\frac{\Sigma(x^2z)}{N}}=1.1065$ mm.

curacy, the diagram should be made on a large scale and be drawn with great care. A better way is to calculate the result numerically. Turning to the diagram, it is evident that the triangles *acb* and *dcm* are similar. Hence $dm : ab :: cd : ca$ and therefore $dm = ab \times \frac{cd}{ca}$. Let N be the total number of observations. Then we have the rule: Multiply the range of the middle class (ab) by $\frac{1}{2}N$ minus the sum of the frequencies of all the lower classes ($=cd$), divide by the frequency of the class (ca) and add the magnitude of the lower limit of its range. The result will be the median magnitude. In our example of the beans, $cd=50-29.9=20.1$ per cent.; $ab=1$ mm.; and $ca=37.2$ per cent. Hence $dm=1\times\frac{20.1}{37.2}=0.54$ mm. The lower limit of class 5 is at 11.5 mm. Therefore the median is $11.5+0.54=12.04$ mm. This method assumes, of course, that the values in the middle class are evenly distributed.

In the same manner may be determined the value of the measurement corresponding to any other point on the scale of frequencies, as at ten per cent., twenty per cent., etc. These values are called *percentile grades* (Galton, 1889), and they have been used extensively for the study of variations of anthrometric measurements; especially by teachers of gymnastics, who use these grades as a means of determining the standing of their pupils before and after training.

We may derive, then, from each collection of data three values, the mean, the mode, and the median, each one of which may serve as a type from which to measure variability. When the deviations from the mean are distributed symmetrically on both sides of it, all three of these values are the same. But they differ when the distribution is skew, as in throwing sixes with dice and in most variations found in nature. In such cases the median always lies between the mean and the mode, and the deviation of the median from the mean is one-third the deviation of the mode. This rule provides an easy way of calculating the *mode*, which, according to Pearson (1902, p. 261), is quite good enough for practical purposes in most cases, and avoids the tedious mathematics of the more exact method. (Compare *a,c,* and *b* in curves of Type III, Fig. 4986.)

The Measure of Variability.—Taking the mean as the type of any given character or organ, the problem is now to find a quantitative expression for the variability of that character in the group of organisms under observation. The first step is to calculate the deviation (x) of each class from the mean (M) by subtracting the values (V) of the classes from the mean. In the case of the beans, the mean is 12.05 mm. Omitting the small fraction for the sake of simplicity, the deviations of the several classes are given in the fourth column of the following table.

Now if the deviation of each class be multiplied by its frequency (z), the results added without regard to signs, and the sum divided by the number of observations, we obtain the *average deviation*. This is represented by the formula $\text{A D}=\frac{\Sigma(xz)}{N}$ and is one measure of variability. But it is rarely used.

The measure of variability most frequently employed at present is what is called the *standard deviation*. This is the square root of the average of the squares of the

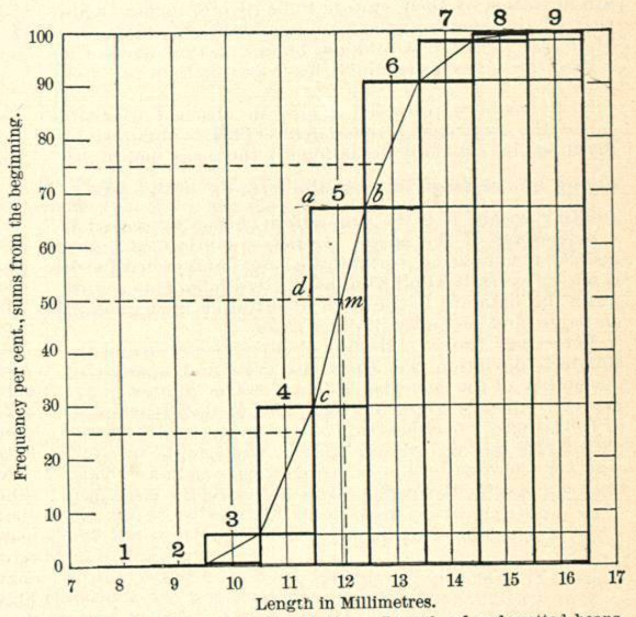

FIG. 4983.—Polygon of Distribution: Length of red-spotted beans. Serial numbers as in Fig. 4982. (Data from De Vries.)

deviations, and is represented by the symbol σ, as in the formula: $\sigma=\sqrt{\frac{\Sigma(x^2z)}{N}}$

The definition and formula indicate the process by which this quantity is found. Take the square of the deviation of each class, multiply it by its frequency, add the results, divide the sum by N, and extract the square root; all of which is performed easily by the aid of a table of logarithms. An example is given in the sixth column of the preceding table.

A third measure of variability, formerly much used, is the *probable deviation*. It might be called the median deviation, because the deviations that are greater and those that are less than it are equal in number. Galton (1889) has pointed out a method by which it may be obtained graphically from a polygon of distribution, as in Fig. 4983. Using the method employed in finding the median, find the values of the percentile grades at twenty-five and seventy-five per cent., and take half their difference. This will be the probable deviation. It is found as easily and perhaps more exactly by multiplying the standard deviation by 0.6745. $PE = 0.6745\,\sigma$.

The probable, or median deviation, corresponds to the so-called "probable error" of the mathematicians, and the standard deviations to the "error of mean squares." If we imagine a polygon of frequency to be a horizontal bar loaded with weights proportional to the ordinates and swinging about the mean, which will be the centre of gravity, then the standard deviation corresponds to what physicists call the "radius of gyration," or "swing radius."

It is found by the theory of probability that when $\sigma = 1$, $AD = 0.7979$, and $PE = 0.6745$. Thus we see that of the three measures of variability, the standard deviation is the largest. It has also the advantage of being the one most frequently used in mathematical investigations.

It has in common with the other two, however, the disadvantage of being expressed in units of measurement—inches, pounds, etc.; so we cannot compare, for instance, the variability of weight with that of stature while using this measure. A similar difficulty arises with organs of different size. In a thousand men from New York City, for example, we should find the standard deviation of their stature to be several inches, while that of the length of the nose would be only a fraction of an inch. But it would not be fair on that account to regard the noses as so much less variable than the statures.

So it has been proposed to use an abstract quantity called the *coefficient of variability*. This is obtained by dividing the standard deviation by the mean and multiplying by one hundred, as in the formula: $CV = 100\,\frac{\sigma}{M}$.

In other words, it is the standard deviation expressed as a percentage of the mean. It is frequently useful as a means of comparing variabilities; but, unfortunately, it is not universally applicable, as it gives fallacious results when the data are in the form of ratios or in degrees of an angle, and probably in other cases.

Theoretical Curves.—Having obtained the mean and the standard deviation, we know the type and amount of variability of the material at hand. The plotted polygon of frequency gives a visual idea of the distribution of frequencies on either side of the mean, and with the aid of the median we are able to determine roughly whether the distribution is symmetrical or skew. Taking these results of observation as premises, the next step

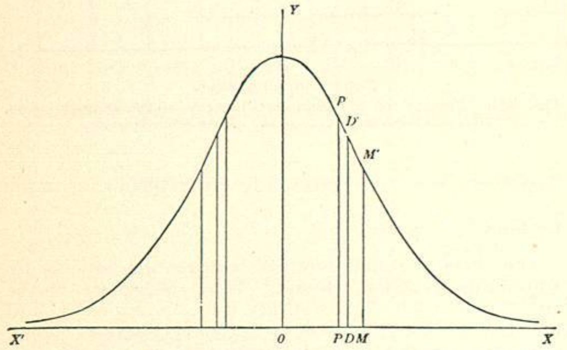

FIG. 4984.—The Normal Curve of Frequency. *YO*, ordinate of the mean; *OP*, "probable" deviation; *OD*, average deviation; *OM*, standard deviation. (From Bartlett.)

is to construct a theory as to the most probable distribution of the variations in the whole population, of which our material is a sample.

In a previous section, we have seen how close the variations in number of spots in throws of dice correspond to the theoretical probability as expressed by the formula $N(p+q)^n$ when this binomial is expanded, and comparison of the polygons (Figs. 4978, 4979, 4981, and 4982) seem to indicate that the same method is applicable with organic variations. It may be used with discrete variations where there are few classes; but in nature N,

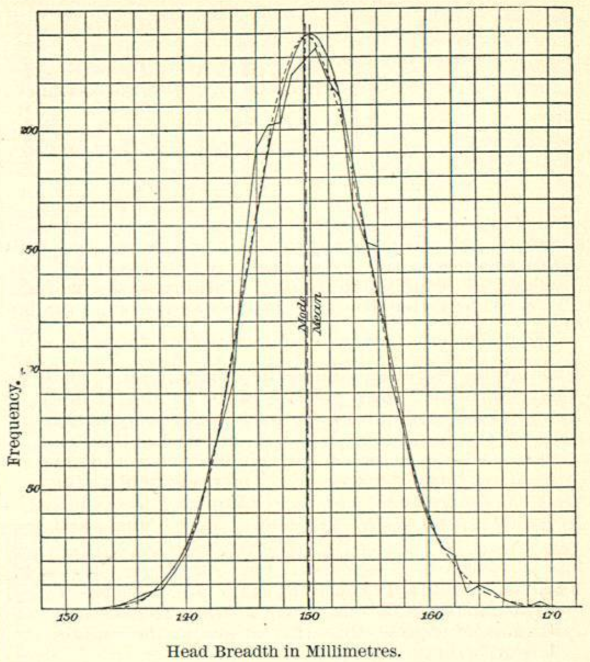

FIG. 4985.—Frequency of the Variations in Breadth of Head of 3,000 criminals. Irregular line, polygon of frequency. Normal curve —; skew curve of Type IV - - - - -. (From Macdonell.)

p, q, and n are all unknown quantities, and with conjacent variations the division into classes is purely arbitrary. So it is better to employ a method that will give a continuous theoretical curve, such as would be obtained if, in the formula $(p+q)^n$, n were infinitely large.

Now it was shown long ago by Quetelet, Galton, and others that variations in human stature and in other characters follow closely the law expressed by the "normal curve of error" of the mathematicians, or what we should call the *normal curve of frequency* (although other curves more often fit the polygons of organic variation). This is a flowing curve (Fig. 4984), symmetrical about the mean, and theoretically without limit at the ends. The standard deviation marks the point of inflection on the sides. The formula for the curve is $y = y_0\,\frac{1}{e^{x^2/2\sigma^2}}$, in which y is any ordinate, x is the corresponding abscissa, or deviation, measured from the ordinate of the mean, or "centroid verticle," y_0 is the height of this ordinate, σ is the standard deviation, and e is the basis of the Napierian system of logarithms, in which $e = 1 + \frac{1}{1} + \frac{1}{1\times 2} + \frac{1}{1\times 2\times 3} + \ldots$ etc. $= 2.7182818$. The meaning of the formula is, of course, that for any value of x the corresponding value of y will satisfy this equation.

If in our material we find the distribution of variations to be very nearly symmetrical, as in the beans, where the difference between mean and median is only 0.01 mm., we may assume that the normal curve will give the correct theoretical values, and may proceed at once to test this by fitting a normal curve to the observed polygon in the following manner: The mean (M) and the standard deviation (σ) have been found, and in doing this we have found also x_1, x_2, x_3, etc., which are the deviations from

the mean of classes 1, 2, 3, etc. Next find the height of the maximum ordinate. This operation is indicated by the formula $y_0 = \frac{N}{\sigma\sqrt{2\pi}}$, in which N = total number of observations and $\pi = 3.14159$. Now turning to Davenport (1899, p. 54) we find a table of the ordinates of the normal curve in which the "argument" is $\frac{x}{\sigma}$ and the corresponding tabular entry is $\frac{y}{y_0}$. So we must find $\frac{x}{\sigma}$, that is, divide the deviation of each class by the standard deviation. Then in the table find the tabular entry corresponding to each value of $\frac{x}{\sigma}$, interpolating, if necessary, and multiply the value thus found by y_0. The result will be the height of the ordinate of the theoretical curve upon the scale of frequency used in constructing the polygon. When these ordinates are plotted and their tops connected by a flowing curve, we have the theoretical curve (as in Fig. 4985), that is, a graphic representation of the distribution of the frequencies of the variations in nature, based upon the sample observed and the assumption that the variations follow the law of probability with $p = q$, and $n = \infty$.

This is the simplest of all the methods for fitting a theoretical curve to an observed polygon. It is especially applicable for discrete variations when the distribution of frequencies is symmetrical. It may be used also, but with less accuracy, for similar polygons of conjacent variations by taking the mid-value of each class as the position of its ordinate.

For greater accuracy with conjacent variation and in all cases of skew variation it is necessary to employ a more complicated procedure, the foundations of which have been laid by Pearson (1896, 1901, 1902). This is the *method of moments*. We return to the analogy between the polygon of frequency and a light rod loaded with balls suspended from it at regular intervals. The tendency of a ball to bend the rod will depend upon the weight of the ball and its distance from the point of support. The product of this distance multiplied by the weight is the *moment*, or first moment, of the ball about the point of support. The square of the distance multiplied by the weight gives the second moment, the cube, the third moment, and so on. Applying this principle to the polygon of frequency, we proceed at once to calculate the first, second, third, and fourth "rough moments" of the polygon as a whole about any convenient ordinate. From these rough moments it is easy to calculate the mean and the moments about the mean. These give us the standard deviation, and by their combination we are able to calculate the skewness and determine the general form of the curve. The actual calculation of the ordinates necessary to plot the curve is quite a different matter.

In practice it is best to use the serial numbers of the classes as their values, then taking the value of a class near the middle as V_m we find the deviations $-x_1$, $-x_2$, $-x_3$, and $+x_1$, $+x_2$, $+x_3$, etc. Then multiplying by their frequencies (z) and adding, being careful to regard the signs, we get a sum which divided by the total number of observations gives $\nu_1 = \frac{\Sigma(z\,x)}{N}$, which is the first rough moment of the curve about V_m with the frequencies of the classes grouped about their midordinates. In the same way we find the second, third, and fourth moments:

$$\nu_2 = \frac{\Sigma(z\,x^2)}{N},\quad \nu_3 = \frac{\Sigma(z\,x^3)}{N},\ \text{and}\ \nu_4 = \frac{\Sigma(z\,x^4)}{N}.$$

Now in the theoretical curve the classes are not represented by rectangles but by trapezia, whose centres of gravity are not on the midordinates (Goodman, 1899, p. 60). This is allowed for and we find the corrected moments about V_m by use of the formulæ (Pearson, 1902*b*, pp. 287–289):

$$\mu_1' = \nu_1$$
$$\mu_2' = \nu_2 - \frac{h^2}{12}$$
$$\mu_3' = \nu_3 - \frac{h^2}{4}\nu_1$$
$$\mu_4' = \nu_4 - \frac{h^2}{2}\nu_2 + \frac{7}{240}h^4.$$

In these formulæ h is the distances apart of the ordinates and, if we use the serial numbers, $h = 1$.

Now we can find the mean from V_m by adding or subtracting μ'_1, after substituting the true values for those derived from the serial numbers, *i.e.*, $M = V_m - \mu'_1$.

Now the moments of the curve about M will be found as follows (*loc. cit.*, p. 283):

$$\mu_1 = 0.$$
$$\mu_2 = \mu_2' - \mu_1'^2$$
$$\mu_3 = \mu_3' - 3\,\mu_1'\,\mu_2' + 2\,\mu_1'^3$$
$$\mu_4 = \mu_4' - 4\,\mu_1'\,\mu_3' + 6\,\mu_1'^2\,\mu_2' - 3\,\mu_1'^4.$$

The square root of the second moment is the standard deviation, $\sigma = \sqrt{\mu_2}$.

Now that we know the mean, or *centroid*, the standard deviation, or swing radius, and the moments of the curve about the centroid, we are prepared to determine the form of the curve.

If $\mu_3 = 0$ and $\mu_4 = 3\,\mu_2^2$, the curve is normal. If these equations are not true, the curve is skew and we may proceed to discover whether it will have a range limited at both ends, limited at one end only, or an unlimited range

TYPES OF FREQUENCY CURVES.

Type.	Description.	Formula.	Criterion κ_1.	Criterion κ_2.	References. Pearson.
Normal..	Symmetrical. Range unlimited. (Figs. 4984 and 4985.)	$y = y_0\,\frac{1}{e^{x^2/2\sigma^2}}$	$\kappa_1 = 0$; $\beta_1 = 0$, $\beta_2 = 3$	$\kappa_2 = 0$	
I........	Skew. Range limited both ends.	$y = y_0\left(1+\frac{x}{a_1}\right)^{m_1}\left(1-\frac{x}{a_2}\right)^{m_2}$	$\kappa_1 < 0$; $\beta_1 > 0$	$\kappa_2 < 0$	'96, p. 367. '01 (b), p. 444.
II.......	Symmetrical. Range limited both ends.	$y = y_0\left(1-\frac{x^2}{a^2}\right)^m$	$\kappa_1 < 0$; $\beta_1 = 0$, β_2 not $= 3$	$\kappa_2 = 0$	'96, p. 372.
III......	Skew. Range limited at one end. Transition form. (Fig. 4986.)	$y = y_0\left(1+\frac{x}{a}\right)^{\gamma a} e^{-\gamma x}$	$\kappa_1 = 0$; $\beta_1 > 0$, $\beta_2 > 3$	$\kappa_2 = \infty$	'93, p. 373.
IV.......	Skew. Range unlimited. (Fig. 4985.)	$y = \frac{y_0}{\left\{1+\left(\frac{x}{a}\right)^2\right\}^m} e^{-\nu\tan^{-1}\left(\frac{x}{a}\right)}$	$\kappa_1 > 0$; $\beta_1 > 0$, $\beta_2 > 3$	$\kappa_2 > 0$ & < 1	'96, p. 376.
V.......	Skew. Range limited at one end. Transition form.	$y = y_0\,x^{-p}e^{-\gamma/x}$	....	$\kappa_2 = 1$	'01 (b), p. 446.
VI......	Skew. Range limited at one end.	$y = y_0\,\frac{(x-a)^{m_1}}{x^{m_2}}$	....	$\kappa_2 > 1$ & $< \infty$	'01 (b), p. 448.