BROOKS

THE NORMAL ELEMENTARY ALGEBRA

THE

NORMAL ELEMENTARY ALGEBRA:

CONTAINING THE

FIRST PRINCIPLES OF THE SCIENCE,

DEVELOPED WITH CONCISENESS AND SIMPLICITY,

FOR

COMMON SCHOOLS, ACADEMIES, SEMINARIES AND NORMAL SCHOOLS.

REVISED EDITION.

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"Mathematical studies cultivate clearness of thought, acuteness of analysis and accuracy of expression."

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PREFACE.

EDUCATION aims at mental culture and practical skill; and for the attainment of both of these objects the mathematical sciences have in all ages been highly valued. They train the mind to logical methods of thought, give vigor and intensity to its operations, and lead to the important habit of resting only in certainty of results; while as instruments of investigation they stand pre-eminent.

Among the three fundamental branches of mathematics, Algebra occupies a prominent position in view of both of these objects. As a method of calculation it is the most powerful of them all, and for giving mental acuteness and the habit of analytic thought it is unequaled. With the advance of education this science is growing in popularity, and is being introduced into our best public schools, as well as academies and seminaries. Many teachers are beginning to see that a knowledge of elementary algebra is worth more than a knowledge of higher arithmetic, and are omitting the arithmetic, when necessary, for the algebra. This has increased the demand for good text-books upon the subject; and to assist in meeting this demand the present work has been prepared. Some of its general and special features will be briefly stated.

GENERAL FEATURES.—This work is not a mere collection of problems and solutions, but the evolution of a carefully-matured plan, the embodiment of an ideal formed by long and thoughtful experience in the schoolroom. Attention is called to its extent, its matter and its method.

Extent.—The work embraces just about as many topics as it is thought the ordinary pupil in elementary algebra should be required to study. These topics have been presented, not superficially, but with comparative thoroughness, so that the knowledge given may be of actual use in calculation, and afford a basis for the study of a higher work if desired. While not presenting quite as much as some teachers might prefer, the author has been careful not to make the work too elementary. Superficial scholarship is one of the growing evils of our country, and teachers and text-books are responsible for it. It should never be forgotten that *it is better to know much of a few things than to know a little of many things.* While endeavoring to avoid superficiality, the author has been careful to so simplify the subject as to render it suitable to those beginning the study. The object has been to hit the golden mean, and thus adapt it to the wants of the majority of pupils and teachers.

Matter.—In the development of the various topics, care has been taken to combine in due proportion theory and practice. The French works on algebra are very complete in the discussion of principles, but are deficient in matter for the attainment of skill in their application. The English works abound with practical examples, but are usually less complete in their theoretical discussions. I have endeavored to combine both of these features, by giving an ample collection of problems, as well as a thorough discussion of principles. The aim has been to make the work both philosophical and practical.

The problems were prepared with especial reference to the principles and methods which they are designed to illustrate. They are often so related that one problem prepares the way for the succeeding one, thus making a problem which alone might be quite difficult, comparatively easy. The miscellaneous examples at the close of the book embrace a choice selection from American and English works, especially from the excellent collection given by Todhunter in his Elementary Algebra.

Method of Treatment.—In the treatment of the various topics I have aimed to simplify the subject without impairing its logical completeness and thoroughness. In this respect the skill of an author is particularly shown, and in this consist the principal merits of a text-book. The difficulty of a subject is not so much in itself as in the manner in which it is presented; special pains have therefore been taken to pass so gradually from the simple to the complex as to make easy what otherwise might be complicated and difficult.

I have also been careful to give conciseness, clearness and simplicity to its methods of explanation. There is a simple and direct method of stating a solution or demonstration that is much more readily comprehended by the learner than a talking about it in a popular sort of style. The scientific method is usually the simplest method. Text-books on algebra have been especially defective in this respect. The methods of mental arithmetic have created a revolution in the forms of explanation in the science of numbers, giving beauty and simplicity to that which was before awkward and complicated. Geometry, coming to us as the product of the Greek mind, is characterized by the simplicity and elegance of its demonstrations; while algebra, mainly the product of modern thought, has been less clear, logical and finished in its methods of development. I have endeavored to make an improvement in this respect, by combining the simple and natural analytical methods of mental arithmetic with the elegance of form and logical exactness of the geometrical methods.

Special Features.—The principal merits of the work are supposed to consist in its methods of treatment, its solutions, discussions and explanations—in brief, in the general spirit that pervades it, giving simplicity and unity to the work as a whole. There are, however, several special features to which the attention of teachers is respectfully invited:

1. The *definition of algebra* seems simpler and more complete than any which the author has met.

2. The *classification of algebraic symbols* under three distinct heads is different from anything heretofore presented.

3. The *method of explaining* addition and subtraction by means of an *auxiliary* quantity is a new feature which merits notice.

4. The new topic, called *Composition*, as a synthetic process correlative to the analytic process of *Factoring*, will no doubt arrest attention.

5. The variety of the cases in factoring, the demonstration of the divisibility of $a^n - b^n$ by $a-b$, and the explanation of the greatest common divisor and least common multiple, solicit notice.

6. The treatment of involution and evolution possesses some points of novelty, and especial attention is called to the second method of cube root.

7. Particular attention is invited to the discussion of the Courier Problem and to the development of the properties of the incomplete quadratic equation.

8. The variety and appropriateness of the problems, the frequent generalization from a particular to a general problem, and the variation of old problems, are designed to add interest to the study and give discipline to the student.

9. Unusual care has been given to the typography of the work, in order to make it attractive and interesting to the pupil. The introduction of the small symbols for addition, subtraction, multiplication and division, as used in the best English works, is regarded as a great improvement.

Encouraged by the approval of many of the best teachers of the country who have used my former works, and yielding to their wishes as well as my own inclination, I have found time and strength, amid the cares and duties of a large institution, to prepare the present work; and I now send it forth, trusting that it may afford discipline and knowledge to many youthful minds of the present generation, and convey a kindly remembrance of my own labors to teachers and pupils of the future.

State Normal School, *Millersville, Pa., May*, 1871. EDWARD BROOKS.

SUGGESTIONS TO TEACHERS.

THE following suggestions are respectfully made for the benefit of young or inexperienced teachers:

1. It will be well to make frequent use of the inductive method of teaching, as suggested in the *Introduction*, leading pupils from the ideas and methods of arithmetic to those of algebra.

2. Drill thoroughly upon the fundamental operations, especially upon the use of the minus sign, the use of exponents in multiplication and division, and the methods of factoring. Slowness here is speed afterward.

3. Where there are two solutions, the teacher may select the one he prefers, and drill the pupils thoroughly upon it before they attempt the other. The first of two given solutions is usually regarded as the simplest, though not always the best.

4. Frequently require pupils to change a particular problem into a general one, as on page 107, and also to make special problems out of general ones. Have pupils to make problems by changing the conditions of given problems, or by using a required condition and requiring some given condition, as in problems on page 111. Pupils should be encouraged to form new problems, and to originate new methods of explanation and solution. We should always aim to make *thinkers* of our pupils rather than *mathematical machines*.

5. *A Shorter Course.*—While this work is the author's ideal of the extent of an elementary algebra, yet it may be used by teachers who desire a shorter course. For such the following omissions may be made without impairing the unity of the subject:

Omit the second methods of Greatest Common Divisor and Least Common Multiple, the Supplement to Simple Equations, Imaginary Quantities, Principles of Quadratics, Quadratics of Two Unknown Quantities, and the Miscellaneous Examples at the close of the work. A still shorter course may be attained by the further omission of the latter half of the examples under each subject.

6. In conclusion, the author suggests to teachers of public schools to give their pupils a course in elementary algebra before completing higher arithmetic. His own experience is, that pupils cannot thoroughly understand arithmetic until they have studied algebra.

THE

NORMAL ELEMENTARY ALGEBRA.

INTRODUCTION.

THE object of the exercises in the *Introduction* is to lead pupils from the ideas and operations of Arithmetic to those of Algebra.

LESSON I.

1. Henry's number of apples, increased by three times his number, equals 24; how many apples has he?

SOLUTION. By arithmetic this problem is solved in the following manner:

Henry's number, plus *three times his number*, equals 24;
hence, 4 times *Henry's number* equals 24,
and once *Henry's number* equals one-fourth of 24, or 6.

ABBREVIATED SOLUTION. If we represent the expression *Henry's number* by some character, as the letter x, the solution will be much shorter; thus:

x plus 3 times x equals 24;
hence, 4 times x equals 24,
and once x equals 6.

ALGEBRAIC SOLUTION. If we use $3x$ and $4x$ to denote "3 *times* x" and "4 *times* x," the sign = for the word "equals," and the sign + for the word "plus," the solution will be purely algebraic; thus:

$$x+3x=24;$$
$$4x=24,$$
$$x=6. \quad \textit{Ans.}$$

SYMBOLS.—It will be seen that this last solution is the same as the first, except that we use *characters* instead of *words*.

These characters are called *symbols*. The method of solving problems by means of symbols is called *Algebra*.

Addition is denoted by the symbol +, read *plus*; *subtraction* is denoted by the symbol −, read *minus*.

The expressions $2x$, $3x$, $4x$, etc., mean 2 *times* x, 3 *times* x 4 *times* x, etc. *One-half* of x, *one-third* of x, *two-thirds* of x, etc., are denoted by $\frac{1}{2}x$, $\frac{1}{3}x$, $\frac{2}{3}x$, etc., or $\frac{x}{2}$, $\frac{x}{3}$, $\frac{2x}{3}$, etc.

The symbol = denotes *equality*, and is read *equals*, or *is equal to*. The expression $x+3x=24$ is called an *equation*.

The pupil will now express the following in algebraic symbols:

2. Three times John's number of apples equals 27.
3. One-half of Mary's number of peaches equals 12.
4. A's number of books, plus three times his number, equals 16.
5. B's number of dollars, minus half his number, equals 18.
6. Two-thirds of C's fortune, minus one-half of his fortune, equals $50.

NOTE.—The pupil may be led to see, from the solution given, which is the *unknown* quantity and which the *known* quantity, and how each is represented.

Also, that the symbols of Algebra are of three classes: symbols of *quantity*, symbols of *operation*, and symbols of *relation*.

LESSON II.

1. John has a certain number of peaches, and James has three times as many, and they both have 40; how many has each?

SOLUTION. Let x equal John's number; then, since James has 3 times as many as John, $3x$ will equal James' number, and since they together have 40, $x+3x$ will equal 40, or, adding, $4x$ will equal 40. If $4x$ equals 40, x will equal one-fourth of 40, which is 10, John's number; and $3x$ will equal 3 times 10, or 30, James' number.

OPERATION.

Let x = John's number.
$3x$ = James' number.
$x+3x=40$
$4x=40$
$x=10$, John's number.
$3x=30$, James' number

2. Mary's age is twice Sarah's, and the sum of their ages is 36 years; what is the age of each?

Ans. Sarah, 12 years; Mary, 24 years.

3. A man bought a coat and a vest for $40, and the coat cost 4 times as much as the vest; required the cost of each.

Ans. Coat, $32; vest, $8.

4. In a mixture of 360 bushels of grain there is 5 times as much wheat as corn; how many bushels of each?

Ans. Wheat, 300 bushels; corn, 60 bushels.

5. Divide the number 144 into two parts, such that the larger part will be 5 times the smaller part. *Ans.* 120; 24.

6. The sum of two numbers is 120, and the larger number is 4 times the smaller number; required the two numbers.

Ans. 24; 96.

7. A man bought a span of horses and a carriage for $1000, paying three times as much for the horses as the carriage; required the cost of each. *Ans.* Horses, $750; carriage, $250.

8. The salary of a clerk for a year was $1500, and he spent five times as much of it as he saved; how much did he save?

Ans. $250.

LESSON III.

1. The difference of two numbers is 24, and the larger equals 4 times the smaller; required the numbers.

SOLUTION. Let x equal the smaller number; then $4x$ will equal the larger number. And since the difference of the two numbers is 24, $4x-x$ will equal 24, or, subtracting, $3x$ will equal 24. If $3x$ equals 24, x will equal one-third of 24, which is 8, the smaller number; and $4x$ will equal 4 times 8, or 32, the larger number.

OPERATION.

Let x = the smaller.
$4x$ = the larger.
$4x-x=24$
$3x=24$
$x=8$, the smaller
$4x=32$, the larger.

2. The difference of two numbers is 28; and 5 times the smaller equals the greater; what are the numbers?

Ans. Smaller, 7; greater, 35.

3. A has 28 cents more than B, and 3 times B's number equals A's; how many has each? *Ans.* A, 42; B, 14.

4. Mary gathered 21 flowers more than her sister; how many did each gather if Mary gathered 4 times as many as her sister? *Ans.* Mary, 28; sister, 7.

5. Seven times a number, diminished by 3 times the number, equals 48; what is the number? *Ans.* 12.

6. Marie has 40 cherries more than Jane, and 5 times Jane's number equals Marie's number; how many has each? *Ans.* Marie, 50; Jane, 10.

7. A bought a house and lot, paying 5 times as much for the house as the lot; what did he pay for each if the house cost \$2560 more than the lot? *Ans.* House, \$3200; lot, \$640.

8. A and B enter into a copartnership, in which A's interest is 6 times as great as B's: A's gain was \$650 more than B's gain; what was the gain of each? *Ans.* A's, \$780; B's, \$130.

LESSON IV.

1. Julia and Anna had 24 oranges, and Julia had one-half as many as Anna; how many had each?

SOLUTION. Let x equal Anna's number; then will $\frac{x}{2}$ equal Julia's number; and since they together have 24, $x+\frac{x}{2}$ equals 24. Adding, x plus $\frac{1}{2}$ of x, or $\frac{3}{2}$ of x, equals 24. If $\frac{3}{2}$ of x equals 24, $\frac{1}{2}$ of x equals $\frac{1}{3}$ of 24, which is 8, Julia's number; and $\frac{2}{2}$ of x, or x, equals 2 times 8, or 16, Anna's number.

OPERATION.

Let x = Anna's number.

$\frac{x}{2}$ = Julia's number.

$$x+\frac{x}{2}=24$$

$$\frac{3x}{2}=24$$

$\frac{x}{2}=8$, Julia's number.

$x=16$, Anna's number.

2. A's money, increased by one-half of his money, equals \$60; what is his money? *Ans.* \$40.

3. What number is that to which if its one-third be added, the sum will be 36? *Ans.* 27.

4. What number is that to which if its two-thirds be added, the sum will be 45? *Ans.* 27.

5. What number is that which being diminished by its three-eighths, the remainder will be 30? *Ans.* 48.

6. Mary's age, diminished by its three-fifths, equals 6 years; how old is Mary? *Ans.* 15 years.

7. If one-half of my age be increased by one-third of my age, the sum will be 40 years; what is my age? *Ans.* 48 years.

8. Four times the distance from Philadelphia to Lancaster, diminished by $2\frac{1}{2}$ times the distance, equals 102 miles; required the distance. *Ans.* 68 miles.

9. Benton lost four-fifths of his money, and then found three-fourths as much as he lost, and then had \$120; how much money had he at first? *Ans.* \$150.

10. Bessie gave three-fourths of her money to the poor, and then found two-thirds as much as she gave away, and then had \$30; how much had she at first? *Ans.* \$40.

LESSON V.

1. A man bought a hat, vest and coat for \$35; the vest cost twice as much as the hat, and the coat cost four times as much as the hat; required the cost of each.

SOLUTION. Let x equal the cost of the hat; then will $2x$ equal the cost of the vest, and $4x$ equal the cost of the coat, and their sum, $x+2x+4x$, will equal the cost of all, or \$35. Adding, we have $7x$ equals \$35. If $7x$ equals \$35, x equals one-seventh of \$35, or \$5, the cost of the hat; $2x$ equals 2 times \$5, or \$10, the cost of the vest; and $4x$ equals 4 times \$5, or \$20, the cost of the coat.

OPERATION.

Let x = cost of the hat.

$2x$ = cost of the vest.

$4x$ = cost of the coat.

$$x+2x+4x=35$$

$$7x=35$$

$x=5$, hat;

$2x=10$, vest;

$4x=20$, coat.

2. Divide the number 105 into three such parts that the first shall be twice the second and the second twice the third. *Ans.* 60; 30; 15.

3. Three men, A, B and C, earned \$216; A earned twice as much as B, and C earned as much as both A and B; how much did each earn? *Ans.* A, \$72; B, \$36; C, \$108.

4. The sum of three numbers is 63; the second is one-half of the first, and the third one-fourth of the first; what are the numbers? *Ans.* 1st, 36; 2d, 18; 3d, 9.

5. A man, with his wife and son, earned \$22 in a week, the man earned twice as much as his wife, and three times as much as his son; what did each earn?

Ans. Man, \$12; wife, \$6; son, \$4.

6. A man bought a horse, a cow and a sheep for \$315; the cow cost 5 times as much as the sheep, and the horse cost three times as much as the cow; required the cost of each.

Ans. Horse, \$225; cow, \$75; sheep, \$15.

7. A tax of \$450 is assessed upon three persons according to the relative value of their property; A is worth two-thirds as much as B, and B is worth three-fourths as much as C; what is each man's tax? *Ans.* A's, \$100; B's, \$150; C's, \$200.

LESSON VI.

1. A being asked how much money he had, replied that three times his money increased by \$8 equals \$80; how much had he?

SOLUTION. Let x equal A's money; then, by the condition of the problem we shall have $3x+8=80$. Now, if $3x$ *increased* by 8 equals 80, $3x$ will equal 80 *diminished* by 8, which is 72; if $3x$ equals 72, x will equal one-third of 72, which is 24. Hence A had 24 dollars.

OPERATION.

Let $x=$ A's money.

$$3x+8=80$$
$$3x=72$$
$$x=24$$

2. If three times a number increased by 12 equals 57, what is that number? *Ans.* 15.

3. If three-fourths of the distance from New York to Troy be diminished by 21 miles, the result will be 90 miles; what is the distance? *Ans.* 148 miles.

4. If A's age be increased by its two-thirds and 7 years more, it will equal 32 years; what is his age? *Ans.* 15 years.

5. If $2\frac{3}{4}$ times the money a boy spent on the Fourth of July be diminished by 40 cents, the result will be \$5.65; how much did he spend? *Ans.* \$2.20.

6. One-half of my fortune, plus one-third of it and \$380 more, equals \$2580; what is my fortune? *Ans.* \$2640.

7. One-third of the trees in an orchard bear apples, one-fourth bear peaches, and the remainder, which is 100, bear plums; required the number of trees in the orchard. *Ans.* 240.

8. If 4 times what Mr. Jones spent during a summer vacation be diminished by three-fifths of the sum spent and \$680 the result will be \$5100; what did he spend? *Ans.* \$1700.

LESSON VII.

1. Anna has 8 oranges more than William, and they together have 36; how many has each?

SOLUTION. Let x equal William's number; then, since Anna has 8 oranges more than William, $x+8$ will equal Anna's number; and since they both have 36, x plus $x+8$ will equal 36. Adding, we have $2x+8=36$. If $2x$ *increased* by 8 equals 36, $2x$ will equal 36 *diminished* by 8, or 28. If $2x$ equals 28, x equals one-half of 28, or 14, William's number; and $x+8$ equals 14+8, or 22, Anna's number.

OPERATION.

Let $x=$ William's number.

$x+8=$ Anna's number.

$$x+x+8=36$$
$$2x+8=36$$
$$2x=28$$
$$x=14, \text{ William's number.}$$
$$x+8=22, \text{ Anna's number.}$$

2. A and B together have \$35, and A's money, plus \$9 equals B's; how much has each? *Ans.* A, \$13; B, \$22.

3. The sum of two numbers is 100, and the smaller number equals the larger diminished by 16; what are the numbers?

Ans. 42; 58.

4. A watch and chain cost \$220, and the chain cost \$20 less than five-sevenths of the cost of the watch; required the cost of each. *Ans.* Watch, \$140; chain, \$80.

5. A house and lot cost \$5800; required the cost of each if the lot cost \$300 more than three-eighths as much as the house. *Ans.* House, \$4000; lot, \$1800.

6. Blanche, Lidie and Kate went a-shopping and spent \$70. Lidie spent \$4 more than Kate, and Blanche spent \$6 more than Kate; how much did each spend?

Ans. Blanche, \$26; Lidie, \$24; Kate, \$20.

7. A, B and C contributed \$125 to a Sabbath-school; A gave \$10 less than twice as much as C, and B gave \$10 more than twice as much as C; what did each contribute?

Ans. A, \$40; B, \$60; C, \$25.

8. A lady bought a hat, cloak and shawl for \$78; what did she pay for each, supposing that the cloak cost twice as much as the hat, plus \$4, and the shawl twice as much as the cloak, lacking \$4? *Ans.* Hat, \$10; cloak, \$24; shawl, \$44.

LESSON VIII.

1. If the height of a tree be increased by its two-thirds and 10 feet more, the sum will be twice the height; what is the height of the tree?

SOLUTION. Let x equal the height of the tree; then, by the condition of the problem, we have the equation $x+\frac{2}{3}x+10=2x$. Adding, we have $\frac{5}{3}x+10=2x$. If $\frac{5}{3}x+10$ equals $2x$, then 10 will equal $2x$ minus $\frac{5}{3}x$, or $\frac{x}{3}$; or $\frac{x}{3}$ will equal 10; hence x equals three times 10, or 30.

OPERATION.

Let x = the height.

$$x+\frac{2x}{3}+10=2x$$
$$\frac{5}{3}x+10=2x$$
$$10=\frac{x}{3}$$
$$\frac{x}{3}=10$$
$$x=30, \text{ height.}$$

2. If twice the length of a pole be increased by two-thirds of its length and 8 feet more, the sum will equal three times its length; what is its length? *Ans.* 24 feet.

3. Three-fourths of Nelson's age, increased by 6 years, equals four-fifths of his age, increased by 5 years; how old is he? *Ans.* 20 years.

4. Five times the money Jennie paid for her bracelets, diminished by \$8, equals three times the money she paid, increased by \$8; what did she pay? *Ans.* \$8.

5. Emma bought a fan, shawl and cloak; the fan cost \$4; the shawl cost \$4 more than two-thirds of the cost of the cloak, and the cloak cost \$4 more than the fan and shawl; required the cost of the shawl and the cloak. *Ans.* Shawl, \$28; cloak, \$36.

6. A's money, plus \$12, equals B's, and B's, plus \$6, equals C's, and the sum of their moneys equals four and one-half times A's money; how much money has each? *Ans.* A, \$20; B, \$32; C, \$38.

7. Three times what it cost Harry to attend college a year

increased by \$50 equals twice the sum obtained by increasing the amount by \$150; what did it cost him? *Ans.* \$250.

8. A man driving his geese to market was met by another, who said, "Good-morning, master, with your hundred geese." He replied, "I have not a *hundred*, but if I had as many more, and half as many more, and two geese and a half, I would have a hundred." How many geese had he? *Ans.* 39.

LESSON IX.

In Problem 1, Lesson I., we supposed Henry's number, plus three times his number, to be equal to *twenty-four*. Suppose now, instead of representing the number *twenty-four* by the figures 2 and 4, we use one of the first letters of the alphabet, as a, to represent it. The problem will then become—

1. Henry's number of apples, plus three times his number, equals a; how many apples has he?

SOLUTION. Let x equal Henry's number; then, by the condition of the problem, we will have $x+3x=a$, or $4x=a$; hence x equals a divided by 4, which we express thus, $\frac{a}{4}$.

OPERATION.

Let x = Henry's number.

$$x+3x=a$$
$$4x=a$$
$$x=\frac{a}{4}$$

Known Numbers.—Now, it is evident that a may represent any other number, as 12, 16, etc. Hence, we see we may represent *known* numbers by *letters* as well as by *figures*.

A number represented by figures expresses a *definite* number of units, and may therefore be called a *definite number*. A number represented by a letter does not express a definite number of units, and may therefore be called an *indefinite number*.

Substitution.—Since a represents any number, let us suppose, as at first, its value to be 24; if we then use 24 for a in the result $x=\frac{a}{4}$, we will have $x=\frac{24}{4}$, or 6, which is the same result as we obtained by using 24 in the solution in Lesson I.

This using some particular value of a general quantity in an expression containing the quantity is called *Substitution.*

PROBLEMS

2. Mary's age is twice Sarah's, and the sum of their ages is a years; how old is each? *Ans.* Sarah, $\frac{a}{3}$; Mary, $\frac{2a}{3}$.

3. Find the age of each when $a=36$, by substituting the value of a in the result. *Ans.* 12 years; 24 years.

4. Divide the number m into two parts, such that the larger part will be 5 times the smaller. *Ans.* $\frac{m}{6}$; $\frac{5m}{6}$.

5. Find the value of each part when $m=144$, by substituting the value of m in the results. *Ans.* 24; 120.

6. The difference of two numbers is a, and 5 times the smaller equals the larger; what are the numbers? *Ans.* $\frac{a}{4}$; $\frac{5a}{4}$.

7. Find the value of each part when $a=24$, by substituting the value of a in the results. *Ans.* 6; 30.

8. What number is that to which if its one-third be added the sum will be b? *Ans.* $\frac{3b}{4}$.

9. Divide the number c into three such parts that the first shall be twice the second, and the second twice the third.

Ans. 1st, $\frac{4c}{7}$; 2d, $\frac{2c}{7}$; 3d, $\frac{c}{7}$.

10. If three times a number increased by n equals a, what is the number? *Ans.* $\frac{a-n}{3}$.

11. The sum of two numbers is a, and the smaller equals the larger diminished by c; what are the numbers?

Ans. $\frac{a+c}{2}$; $\frac{a-c}{2}$.

12. One-half the length of a pole is in the mud, one-third in the water, and h feet in the air; what is the length of the pole? *Ans.* $6h$.

NOTE.—Let the pupil give special values to the general quantities in each of the above problems, and find the results by Substitution.

ELEMENTARY ALGEBRA.

SECTION I.

DEFINITIONS AND EXPLANATIONS.

1. Mathematics is the science of quantity. It treats of the properties and relations of quantity.

2. Quantity is anything that can be measured. It is of two kinds, *Number* and *Extension.*

3. Arithmetic is the science of *Number; Geometry* is the science of *Extension.*

4. Algebra is a method of investigating quantity by means of general characters called *symbols.*

5. Algebraic Symbols are the characters used to represent quantities, their relations and the operations performed upon them.

6. The **Symbols of Algebra** are of three kinds, namely—

1. Symbols of Quantity; 2. Symbols of Operation;

3. Symbols of Relation.

NOTES.—1. With beginners we regard Algebra as restricted to *numbers,* or as a kind of *general Arithmetic.* They may afterward be led to see how general symbols introduce ideas not found in Arithmetic; and eventually, that Algebra is a general method of investigation that may be applied to all kinds of quantity.

2. Some writers divide Algebra into *Arithmetical* Algebra and *Symbolical* Algebra. Newton called it *Universal Arithmetic,* and many writers speak of it as *General Arithmetic.* D'Alembert divides Arithmetic into *Numérique,* Special Arithmetic, and *Algebra,* General Arithmetic.

PROBLEMS

2. Mary's age is twice Sarah's, and the sum of their ages is a years; how old is each? *Ans.* Sarah, $\frac{a}{3}$; Mary, $\frac{2a}{3}$.

3. Find the age of each when $a=36$, by substituting the value of a in the result. *Ans.* 12 years; 24 years.

4. Divide the number m into two parts, such that the larger part will be 5 times the smaller. *Ans.* $\frac{m}{6}$; $\frac{5m}{6}$.

5. Find the value of each part when $m=144$, by substituting the value of m in the results. *Ans.* 24; 120.

6. The difference of two numbers is a, and 5 times the smaller equals the larger; what are the numbers? *Ans.* $\frac{a}{4}$; $\frac{5a}{4}$.

7. Find the value of each part when $a=24$, by substituting the value of a in the results. *Ans.* 6; 30.

8. What number is that to which if its one-third be added the sum will be b? *Ans.* $\frac{3b}{4}$.

9. Divide the number c into three such parts that the first shall be twice the second, and the second twice the third.

Ans. 1st, $\frac{4c}{7}$; 2d, $\frac{2c}{7}$; 3d, $\frac{c}{7}$.

10. If three times a number increased by n equals a, what is the number? *Ans.* $\frac{a-n}{3}$.

11. The sum of two numbers is a, and the smaller equals the larger diminished by c; what are the numbers?

Ans. $\frac{a+c}{2}$; $\frac{a-c}{2}$.

12. One-half the length of a pole is in the mud, one-third in the water, and h feet in the air; what is the length of the pole? *Ans.* $6h$.

NOTE.—Let the pupil give special values to the general quantities in each of the above problems, and find the results by Substitution.

ELEMENTARY ALGEBRA.

SECTION I.

DEFINITIONS AND EXPLANATIONS.

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SYMBOLS OF QUANTITY.

7. A **Symbol of Quantity** is a character used to represent a quantity.

8. The **Symbols of Quantity** generally used are the *figures* of arithmetic and the *letters* of the alphabet.

9. Known Quantities are represented by *figures* and the *first* letters of the alphabet, as 1, 2, 3, etc., and a, b, c, etc.

10. Unknown Quantities are usually represented by the *final* letters of the alphabet, as x, y, z, v, etc.

11. The **Symbol** 0, called *zero*, denotes the absence of quantity, or that which is less than any assignable quantity.

12. The **Symbol** ∞, called *infinity*, denotes that which is greater than any assignable quantity.

13. Accents are small marks used to denote different quantities which occupy similar positions in an operation; as a', a'', a''', etc. These are read *a prime*, *a second*, etc.

14. Subscript figures are sometimes used for the same purpose; as a_1, a_2, a_3, etc. These are read *a sub. one*, *a sub. two*, etc.

15. The **Sign of Continuation** is .... It denotes that the quantities are continued by the same law, and is read *and so on.* Thus, a, $2a$, $3a$ ...., means a, $2a$, $3a$, $4a$, $5a$, etc.

16. *Quantities* represented by *letters* are called *Literal Quantities.* Quantities represented by *figures* are called *Numerical Quantities.*

NOTE.—These symbols are the *representatives* of quantities, but for convenience we speak of them as quantities, meaning the quantities which they represent. Thus we say, the quantity a, the quantity b, and also a and b; as, add a to b; subtract a from b, etc.

SYMBOLS OF OPERATION.

17. A **Symbol of Operation** is a character used to indicate the operations of quantities.

18. The **Sign of Addition** is +, called *plus.* Thus, $a+b$ indicates the addition of a and b, and is read *a plus b.*

19. The **Sign of Subtraction** is −, called *minus.* Thus, $a-b$ denotes the subtraction of b from a, and is read *a minus b.*

20. The **Sign of Multiplication** is ×, read *into*, *times* or *multiplied by.* Thus, $a\times b$ denotes that a is to be multiplied by b, and is read *a into b*, or *a times b*, or *a multiplied by b.*

Multiplication is also denoted by a simple point; thus, $a.b$ denotes the same as $a\times b$. With letters the sign is usually omitted; thus, $2ab$ denotes the same as $2\times a\times b$.

The *Coefficient* of a quantity is a number written before it to show how many times the quantity is taken. Thus, in $3ab$ the 3 is the coefficient, and shows that ab is taken 3 times; in ax, the a is the coefficient of x, showing that x is taken a times. When the coefficient is expressed by a figure, it is called a *numerical* coefficient; when it is expressed by a letter, it is called a *literal* coefficient.

21. The **Sign of Division** is ÷, read *divided by.* Thus, $a\div b$ denotes that a is to be divided by b, and is read *a divided by b.*

Division is also indicated by writing the dividend above and the divisor below a short horizontal line, as in a fraction, as $\frac{a}{b}$. The expressions $a\lfloor b$ and $a(b$ also denote the division of a by b.

22. The **Sign of Involution**, called the *Exponent*, is a number written at the right of and above a quantity to indicate its power.

The *Power* of a quantity is the product obtained by using the quantity as a factor any number of times. Thus, $a\times a$ is the second power of a; $a\times a\times a$ is the third power of a, etc.

The *Exponent* of a quantity is the number which indicates how often the quantity is used as a factor. Thus, in a^3 the ^{3} indicates that a is used as a factor *three* times; a^3 is equivalent to $a\times a\times a$. a^2 is read "*a square*," or "*a second power*;" a^3 is read "*a cube*," or "*a third power*," or "*a third*;" a^n is read "*a nth power*," or "*a nth*."

When the exponent is expressed by a *figure*, it is called a *numerical* exponent; when it is expressed by a *letter*, it is called a *literal* exponent. When no exponent is written, the exponent 1 is understood.

23. The **Sign of Evolution** is $\sqrt{\ }$, called the *Radical Sign.* It indicates that some root of the quantity before which it is placed is to be extracted. Thus, $\sqrt[2]{a}$, $\sqrt[3]{a}$, $\sqrt[4]{a}$ indicate, respectively, the *square root*, the *cube root* and the *fourth root* of a.

The *Index* of the root is the number written in the angle of the radical sign to indicate the required root. When no index is written, 2 is understood; thus, $\sqrt{a}$ is the same as $\sqrt[2]{a}$.

A *Fractional Exponent* is also used to indicate some root of a quantity. Thus, $a^{\frac{1}{2}}$ indicates the square root of a, $a^{\frac{1}{3}}$ the cube root of a, etc.

24. The **Signs of Aggregation** are the *Vinculum*, ——; the *Bar*, |; the *Parentheses*, ( ); the *Brackets*, [ ], and the *Braces*, { }. These indicate that the quantities connected or enclosed are to be subjected to the same operation. Thus, $\overline{a+b}\times c$; $\left.\begin{matrix}+a\\+b\end{matrix}\right|c$; $(a+b)c$; $[a+b]c$; $\{a+b\}c$, each indicates that $a+b$ is to be multiplied by c.

SYMBOLS OF RELATION.

25. A **Symbol of Relation** is a character used to indicate the relation of quantities.

26. The **Sign of Equality** is =, read *equals* or *equal to.* Thus, $x=a$ indicates the *equality* of x and a, and is read *x is equal to a*, or *x equals a.*

27. The **Sign of Ratio** is :, read *to* or *is to.* Thus, $a:b$ indicates the *ratio* of a to b, and is read *the ratio of a to b.*

28. The **Sign of Equality of Ratios** is : :, read *equals* or *as.* Thus, $a:b::c:d$ indicates the *equality of the ratios* of $a:b$ and $c:d$, and is read *the ratio of a to b equals the ratio of c to d*, or *a is to b as c is to d.*

29. The **Signs of Inequality** are >, read *is greater than*, and <, read *is less than.* Thus, $a>b$ and $a<b$ indicate the *inequality* of a and b; $a>b$ is read *a is greater than b*, and $a<b$ is read *a is less than b.*

30. The **Signs of Deduction** are ∴, read *therefore* or *hence*, and ∵, read *since* or *because.*

NOTE.—The signs of Deduction are used when the *relation* is inferred from some previous *relation.* It is evident, therefore, that they may be classed with Symbols of Relation.

ALGEBRAIC EXPRESSIONS.

31. An **Algebraic Expression** is the expression of a quantity by means of algebraic symbols. Thus, $a+3b-c$.

32. The **Terms** of an algebraic expression are the parts connected by the signs + and −. Thus, in $a+3b-c$ the terms are a, $3b$ and $-c$.

33. A **Positive Term** is one having the plus sign prefixed to it, as $+3a$. When no sign is expressed the sign + is understood.

34. A **Negative Term** is one having the minus sign prefixed to it; as $-3a$. This sign should not be omitted.

35. **Similar** or **Like Terms** are those which contain the same letters affected by the same exponents; as, $3ab^2$ and $-5ab^2$.

36. **Dissimilar** or **Unlike Terms** are those which contain different letters or exponents; as, $3ab^2$ and $-5a^2b^3c$.

37. A **Monomial** is an algebraic expression consisting of *one* term; as, a, $4a$, $5a^3$, etc.

38. A **Polynomial** is an algebraic expression consisting of *two* or *more* terms; as $a+b$, $a+b+c+d+e$, etc.

39. A **Binomial** is a polynomial consisting of *two* terms; as, $a+b$ and $3a+4b^3$.

40. A **Trinomial** is a polynomial consisting of *three* terms; as, $a+2ab+c$.

41. The **Degree** of a term is determined by the number of literal factors it contains. Thus, $2a$ is of the *first* degree, $3a^2$ or $3ab$ of the *second* degree.

42. **Homogeneous Terms** are those which are of the same degree. Thus, $3abc$ and $5ab^2$ are homogeneous.

43. A **Polynomial** *is homogeneous* when all of its terms are of the *same degree;* as, $a^4-4a^3b+a^2b^2$.

ALGEBRAIC LANGUAGE.

44. Algebraic Language is a method of expressing mathematical ideas by means of algebraic symbols.

45. Numeration is the art of translating algebraic expressions into common language.

46. Notation is the art of expressing mathematical ideas in algebraic language.

EXERCISES IN NUMERATION.

1. Read $a+b$.
2. Read a^2+2ab.
3. Read $(a+b)c$.
4. Read $\sqrt{(a+b)}$.
5. Read $2(a+b^3)$.
6. Read $x^2+2xy+y^2$.
7. Read $(a+x)(a-x)$.
8. Read $\frac{a-b}{a+b}\times\frac{a-x}{a+x}$.
9. Read $\sqrt{a+(x-z)^2}$.
10. Read $4\sqrt[3]{a+\sqrt{b^2-\sqrt[4]{c}}}$.

EXERCISES IN NOTATION.

Express in algebraic language—

1. The sum of a and b. *Ans.* $a+b$.
2. Three times b subtracted from a. *Ans.* $a-3b$.
3. The sum of a and b, minus c. *Ans.* $a+b-c$.
4. The product of a and b, minus c squared. *Ans.* $ab-c^2$.
5. The sum of a and b, multiplied by c. *Ans.* $(a+b)c$.
6. The square of m, minus m into n. *Ans.* m^2-mn.
7. The sum of a and b, into the difference of a and b. *Ans.* $(a+b)(a-b)$.
8. The square of a, plus the square root of a. *Ans.* $a^2+\sqrt{a}$.
9. The square of the sum of a and b. *Ans.* $(a+b)^2$.

10. Four times a square into b, minus three times c square into x cube. *Ans.* $4a^2b-3c^2x^3$.
11. The square of a plus b, divided by a minus b, plus four times a into b square. *Ans.* $\frac{(a+b)^2}{a-b}+4ab^2$.
12. The sum of a times x, and the square of b, divided by a minus x. *Ans.* $\frac{ax+b^2}{a-x}$.
13. The sum of the squares of b and c, divided by the difference of three times a and twice c. *Ans.* $\frac{b^2+c^2}{3a-2c}$.
14. The cube of $a-x$, diminished by the square root of a plus x. *Ans.* $(a-x)^3-\sqrt{a+x}$.
15. The cube of a, minus x, diminished by the sum of a and the square root of x. *Ans.* $a^3-x-(a+\sqrt{x})$.
16. A trinomial with its second term negative, and twice the product of the other two terms.
17. A homogeneous trinomial of the fifth degree, with the second term negative.

NUMERICAL VALUES.

47. The **Numerical Value** of an algebraic expression is the result obtained by substituting for its letters definite numerical values, and then performing the operations indicated.

1. Find the numerical value of $(a^2-ab)c$ when $a=5$, $b=4$, and $c=3$.

SOLUTION. Substituting for a, b and c their assigned values, we have $(5^2-5\times4)\times3$; performing a part of the operations indicated, we have $(25-20)\times3$, which equals 5×3, or 15.

OPERATION.

$(a^2-ab)c=(5^2-5\times4)\times3$
$=(25-20)\times3=5\times3=15.$

EXAMPLES.

Find the numerical value of the following expressions when $a=6$, $b=5$, $c=4$, $m=3$, $n=2$:

2. a^2-ab. *Ans.* 6.
3. $(a^2-bc)n$. *Ans.* 32.
4. $ab+3a^2-5cn$. *Ans.* 98.

5. $(a+b)(a-b)$. *Ans.* 11.

6. $(a+b)m-(a-b)n$. *Ans.* 31.

7. $(a^2-b^2)(c-n)$. *Ans.* 22.

8. $\left(\frac{a+c}{a-n}\right)(b+m)$. *Ans.* 20.

9. $(m^2+n)\times(m^2-n)$. *Ans.* 77.

10. $\overline{m+c}\times\overline{a-c+n}$. *Ans.* 28.

11. $\overline{a+b+c}\div\overline{a-m}$. *Ans.* 5.

12. $\sqrt{(a+b)^2-2n}$. *Ans.* 7.

13. $\sqrt{a+2b+3mn}$. *Ans.* 22.

14. $(2a+2b+3)^{\frac{1}{2}}$. *Ans.* 5.

15. $\frac{ab+bc+mn}{a+2b-nc}$. *Ans.* 7.

POSITIVE AND NEGATIVE QUANTITIES.

48. A quantity with the *plus* sign prefixed is called an *Additive* or *Positive* quantity; a quantity with the *minus* sign prefixed is called a *Subtractive* or *Negative* quantity.

A *Positive Quantity* indicates *addition*, or that, when used, something is to be *increased* by it. A *Negative Quantity* indicates *subtraction*, or that, when used, something is to be *diminished* by it.

Positive and *Negative* quantities, being thus *opposite* in meaning, may be conveniently used to represent quantities reckoned in *opposite directions*.

Thus, if we use + to represent a person's *gains* in business, we may use − to represent his *losses*; *north* latitude may be represented by +, *south* latitude by −; *future* time by +, *past* time by −, etc.

The symbols + and − may therefore indicate the *nature* of the quantities to which they are prefixed, as well as the *operations* to be performed upon them.

49. The *Absolute Value* of a quantity is its value taken independently of the sign prefixed to it. Two quantities are evidently *equal* when they have the *same absolute value* and the *same sign*.

If I take any number, as 8, and *increase* it by 5, and then *diminish* it by 5, the value of 8 will remain unchanged; hence I may infer that *uniting* a positive and a negative quantity of the *same absolute* value gives *nothing* for the result.

If I unite 8 with +4, the result is 12; and if I unite 8 with −6, the result is 2; hence, since +4 united with 8 gives a *greater* result than −6 united with 8, I may infer that +4 is greater than −6, and in general that a *positive* quantity in Algebra may be regarded as greater than a *negative* quantity.

50. The above explanations may be formally stated in the following principles.

PRINCIPLES.

1. *A Positive quantity indicates that, when used, some quantity is to be* INCREASED *by it, and a Negative quantity that some quantity is to be* DIMINISHED *by it.*

2. *Positive and Negative quantities are sometimes used to indicate quantities reckoned in opposite directions.*

3. *A Positive and a Negative quantity of the same absolute value, united, amount to nothing.*

4. *In Algebra a Positive quantity is regarded as greater than a Negative quantity, whatever may be their absolute values.*

ALGEBRAIC REASONING.

51. All **Reasoning** is *comparison*. The reasoning in Algebra consists principally of the comparison of *equals*.

This **comparison** gives rise to the *equation*. The *equation* is therefore the fundamental idea in Algebra; it is the basis of all its investigations.

Comparison is controlled by certain laws called *axioms*, and gives rise to certain operations called *processes*.

52. **Algebraic Reasoning** is employed in the *solution* of *problems* and the *demonstration* of *theorems*.

53. A **Problem** is a question to be solved. A *solution* of a problem is the process of obtaining a required result.

54. A **Theorem** is a truth to be demonstrated. A *demon-*

stration of a theorem is a course of reasoning employed in establishing its truth.

55. An **Axiom** is a self-evident truth. *Axioms* are the laws which control the reasoning processes.

AXIOMS.

1. If equals be added to equals, the sums will be equal.
2. If equals be subtracted from equals, the remainders will be equal.
3. If equals be multiplied by equals, the products will be equal
4. If equals be divided by equals, the quotients will be equal.
5. If a quantity be both increased and diminished by another, the value of the former will not be changed.
6. If a quantity be both multiplied and divided by another, the value of the former will not be changed.
7. Quantities which are equal to the same quantity are equal to each other.
8. Like powers of equal quantities are equal. Like roots of equal quantities are equal.

REVIEW QUESTIONS.

NOTE.—These REVIEW QUESTIONS are simply suggestive to the teacher, who can extend them as fully as is deemed desirable.

Define Mathematics. Quantity. Arithmetic. Geometry. Algebra. Symbols of Algebra. State the classes of symbols.

Define a Symbol of Quantity. Name Symbols of Quantity. Of Known Quantities. Of Unknown Quantities. Use of 0; of ∞. Of Accents. Of Subscript figures. The sign of continuation.

Define a Symbol of Operation. Explain the sign of Addition, Subtraction, etc. Define Coefficient. Power. Exponent. Index.

Define a Symbol of Relation. Explain the sign of Equality, etc.

Define an Algebraic Expression, the Terms, etc.

Define Algebraic language. Numeration. Notation. Numerical Value. State principles of positive and negative quantities. Define Reasoning. A Problem. A Theorem. An Axiom. Enunciate the Axioms.

SECTION II.

FUNDAMENTAL OPERATIONS.

ADDITION.

56. Addition is the process of finding the sum of two or more algebraic quantities.

57. The **Sum** of several algebraic quantities is a single quantity equal in value to the several quantities united.

NOTE.—The symbol + was introduced by *Stifelius*, a German mathematician, in a work published in 1544.

CASE I.

58. To add when the terms are similar.

CLASS I. When the terms have the same sign.

1. Find the sum of $2a$, $3a$ and $5a$.

OPERATION.

$$\begin{array}{r} 2a \\ 3a \\ 5a \\ \hline 10a \end{array}$$

SOLUTION. $5a$, plus $3a$, are $8a$; and $8a$, plus $2a$, are $10a$. Hence the sum of $2a$, $3a$ and $5a$ is $10a$.

Rule.—*Add the coefficients, and prefix the sum with its proper sign to the common literal part.*

EXAMPLES.

(2.)	(3.)	(4.)	(5.)	(6.)
$3a$	$5x$	$-5ax$	$4a^2c$	$-9a^2b^3c$
$4a$	$3x$	$-7ax$	$5a^2c$	$-18a^2b^3c$
$5a$	$7x$	$-6ax$	$12a^2c$	$-a^2b^3c$
$7a$	$8x$	$-8ax$	$15a^2c$	$-6a^2b^3c$
$19a$	$23x$	$-26ax$	$36a^2c$	$-34a^2b^3c$

7. Find the sum of $4a$, $6a$ and $7a$. *Ans.* $17a$.
8. Find the sum of $-2a$, $-3a$ and $-5a$. *Ans.* $-10a$.
9. Find the sum of $3ab$, $5ab$, $6ab$ and $8ab$. *Ans.* $22ab$.
10. Find the sum of $-3ac$, $-4ac$, $-7ac$ and $-9ac$. *Ans.* $-23ac$.

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$7a$	$8x$	$-8ax$	$15a^2c$	$-6a^2b^3c$
$19a$	$23x$	$-26ax$	$36a^2c$	$-34a^2b^3c$

7. Find the sum of $4a$, $6a$ and $7a$. *Ans.* $17a$.
8. Find the sum of $-2a$, $-3a$ and $-5a$. *Ans.* $-10a$.
9. Find the sum of $3ab$, $5ab$, $6ab$ and $8ab$. *Ans.* $22ab$.
10. Find the sum of $-3ac$, $-4ac$, $-7ac$ and $-9ac$. *Ans.* $-23ac$.

11. Find the sum of $5a^2b^3$, $7a^2b^3$, $9a^2b^3$ and $10a^2b^3$. *Ans.* $31a^2b^3$.

12. Find the sum of $6x^3y^5$, $7x^3y^5$, $9x^3y^5$ and $12x^3y^5$. *Ans.* $34x^3y^5$.

13. Find the sum of $-5abc$, $-4abc$, $-7abc$ and $-9abc$. *Ans.* $-25abc$.

59. CLASS II. When the terms have different signs.

1. Find the sum of $7a$ and $-4a$.

SOLUTION. $7a$ is equal to $3a+4a$. Now, $-4a$ united with $+4a$, a part of $7a$ is equal to nothing, Prin. 3, Art. 50; therefore $-4a$ added to $3a+4a$ equals $3a$. Hence, $-4a$ added to $7a$ equals $3a$.

OPERATION
$7a$
$-4a$
$3a$

SOLUTION 2D. Plus $7a$ may indicate some quantity *increased* by $7a$, and $-4a$ may indicate some quantity *diminished* by $4a$. A quantity *increased* by $7a$ and then *diminished* by $4a$ is evidently *increased* by $3a$; hence the sum of $7a$ and $-4a$ is *plus* $3a$.

2. Find the sum of $-8a$, $4a$, $-7a$ and $9a$.

SOLUTION. The sum of the positive quantities, $9a$ and $4a$, is $13a$; and the sum of the negative quantities, $-7a$ and $-8a$, is $-15a$. Now, $-15a=-13a-2a$; $+13a$ united to $-13a$ is equal to nothing, and there remains $-2a$. Hence the sum is $-2a$.

OPERATION
$-8a$
$4a$
$-7a$
$9a$
$-2a$

SOLUTION 2D. The latter part may be given thus: Any quantity *increased* by $13a$ and then *diminished* by $15a$ is evidently *diminished* by $2a$; hence the sum is *minus* $2a$.

Rule.—I. *Find the sum of the coefficients of the positive and negative terms separately.*

II. *Take the difference of these sums, and prefix it, with the sign of the greater, to the common literal part.*

EXAMPLES.

(3.)	(4.)	(5.)	(6.)	(7.)
$+7ax$	$-5a^2c^3$	$-7z^2$	$+27xy$	$-2x^2y^3z$
$-9ax$	$+3a^2c^3$	$-z^2$	$-34xy$	$-12x^2y^3z$
$+8ax$	$-9a^2c^3$	$+9z^2$	$-150xy$	$+x^2y^3z$
$-3ax$	$+4a^2c^3$	$-5z^2$	$+27xy$	$+28x^2y^3z$
$+3ax$	$-7a^2c^3$	$-4z^2$		

Find the sum—

8. Of $6ab$, $-5ab+8ab$ and $-3ab$. *Ans.* $6ab$.

9. Of $3cd$, $-6cd$, $-7cd$, $+8cd$ and $-4cd$. *Ans.* $-6cd$.

10. Of $7xy$, $+8xy$, $-9xy$, $+3xy$ and $-4xy$. *Ans.* $5xy$.

11. Of $5an$, $+7an$, $-12an$, $+15an$, $-19an$. *Ans.* $-4an$.

12. Of $7a^2b$, $-9a^2b$, $+10a^2b$, $+12a^2b$, $-30a^2b$. *Ans.* $-10a^2b$.

13. Of $12a^2c^3$, $-6a^2c^3$, $+a^2c^3$, $-15a^2c^3$, $+7a^2c^3$. *Ans.* $-a^2c^3$.

14. Of $15xy^2z$, $-19xy^2z$, $+12xy^2z$, $-10xy^2z$, $-15xy^2z$ and $+22xy^2z$. *Ans.* $5xy^2z$.

15. Of $5ac^3b^5$, $+6ac^3b^5$, $-7ac^3b^5$, $+8ac^3b^5$, $-17ac^3b^5$, $-4ac^3b^5$, $+5ac^3b^5$. *Ans.* $-4ac^3b^5$.

16. Of $21am^2nx^3$, $-19am^2nx^3$, $-21am^2nx^3$, $+25am^2nx^3$, $+19am^2nx^3$, $-25am^2nx^3$. *Ans.* 0.

CASE II.

60. To add when the terms are dissimilar.

1. Find the sum of $3a$, $4b$ and $-ab$.

SOLUTION. Since the quantities are dissimilar, we cannot unite them into one sum by adding their coefficients; we therefore indicate the addition by writing them one after another with their respective signs. We thus have $3a+4b-ab$.

OPERATION.
$3a$
$4b$
$-ab$
$3a+4b-ab$

2. Find the sum of $2a+3ab$, $3a-4ab+5b$ and $5ab-7b$.

SOLUTION. We write the similar terms in the same column for convenience in adding, and begin at the left to add: $3a$ and $2a$ are $5a$, which we write under the column added; $5ab$, $-4ab$, $+3ab$ are $+4ab$, which we write under the column added; $-7b$, $+5b$ equals $-2b$, which we write under the column added. Hence the sum is $5a+4ab-2b$.

OPERATION.
$2a+3ab$
$3a-4ab+5b$
$5ab-7b$
$5a+4ab-2b$

Rule.—I. *Write similar terms, with their proper signs, in the same column.*

II. *Add each column separately, and connect the results with their proper signs.*

EXAMPLES.

(3.)	(4.)	(5.)
$2a+3b$	$3x-5xy$	$12ab^2+28cx^3$
$5a-7b$	$7x+8xy$	$-ab^2+25cx^3$
$a+9b$	$a-9x-6xy$	$24ab^2-23cx^3$
$3a-8b$	$4a-5x+7xy$	$-35ab^2-17cx^3$
$11a-3b$	$5a-4x+4xy$	$+13cx^3$

6. Find the sum of $3ac-5ax$, $7ac+6ax$, $5ac-12ax$ and $9ac+15ax$. *Ans.* $24ac+4ax$.

7. Find the sum of $5ab+12bc-7cd$, $9ab-18bc+11cd$ and $17ab-15bc+13cd$. *Ans.* $31ab-21bc+17cd$.

8. Find the sum of $3ax-2b^2c$, $5ax+7c^3$, $9b^2c-12c^3$, $8ax+15c^3$ and $14b^2c-18c^3$. *Ans.* $16ax+21b^2c-8c^3$.

9. Find the sum of $m+3n^2-5mn$, $3m-8n^2$, $7n^2-8mn$, $19m+27mn$ and $16n^2-17mn$. *Ans.* $23m-3mn+18n^2$.

10. Find the sum of $a+2b+3c$, $2a-b-2c$, $b-a-c$ and $c-a-b$. *Ans.* $a+b+c$.

11. Find the sum of $3a-4p+q$, $7p+3q-6$, $9a-7+3p$ and $9q-12+11p$. *Ans.* $12a+17p+13q-25$.

12. Find the sum of $a+b-c$, $a-b+c$, $a+c+b$ and $b-a+c$. *Ans.* $2a+2b+2c$.

13. Find the sum of $4a+7a^2c-8m^3$, $7a+16m^3$, $15a^2c-20m^3+17$ and $12m^3-5-22a^2c$. *Ans.* $11a+12$.

14. Add $34ax^3-16ay^2$, $-25ax^3-13ay^3+14ay^2$, $16+15ay^3$, $15ay^2-16$ and $22ax^3+7ay^2-11ay^3$. *Ans.* $31ax^3+20ay^2-9ay^3$.

15. Add $12x+9y-6z$, $5a-12y+13x$, $7y-16x+10z$ and $10x-5a+12z$. *Ans.* $19x+4y+16z$.

16. Add x^n-ax^2+3b, $3ax^2-2b+y^{2n}$, $5x^n+4b-3y^{2n}$, $7b-4x^n+y^{2n}-3ax^2$. *Ans.* $2x^n-ax^2-y^{2n}+12b$.

17. Add $5a-9b+5c+3-d$, $a-3b-8-d$, $3a+2b-3c+4+5d$, $2a+5c-6-3d$. *Ans.* $11a-10b+7c-7$.

18. Add $x^4-4x^3y+6x^2y^2-4xy^3+y^4$, $4x^3y-12x^2y^2+12xy^3-4y^4$, $6x^2y^2-12xy^3+6y^4$ and $4xy^3-4y^4$. *Ans.* x^4-y^4.

19. Add $a^3+ab^2+ac^2-a^2b-abc-a^2c$, $a^2b+b^3+bc^2-ab^2-b^2c-abc$ and $a^2c+b^2c+c^3-abc-bc^2-ac^2$. *Ans.* $a^3+b^3+c^3-3abc$.

20. Add $4ab-3mn+10am-6an$, $7mn-7am+4an$, $3ab+7an+9$, $6-4mn-3am-5n^2$ and $4n^2-15-2m^2$.
Ans. $7ab-2m^2-n^2+5an$.

FACTORED FORMS.

61. Similar quantities in any form may be added by taking the algebraic sum of their coefficients.

EXAMPLES.

(1.)	(2.)	(3.)	(4.)
$6\sqrt{7}$	$8(a-b)$	$-12\sqrt{a+b}$	$5(m-n+2)$
$4\sqrt{7}$	$-5(a-b)$	$15\sqrt{a+b}$	$7(m-n+2)$
$5\sqrt{7}$	$6(a-b)$	$-18\sqrt{a+b}$	$-9(m-n+2)$
$15\sqrt{7}$	$9(a-b)$	$-15\sqrt{a+b}$	$3(m-n+2)$

5. What is the sum of $5(x-y)$, $-12(x-y)$, $3(x-y)$, $10(x-y)$ and $-14(x-y)$? *Ans.* $-8(x-y)$.

6. What is the sum of $7(a-b)^3$, $-9(a-b)^3$, $+12(a-b)^3$, $+16(a-b)^3$, $-18(a-b)^3$? *Ans.* $8(a-b)^3$.

7. Find the sum of $3\sqrt{a+x}$, $5\sqrt{a+x}$, $-7\sqrt{a+x}$, $+8\sqrt{a+x}$, $-5\sqrt{a+x}$ and $12\sqrt{a+x}$. *Ans.* $16\sqrt{a+x}$.

8. Add $4ax+7(a^2-b^2)$, $6ax-5(a^2-b^2)$, $+3(a^2-b^2)-5ax$, $12(a^2-b^2)-7ax$, $16(a^2-b^2)+9ax$, $-33(a^2-b^2)$. *Ans.* $7ax$.

9. Add $2a^2-3(a+x)$, $5a^2+6(x-y)^2$, $4a^2-7(x-y)^2$, $9(a+x)-6a^2$, $3(a+x)-9(x-y)^2$, $a^2-(a+x)+(x-y)^2$.
Ans. $6a^2+8(a+x)-9(x-y)^2$.

62. Dissimilar Terms having a common factor may be added by taking the algebraic sum of the dissimilar parts, enclosing it in a parenthesis, and affixing the common factor.

1. Find the sum of $ax+bx-cx$.

OPERATION.

SOLUTION. a times x, $+b$ times x, $-c$ times x, equals $(a+b-c)$ times x; hence the sum is $(a+b-c)x$.

$$\begin{array}{r} ax \\ bx \\ -cx \\ \hline (a+b-c)x. \end{array}$$

EXAMPLES.

2. Find the sum of $ax^3-bx^3+cx^3$. *Ans.* $(a-b+c)x^3$.

3. Find the sum of $az^5-mz^5+nz^5-qz^5$.
Ans. $(a-m+n-q)z^5$.

4. Find the sum of $2ax-2bx+(a-b)x$. *Ans.* $3(a-b)x$.

5. Find the sum of $4ax+3x+2ax-5x+bx-5ax+2x-2bx$.
Ans. $(a-b)x$.

6. Find the sum of $3ay-2by+(a+2b+c)y$. *Ans.* $(4a+c)y$.

7. Find the sum of $3an-5am+2an-3bn+3am-5m+6bn+2am-3bn+5m+cn$. *Ans.* $(5a+c)n$.

SUBTRACTION.

63. Subtraction is the process of finding the difference of two algebraic quantities.

64. The **Subtrahend** is the quantity to be subtracted.

65. The **Minuend** is the quantity from which the subtrahend is to be subtracted.

66. The **Difference** or **Remainder** is a quantity which, added to the subtrahend, will equal the minuend.

NOTE.—The symbol − was introduced by *Stifelius*, a German mathematician, in a work published in 1544.

CASE I.

67. To subtract when all the terms are positive.

1. Subtract $4a$ from $7a$.

SOLUTION. 4 times a quantity subtracted from 7 times the quantity equals 3 times the quantity; hence, $4a$ subtracted from $7a$ equals $3a$.

OPERATION.
$$\begin{array}{r} 7a \\ 4a \\ \hline 3a \end{array}$$

2. Subtract $7a$ from $4a$.

SOLUTION. $4a$ equals $7a-3a$; $7a$ subtracted from $7a-3a$ leaves $-3a$; hence $7a$ subtracted from $4a$ equals $-3a$.

OPERATION.
$$\begin{array}{rr} 4a = & 7a-3a \\ 7a = & 7a \\ \hline -3a & -3a \end{array}$$

SOLUTION 2D. Plus $4a$ may indicate some quantity *increased* by $4a$, and $+7a$ may indicate some quantity increased by $7a$. A quantity *increased by* $4a$ is evidently $3a$ *less* than the quantity *increased by* $7a$; hence, $7a$ subtracted from $4a$ equals *minus* $3a$.

3. Subtract $b+c$ from a.

SOLUTION. Subtracting b from a, we have the remainder $a-b$; but we wish to subtract b *increased* by c from a, hence the true remainder will be $a-b$ diminished by c, or $a-b-c$.

OPERATION.
$$\begin{array}{l} a \\ b+c \\ \hline a-b-c \end{array}$$

Rule.—*Change the signs of the subtrahend and proceed as in addition.*

NOTE.—Signs of terms are said to be changed when, being *plus*, they are changed to *minus*, or being −, they are changed to +.

EXAMPLES.

(4.)	(5.)	(6.)	(7.)	(8.)
$12a$	$15x^2y$	$21m^2n^2$	$5a^2+3b$	$3a+4b$
$9a$	$9x^2y$	$28m^2n^2$	$3a^2+7b$	$9b+2c$
$3a$	$6x^2y$	$-7m^2n^2$	$2a^2-4b$	$3a-5b-2c$

9. From $19ab$ take $12ab$. *Ans.* $7ab$.
10. From $21ac^2$ take $16ac^2$. *Ans.* $5ac^2$.
11. From $10axy$ take $17axy$. *Ans.* $-7axy$.
12. From $12m^2n^3$ take $18m^2n^3$. *Ans.* $-6m^2n^3$.
13. From $4a^2+6b$ take $12b$. *Ans.* $4a^2-6b$.
14. From $7a+5c$ take $10c$. *Ans.* $7a-5c$.
15. From $3x^2+2y^2$ take $4x^2+y^2$. *Ans.* $-x^2+y^2$.
16. From $2a+3b$ take $a+2b$. *Ans.* $a+b$.
17. From $4a+2b$ take $2a+3b$. *Ans.* $2a-b$.
18. From $a^2+4ab+b^2$ subtract $a^2+2ab+b^2$. *Ans.* $2ab$.
19. From a^2+b^2 subtract $a^2+2ab+b^2$. *Ans.* $-2ab$.
20. From $4a+2b$ subtract $3a+4b+2c$. *Ans.* $a-2b-2c$.
21. From $7a^2b+3ac$ subtract $5ac+4a^2b$. *Ans.* $3a^2b-2ac$.
22. From $ab+bc+cd$ subtract $bc+2cd+c$. *Ans.* $ab-cd-c$.

CASE II.

68. To subtract when one or more terms are negative.

1. Subtract $-c$ from $+a$.

SOLUTION. a equals $a+c-c$, since increasing and diminishing a quantity by the same quantity does not change its value. Now, $-c$ subtracted from $a+c-c$, leaves $a+c$. Hence, $-c$ subtracted from $+a$ leaves $a+c$.

OPERATION.
$$\begin{array}{rr} +a= & a+c-c \\ -c= & -c \\ \hline a+c & a+c \end{array}$$

SOLUTION 2D. The difference between any quantity *increased* by a and *diminished* by c is evidently the *sum* of a and c; hence, $-c$ subtracted from $+a$ equals $a+c$.

OPERATION.
$$\begin{array}{r} +a \\ -c \\ \hline a+c \end{array}$$

2. Subtract $b-c$ from a.

SOLUTION. Subtracting b from a, we have the remainder $a-b$; but we wish to subtract b diminished by c from a; we have therefore subtracted c too much, consequently the remainder, $a-b$, is c too small; hence the true remainder is $a-b$ increased by c, or $a-b+c$.

OPERATION.
$$\begin{array}{l} a \\ b-c \\ \hline a-b+c \end{array}$$

Rule.—I. *Write the subtrahend under the minuend, placing similar terms one under another.*

II. *Conceive the signs of the subtrahend to be changed, and then proceed as in addition.*

EXAMPLES.

(3.)	(4.)	(5.)	(6.)
$9a^2b$	$5a^2$	$-2a$	$7m^2-3n$
$-6a^2b$	$3a^2-2b$	$a-b$	$-4m^2-6n+c$
$15a^2b$	$2a^2+2b$	$-3a+b$	$11m^2+3n-c$

(7.)	(8.)	(9.)
$2a^2-5a+6b$	$c-5m+4$	$ax^2-2ac+3\frac{1}{2}$
$-a^2-3a+4b$	$-3c-7m-2$	$ax^2-5ac+2\frac{1}{4}-z^2$
$3a^2-2a+2b$	$4c+2m+6$	$3ac+1\frac{1}{4}+z^2$

10. From $a+b$ take $a-b$. *Ans.* $2b$.
11. From $10a$ take $-10a$. *Ans.* $20a$.
12. From $a-b$ take $b-a$. *Ans.* $2a-2b$.
13. From $a+2b$ take $a-b$. *Ans.* $3b$.
14. From $5m-5n$ take $4m+6n$. *Ans.* $m-11n$.
15. From $1+a^2x^2$ take $1-a^2x^2$. *Ans.* $2a^2x^2$.
16. From $4a^m-3b^n$ take $2a^m-5b^n$. *Ans.* $2a^m+2b^n$.
17. From $a^2+2ab+b^2$ take $a^2-2ab+b^2$. *Ans.* $4ab$.
18. From a^2-b^2 take $a^2-2ab+b^2$. *Ans.* $2ab-2b^2$.
19. From $3a+c+d-f-8$ take $c+3a-d$. *Ans.* $2d-f-8$.
20. From $4ab+3b^2-2c$ take $4ab-2b^2-3d$. *Ans.* $5b^2-2c+3d$
21. From $7am-3bc-c^2$ take $5am-2c^2-3bc-5x^3$. *Ans.* $2am+c^2+5x^3$.
22. From $2a+2b-3c-8$ take $3c+4b-3a-5$. *Ans.* $5a-2b-6c-3$.
23. From $a^3+3a^2b+3ab^2+b^3$ take $a^3-3a^2b+3ab^2-b^3$. *Ans.* $6a^2b+2b^3$.
24. From $a^2-3ab-b^2+bc-2c^2$ take $a^2-5ab+5bc-3b^2-2c^2$. *Ans.* $2ab+2b^2-4bc$.

FACTORED FORMS.

69. Similar quantities in any form may be subtracted by taking the algebraic difference of their coefficients.

EXAMPLES.

(1.)	(2.)	(3.)	(4.)
$9\sqrt{6}$	$12(a-b)$	$15\sqrt{a+b}$	$-7(a-b+4)$
$5\sqrt{6}$	$7(a-b)$	$-7\sqrt{a+b}$	$-12(a-b+4)$
$4\sqrt{6}$	$5(a-b)$	$22\sqrt{a+b}$	$5(a-b+4)$

5. From $5(x^2-y^2)$ take $-7(x^2-y^2)$. *Ans.* $12(x^2-y^2)$.
6. From $-6(a^2-b^2)$ take $12(a^2-b^2)$. *Ans.* $-18(a^2-b^2)$.
7. From $6a^2(a-b)$ take $-4a^2(a-b)$. *Ans.* $10a^2(a-b)$.
8. Find the value of $5\sqrt{2}-7\sqrt{2}+6\sqrt{2}$. *Ans.* $4\sqrt{2}$.
9. From $-5x^2(c-d)$ take $-12x^2(c-d)$. *Ans.* $7x^2(c-d)$.
10. Find the value of $7c^2(m-n)-13c^2(m-n)+12c^2(m-n)$. *Ans.* $6c^2(m-n)$.

70. Dissimilar terms having a common factor may be subtracted by taking the algebraic difference of the dissimilar parts, enclosing it in a parenthesis and affixing the common part.

1. From ax subtract cx.

SOLUTION. a times x minus c times x is evidently equal to $(a-c)$ times x, which is expressed thus $(a-c)x$.

OPERATION

$$\begin{array}{r} ax \\ cx \\ \hline (a-c)x \end{array}$$

EXAMPLES.

(2.)	(3.)	(4.)	(5.)
ax^2	mz^2	axy	az
bx^2	$-nz^2$	cxy	z
$(a-b)x^2$	$(m+n)z^2$	$(a-c)xy$	$(a-1)z$

6. From $5az$ take baz. *Ans.* $(5-b)az$.
7. From cax take $-3ax$. *Ans.* $(c+3)ax$.

8 From az take $bz-3z$. *Ans.* $(a-b+3)z$.

9. From $4n^2c+3c$ take $7c-4ac$. *Ans.* $(n^2-1+a)4c$.

10. From $an+cn+dn$ take $n+an+dn$. *Ans.* $(c-1)n$.

11. From $(6a+2x)cd$ take $4acd+2cdx$. *Ans.* $2acd$.

12. From $5a^2+10b^2$ take $-3a^2+2b^2$. *Ans.* $8(a^2+b^2)$.

13 From $6ay-3my$ subtract $-5my+6cy$. *Ans.* $(3a-3c+m)2y$.

14. From $6\sqrt{c}-a\sqrt{c}+b\sqrt{c}$ subtract $2a\sqrt{c}+b\sqrt{c}-2\sqrt{c}-ax\sqrt{c}$. *Ans.* $(8-3a+ax)\sqrt{c}$.

USE OF THE PARENTHESIS.

71. The **Parenthesis** is frequently used in Algebra: we will therefore now explain its use in Addition and Subtraction.

The plus sign before a parenthesis indicates that the quantity within the parenthesis is to be *added*, and the *minus* sign indicates that it is to be *subtracted*.

PRIN. 1. *A parenthesis with the plus sign before it may be removed from a quantity without changing the signs of its terms.*

Thus, $a+(b-c+d)$ is equal to $a+b-c+d$.

PRIN. 2. *A quantity may be enclosed in a parenthesis preceded by a plus sign without changing the signs of its terms.*

Thus, $a+b-c+d-e$ is equal to $a+(b-c+d-e)$, or to $a+b+(-c+d-e)$, etc.

PRIN. 3. *A parenthesis preceded by the minus sign may be removed from a quantity if the signs of all its terms be changed.*

This is evident from the rule for subtraction. Thus, $a-(b-c+d)$ is equal to $a-b+c-d$.

PRIN. 4. *A quantity may be enclosed in a parenthesis preceded by the minus sign if the signs of all its terms be changed.*

This is evident from the principles of subtraction, and also from the previous principle. Thus, $a-b+c-d$ is equal to $a-(b-c+d)$; or to $a-b-(-c+d)$, etc.

EXAMPLES.

Find the value—

1. Of $-(-a^2)$ and $x-(a-b)$. *Ans.* a^2; $x-a+b$.

2. Of $+(-ab)$ and $a-(b-c+d)$. *Ans.* $-ab$; $a-b+c-d$.

3. Of $-(b^2-a^2)$ and $3c-(2c-5)$. *Ans.* a^2-b^2; $c+5$.

4. Of $4a-5b-(a-5b+3c)$. *Ans.* $3(a-c)$.

5. Of $5a-2b-3c-(-5c+2a-2b)$. *Ans.* $3a+2c$.

6. Put in a parenthesis preceded by a plus sign the last three terms of $a+2b-3c+d-4$. *Ans.* $a+2b+(-3c+d-4)$.

7. Put in a parenthesis preceded by a minus sign the last three terms of $3a-4b+5c-7d$. *Ans.* $3a-(4b-5c+7d.)$

8. Find the value of $2a-(b+c-d+e-f)$ plus $2b-(a-c+d-e+g)$. *Ans.* $a+b-g+f$.

72. Expressions sometimes occur containing more than one pair of brackets, as $a-\{b-(c-d)\}$.

Such brackets may be removed in succession, *beginning*, for convenience, *with the inside pair.*

NOTE.—Brackets may also be removed by beginning with the *outer pair*, or with *any pair.*

Find the value—

1. Of $a-\{b+(c-d)\}$.

SOLUTION. $a-\{b+(c-d)\}=a-\{b+c-d\}=a-b-c+d$.

2. Of $a-\{b-(c-d)\}$. *Ans.* $a-b+c-d$.

3. Of $a-\{b-c-(d-e)\}$. *Ans.* $a-b+c+d-e$.

4. Of $2a-\{b-(a-2b)\}$. *Ans.* $3a-3b$.

5. Of $3a-\{b+(2a-b)-(a-b)\}$. *Ans.* $2a-b$.

6. $7a-[3a-\{4a-(5a-2a)\}]$. *Ans.* $5a$.

7. Of $6a-[4b-\{4a-(6a-4b)\}]$. *Ans.* $4a$.

8. Of $a-[2b+\{3c-3a-(a+b)\}+\{2a-(b+c)\}]$. *Ans.* $3a-2c$.

REMARKS UPON ADDITION AND SUBTRACTION.

1. Addition and Subtraction may also be explained by regarding the positive and negative quantities as representing, respectively, *gain* and *loss* in business, distance *north* and *south*, etc. But these illustrations, though they may aid the beginner, are not sufficiently general to be embodied in a solution.

2. The problem, subtract $7a$ from $4a$, may be explained by the following method: $7a$ equals $4a+3a$; subtracting $4a$ from $4a$, nothing remains, and there is still $3a$ to be subtracted, which we may represent by writing $-3a$. This method, however, is not general; it will not explain several cases, such as $-7a$ from $3a$, nor the general problem, subtract $-c$ from a.

3. Special attention is invited to the method of explaining Addition and Subtraction given in the "Solution 2d" of Articles 67 and 68. The peculiarity of the method consists in regarding a *positive* term as indicating that *some quantity* is *increased* by the term, and a *negative* term as indicating that *some quantity* is *diminished* by that term, or in using an *auxiliary quantity*.

Thus, to subtract $-2a$ from $+3a$, we regard $+3a$ as indicating that *some quantity* is to be *increased* by $3a$, and $-2a$ that *some quantity* is to be *diminished* by $2a$; then since *a quantity increased*, by $3a$ is *greater* than the quantity *diminished* by $2a$, by the sum of $3a$ and $2a$, or $5a$, we infer that $-2a$ taken from $+3a$ leaves $+5a$. Hence we use "*a quantity*" as *auxiliary*.

The same idea is presented in the following form of statement: The *difference* between a quantity *increased* by $3a$ and *diminished* by $2a$ is evidently the sum of $3a$ and $2a$, or $5a$; hence $-2a$ subtracted from $3a$ leaves $+5a$. The *plus* sign before the remainder will show that the minuend is *greater* than the subtrahend; the *minus* sign before the remainder will show that the *minuend* is *less* than the subtrahend.

4. This method enables us to give a simple explanation to each of the eight possible cases in the subtraction of monomials. It will be well to have the pupils explain each of the cases given below:

(1.)	(2.)	(3.)	(4.)	(5.)	(6.)	(7.)	(8.)
$7a$	$4a$	$-7a$	$-4a$	$-7a$	$7a$	$4a$	$-4a$
$4a$	$7a$	$-4a$	$-7a$	$4a$	$-4a$	$-7a$	$7a$
$3a$	$-3a$	$-3a$	$+3a$	$-11a$	$11a$	$11a$	$-11a$

MULTIPLICATION.

73. **Multiplication** is the process of taking one quantity as many times as there are units in another.

74. The **Multiplicand** is the quantity to be multiplied.

75. The **Multiplier** is the quantity by which we multiply.

76. The **Product** is the result obtained by multiplying.

77. The Multiplicand and Multiplier are called *factors* of the product.

NOTE.—The symbol × was introduced by *Wm. Oughtred*, an English mathematician, born in 1574.

PRINCIPLES.

1. *The product of two or more quantities is the same in whatever order the factors are arranged.*

Thus, a times b is the same as b times a, as may be seen by assigning special values to the letters; and the same is true of any number of quantities.

2. *Multiplying any factor of a quantity multiplies the quantity.*

Thus, 4 times the quantity 2×3 equals $4\times2\times3$, which is 8×3, or $2\times3\times4$, which is 2×12. Thus, also, 3 times $2a$ is $6a$; 4 times $3ab$ is $12ab$.

3. *The exponent of a quantity in the product is equal to the sum of its exponents in the two factors.*

Thus, $a^2\times a^3$ equals a^5, since a used as a factor *twice*, multiplied by a used as a factor *three* times, equals a used as a factor *five* times. It may also be seen thus: $a^2\times a^3=aa\times aaa$, which equals $aaaaa$, which equals a^5.

4. *The product of two factors having* LIKE *signs is positive, and the product of two factors having* UNLIKE *signs is negative.*

To prove this, multiply b by a; $-b$ by a; b by $-a$, and $-b$ by $-a$.

First, $+b$, taken any number of times, as a times, is evidently $+ab$.

Second, $-b$ taken once is $-b$; taken twice, is $-2b$, etc.; hence, $-b$, taken any number of times, as a times, is $-ab$.

OPERATION.

$+b$	$-b$	$+b$	$-b$
$+a$	$+a$	$-a$	$-a$
$+ab$	$-ab$	$-ab$	$-(-ab)$
			$=+ab$

Third, b multiplied by $-a$ means that b is to be taken *subtractively* a times; b taken a times is ab, and taken *subtractively* is $-ab$.

Fourth, $-b$ multiplied by $-a$ means that $-b$ is to be taken *subtractively* a times; $-b$ taken a times is $-ab$, and used *subtractively* is $-(-ab)$, which by the principles of subtraction is $+ab$.

Hence we infer that *the product of quantities having* LIKE *signs is* PLUS, *and having* UNLIKE *signs is* MINUS.

CASE I.

78. To multiply a monomial by a monomial.

1. Multiply $3b$ by $2a$.

SOLUTION. To multiply $3b$ by $2a$, we multiply by 2 and by a. $3b$ multiplied by 2 and by a equals $3b \times 2 \times a$, which, since the product is the same in whatever order the factors are placed, equals $3 \times 2 \times a \times b$, which equals $6ab$. Therefore, $3b$ multiplied by $2a$ is $6ab$.

OPERATION.

$$\begin{array}{r} 3b \\ 2a \\ \hline 6ab \end{array}$$

2. Multiply $4a^3$ by $3a^2$.

SOLUTION. To multiply $4a^3$ by $3a^2$, we may multiply one factor by 3 and the other factor by a^2 (Prin. 2). 3 times 4 are 12, and a^2 times a^3 is a^5 (Prin. 3). Therefore $4a^3$ multiplied by $3a^2$ equals $12a^5$.

OPERATION

$$\begin{array}{r} 4a^3 \\ 3a^2 \\ \hline 12a^5 \end{array}$$

Rule.—I. *Multiply the coefficients of the two factors together.*

II. *To this product annex all the letters of both factors, giving each letter an exponent equal to the sum of its exponents in the two factors.*

III. *Make the product positive when the factors have like signs, and negative when they have unlike signs.*

EXAMPLES.

(3.)	(4.)	(5.)	(6.)
$5a$	$-6a^2$	$5a^3$	$-6c^2b$
$2b$	$3a$	$4a^2$	$-3c$
$10ab$	$-18a^3$	$20a^5$	$+18c^3b$

(7.)	(8.)	(9.)	(10.)
$8ab^2$	$12ax$	$15m^2n$	$-5a^3c$
$3a^2b$	$4cx$	$-5an$	$6c^2d$
$24a^3b^3$	$48acx^2$	$-75am^2n^2$	$30a^3c^3d$

11. Multiply $7m^5n^3$ by $-5n^4x$. *Ans.* $-35m^5n^7x$.
12. Multiply $12a^2x^3y^2$ by $7a^3c^2xy$. *Ans.* $84a^5c^2x^4y^3$.
13. Multiply $-9a^3b^4c^5$ by $-7a^2b^3x^2$ *Ans.* $63a^5b^7c^5x^2$.
14. Multiply $4(a+b)^2$ by $2a$. *Ans.* $8a(a+b)^2$.
15. Multiply $-a(a-x)$ by b. *Ans.* $-ab(a-x)$.
16. Multiply $(a+b)^3$ by $(a+b)^2$. *Ans.* $(a+b)^5$.
17. Multiply $a(x-y)^2$ by $2(x-y)$. *Ans.* $2a(x-y)^3$.
18. Multiply $-5x(m-n)^3$ by $-3x(m-n)^5$. *Ans.* $15x^2(m-n)^8$.
19. Multiply a^m by a^n. *Ans.* a^{m+n}.
20. Multiply b^{2n} by b^n. *Ans.* b^{3n}.
21. Multiply c^m by c^2. *Ans.* c^{m+2}.
22. Multiply d^{3n} by d^{n+2}. *Ans.* d^{4n+2}.
23. Multiply $(a-x)^m$ by $(a-x)^{-n}$. *Ans.* $(a-x)^{m-n}$.
24. Multiply $-3a^2(l^2-m)^n$ by $-2a^3(l^2-m)^{-3}$. *Ans.* $6a^5(l^2-m)^{n-3}$.

CASE II.

79. To multiply a polynomial by a monomial.

1. Multiply $a-b$ by c.

SOLUTION. To multiply $a-b$ by c we must multiply each term by c. c times a is ac, and c times $-b$ is $-bc$. Hence, $a-b$ multiplied by c is $ac-bc$.

OPERATION.

$$\begin{array}{l} a-b \\ c \\ \hline ac-bc \end{array}$$

Rule.—*Multiply each term of the multiplicand by the multiplier, and connect the products by their proper signs.*

EXAMPLES.

(2.)	(3.)	(4.)
$7a^2-3b$	$6ax-5c^2y$	$3m^6-4n^3+7$
$3a$	$3ac$	$-2mn$
$21a^3-9ab$	$18a^2cx-15ac^3y$	$-6m^7n+8mn^4-14mn$

(5.)	(6.)	(7.)
$4a^n-3ab^n$	$3c^m-4bc+5d^n$	$4x^n-5xy^n$
$2a^n$	$3cd$	$3x^3y^{-2}$
$8a^{2n}-6a^{n+1}b^n$	$9c^{m+1}d-12bc^2d+15cd^{n+1}$	$12x^{n+3}y^{-2}-15x^4y^{n-2}$

8. Multiply $5ax^2-3x^3y$ by $-6a^2x$. *Ans.* $-30a^3x^3+18a^2x^4y$.
9. Multiply $11m^2-3$ by -5. *Ans.* $-55m^2+15$.
10. Multiply $a^2-2ab+b^2$ by ab. *Ans.* $a^3b-2a^2b^2+ab^3$.
11 Multiply $3a^n-4b^m$ by $a^{2n}b^{3m}$. *Ans.* $3a^{3n}b^{3m}-4a^{2n}b^{4m}$.
12. Multiply $a^{n-1}b-b^{n-2}c$ by ab^2. *Ans.* $a^nb^3-ab^nc$.
13. Multiply $5x^3-7m^3x+3\frac{1}{2}$ by $4mx$. *Ans.* $20mx^4-28m^4x^2+14mx$.

CASE III.

80. To multiply a polynomial by a polynomial.

1. Multiply $2a-b$ by $a+2b$.

SOLUTION. $a+2b$ times $2a-b$ equals a times $2a-b$ plus $2b$ times $2a-b$. a times $2a-b$ equals $2a^2-ab$; $2b$ times $2a-b$ equals $4ab-2b^2$. Adding the partial products, we have $2a^2+3ab-2b^2$. Therefore, etc.

OPERATION.

$$\begin{array}{r} 2a-b \\ a+2b \\ \hline 2a^2-ab \\ +4ab-2b^2 \\ \hline 2a^2+3ab-2b^2 \end{array}$$

Rule.—*Multiply each term of the multiplicand by each term of the multiplier, and add the partial products.*

EXAMPLES.

(2.)
$$\begin{array}{r} 2a-3b \\ a-b \\ \hline 2a^2-3ab \\ -2ab+3b^2 \\ \hline 2a^2-5ab+3b^2 \end{array}$$

(3.)
$$\begin{array}{r} a+b \\ a+b \\ \hline a^2+ab \\ +ab+b^2 \\ \hline a^2+2ab+b^2 \end{array}$$

(4.)
$$\begin{array}{r} a-b \\ a+b \\ \hline a^2-ab \\ +ab-b^2 \\ \hline a^2 \quad -b^2 \end{array}$$

(5.)
$$\begin{array}{r} a^2+ab+b^2 \\ a-b \\ \hline a^3+a^2b+ab^2 \\ -a^2b-ab^2-b^3 \\ \hline a^3 \qquad -b^3 \end{array}$$

(6.)
$$\begin{array}{r} a^n-b^n \\ a^2-b^2 \\ \hline a^{n+2}-a^2b^n \\ -a^nb^2+b^{n+2} \\ \hline a^{n+2}-a^nb^2-a^2b^n+b^{n+2} \end{array}$$

7. Multiply $3a-2b$ by $2a-3b$. *Ans.* $6a^2-13ab+6b^2$.
8. Multiply a^2-b^2 by a^2+b^2. *Ans.* a^4-b^4.

9. Multiply $3x-6y$ by $2x+4y$. *Ans.* $6x^2-24y^2$.
10. Multiply c^2+cd+d^2 by $c-d$. *Ans.* c^3-d^3.
11. Multiply x^3+y^3 by x^3-y^3. *Ans.* x^6-y^6.
12. Multiply $4x-3y$ by $4x+3y$. *Ans.* $16x^2-9y^2$.
13. Multiply $x^4-x^3z+x^2z^2-xz^3+z^4$ by $x+z$. *Ans.* x^5+z^5.
14. Multiply $a^{n-2}-b^{n-2}$ by a^2+b^2. *Ans.* $a^n-a^2b^{n-2}+a^{n-2}b^2-b^n$.
15. Multiply $a^2x^3+x^2y^3$ by $a^2x^3-x^2y^3$. *Ans.* $a^4x^6-x^4y^6$.
16. Multiply $x^{\frac{1}{2}}-y^{\frac{1}{2}}$ by $x^{\frac{1}{2}}+y^{\frac{1}{2}}$. *Ans.* $x-y$.
17. Multiply $3\frac{1}{2}a^2+5\frac{1}{2}c^3$ by $2a^2+4c^3$. *Ans.* $7a^4+25a^2c^3+22c^6$.
18. Multiply c^2+cd-d^2 by $c-d$. *Ans.* $c^3-2cd^2+d^3$.
19. Multiply $a^2-3ab+4ab^2$ by $a^2+3ab-4ab^2$. *Ans.* $a^4-9a^2b^2+24a^2b^3-16a^2b^4$.
20. Multiply n^2+np+p^2 by n^2-np+p^2. *Ans.* $n^4+n^2p^2+p^4$.
21. Multiply $a^2+2ab+b^2$ by $a^2-2ab+b^2$. *Ans.* $a^4-2a^2b^2+b^4$.
22. Multiply a^m+b^n by a^m-b^n. *Ans.* $a^{2m}-b^{2n}$.
23. Multiply a^n-b^m by a^n-b^m. *Ans.* $a^{2n}-2a^nb^m+b^{2m}$.
24. Multiply $m^3+m^2n+mn^2+n^3$ by $m-n$. *Ans.* m^4-n^4.
25. Multiply $a^3+3a^2b+3ab^2+b^3$ by $a^3-3a^2b+3ab^2-b^3$. *Ans.* $a^6-3a^4b^2+3a^2b^4-b^6$.
26. Multiply $a^4-a^3+a^2-a+1$ by $a+1$. *Ans.* a^5+1.
27. Multiply $1+c$, $1-c$, $1+c+c^2$ and $1-c+c^2$. *Ans.* $1-c^6$.

EXPANDING EXPRESSIONS.

81. An algebraic expression is *expanded* when the multiplication indicated is performed.

1. Expand $(a-x)(a-x)$. *Ans.* $a^2-2ax+x^2$.
2. Expand $(2a^2-3b^n)(3a^2+4b^n)$. *Ans.* $6a^4-a^2b^n-12b^{2n}$.
3. Expand $(a^n+b^n)(a^m+b^m)$. *Ans.* $a^{m+n}+a^mb^n+a^nb^m+b^{m+n}$.
4. Expand $(a-2)(a-3)(a+2)(a+3)$. *Ans.* a^4-13a^2+36.
5. Expand $(a+b)(a-b)(a+b)(a-b)$. *Ans.* $a^4-2a^2b^2+b^4$.
6. Expand $(1+a)(1+a^4)(1-a+a^2-a^3)$. *Ans.* $1-a^8$.
7. Expand $(a^2+a+1)(a^2+a+1)(a-1)(a-1)$. *Ans.* a^6-2a^3+1.

DIVISION.

82. Division is the process of finding how many times one quantity is contained in another.

83. The **Dividend** is the quantity to be divided.

84. The **Divisor** is the quantity by which we divide.

85. The **Quotient** is the result obtained by the division.

86. The **Remainder** is the quantity which is sometimes left after dividing.

NOTE.—The symbol of division, $\div$, was introduced by *Dr. John Pell*, an English mathematician, born in 1610.

PRINCIPLES.

1. *Taking a factor out of a quantity divides the quantity by that factor.*

Thus, taking the factor a out of $4ab$, we have $4b$, which is the quotient of $4ab$ divided by a, since $4b$ multiplied by a is $4ab$.

2. *The exponent of a quantity in the quotient equals its exponent in the dividend diminished by its exponent in the divisor.*

Thus, a^6 divided by a^4 equals a^{6-4}, or a^2; since a^4 multiplied by a^2 equals a^{4+2}, or a^6.

3. *The quotient is* POSITIVE *when the dividend and divisor have* LIKE *signs, and* NEGATIVE *when they have* UNLIKE *signs.*

Thus, $+ab \div +b = +a$, since $+a \times +b = +ab$;
$-ab \div -b = +a$, since $+a \times -b = -ab$;
$+ab \div -b = -a$, since $-a \times -b = +ab$;
$-ab \div +b = -a$, since $-a \times +b = -ab$.

87. This principle, and the corresponding one in multiplication, may be briefly stated thus:

LIKE *signs give* PLUS, and UNLIKE *signs give* MINUS.

CASE I.

88. To divide a monomial by a monomial.

1. Divide $12ab$ by $4a$.

SOLUTION. To divide $12ab$ by $4a$, we divide by 4 and a. Dividing $12ab$ by 4 and a, by taking out the factors 4 and a (Prin. 1), we have $3b$. Hence, $12ab$ divided by $4a$ equals $3b$.

OPERATION.
$$12ab \div 4a = 3b$$

2. Divide $20a^5b$ by $5a^3$.

SOLUTION. To divide $20a^5b$ by $5a^3$, we divide by 5 and a^3. Dividing 20 by 5, we have 4; dividing a^5 by a^3, we have a^2 (Prin. 2). Hence the quotient is $4a^2b$.

OPERATION.
$$20a^5b \div 5a^3 = 4a^2b$$

Rule.—I. *Divide the coefficient of the dividend by the coefficient of the divisor for the coefficient of the quotient.*

II. *Write the letters of the dividend in the quotient, giving each an exponent equal to its exponent in the dividend minus its exponent in the divisor.*

III. *Make the quotient positive when the two terms have like signs, and negative when they have unlike signs.*

NOTE.—An equal literal factor in dividend and divisor is suppressed in the quotient, since it is canceled by the division.

EXAMPLES.

3. Divide $12x^3$ by $4x^2$. *Ans.* $3x$.
4. Divide $20ab^3$ by $5b^2$. *Ans.* $4ab$.
5. Divide $4abc^2$ by $2ac$. *Ans.* $2bc$.
6. Divide $8a^3b^2$ by $4ab^2$. *Ans.* $2a^2$.
7. Divide $9m^2n^5$ by $3mn^2$. *Ans.* $3mn^3$.
8. Divide $-15ay^3$ by $3ay$. *Ans.* $-5y^2$.
9. Divide $-ab^5c^3$ by ab^2c. *Ans.* $-b^3c^2$.
10. Divide $16a^7b^5$ by $-8a^3b^4$. *Ans.* $-2a^4b$.
11. Divide $-24x^6z$ by $-6x^3$. *Ans.* $4x^3z$.

12. Divide $-15c^7d^8$ by $-5c^2d^5$. *Ans.* $3c^5d^3$.

13 Divide $14a^3b^5c$ by $7b^3c$. *Ans.* $2a^3b^2$.

14. Divide a^m by a^n. *Ans.* a^{m-n}.

15. Divide a^{m+n} by a^n. *Ans.* a^m.

16. Divide a^{m-n} by a^n. *Ans.* a^{m-2n}.

17. Divide a^{2m} by a^n. *Ans.* a^{2m-n}.

18. Divide $14a^{m+n}$ by $-2a^{m-n}$. *Ans.* $-7a^{2n}$.

19. Divide $-24a^{p+1}$ by $6a^{p-1}$. *Ans.* $-4a^2$.

20. Divide $a^{3m}b^{4n}$ by a^mb^n. *Ans.* $a^{2m}b^{3n}$.

21. Divide $84a^5c^6$ by $7a^nc^x$. *Ans.* $12a^{5-n}c^{6-x}$.

22. Divide $(a+b)^5$ by $(a+b)^3$: *Ans.* $(a+b)^2$.

23. Divide $(a-c)^{2n}$ by $(a-c)^n$. *Ans.* $(a-c)^n$.

24. Divide $12a^2b^3(a-b)^{n+2}$ by $4b^5(a-b)^{n-2}$.
Ans. $3a^2b^{-2}(a-b)^4$.

CASE II.

89. To divide a polynomial by a monomial,

1. Divide $8a^4-16a^3b+12a^2c^5$ by $4a^2$.

SOLUTION. $4a^2$ is contained in $8a^4$, $2a^2$ times; $4a^2$ is contained in $-16a^3b$, $-4ab$ times; $4q^2$ is contained in $12a^2c^5$, $3c^5$ times. Hence, the quotient is $2a^2-4ab+3c^5$.

OPERATION.

$$4a^2)\underline{8a^4-16a^3b+12a^2c^5}$$
$$2a^2-4ab+3c^5$$

Rule.—*Divide each term of the dividend by the divisor and connect the results by their proper signs.*

EXAMPLES.

2. Divide $6a^3b-9ab^4$ by $3ab$. *Ans.* $2a^2-3b^3$.

3. Divide $8a^4c-12a^2c^5$ by $4ac$. *Ans.* $2a^3-3ac^4$.

4. Divide $9ab^5-15ab^3c^2$ by $3ab^3$. *Ans.* $3b^2-5c^2$.

5. Divide $6a^5c^2-12a^8c$ by $3a^4c$. *Ans.* $2ac-4a^4$.

6. Divide $abc-5a^2b^3c^4$ by abc. *Ans.* $1-5ab^2c^3$.

7. Divide $4a^{2n}-8a^{3n}$ by $2a^n$. *Ans.* $2a^n-4a^{2n}$.

8. Divide $16ab^3x^3-20b^2x^5z$ by $4b^2x^3$. *Ans.* $4ab-5x^2z$.

9. Divide $18a^5x^6-27a^6x^8-9a^3x^6$ by $9a^3x^6$.
Ans. $2a^2-3a^3x^2-1$.

10. Divide $-16x^3+24z^5-48$ by -8. *Ans.* $2x^3-3z^5+6$.

11. Divide $12a^3b-18a^2b^3+6a^2b$ by $6a^{-2}b$.
Ans. $2a^5-3a^4b^2+a^4$.

12. Divide $a^{2n}b^3-a^nb^4+a^{n+2}b^3$ by a^nb^3. *Ans.* a^n-b+a^2.

13. Divide $a^nb^m-a^{n+3}b^{m-2}$ by a^nb^m. *Ans.* $1-a^3b^{-2}$.

14. Divide $16(a-x)-24(c-z)$ by 8.
Ans. $2(a-x)-3(c-z)$.

15. Divide $7(m+n)-14a(m+n)$ by $(m+n.)$ *Ans.* $7-14a$.

16. Divide $(r-s)^2-(r-s)^4$ by $(r-s)$. *Ans.* $(r-s)-(r-s)^3$.

17. Divide $a(c-d)-b(c-d)$ by $c-d$. *Ans.* $a-b$.

18. Divide $4a(a-c)+(a-c)^2$ by $a-c$.
Ans. $4a+(a-c)$, or $5a-c$.

19. Divide $5x(x-y)^2-2(x-y)^3$ by $(x-y)^2$.
Ans. $5x-2(x-y)$ or $3x+2y$.

20. Divide $2a(1+c)^3-2ac(1+c)^2$ by $(1+c)^2$. *Ans.* $2a$.

CASE III.

90. To divide a polynomial by a polynomial.

1. Divide $a^2+2ab+b^2$ by $a+b$.

SOLUTION. We write the divisor at the right of the dividend, both being arranged according to the powers of a, and commence at the left to divide. Since the first term of the dividend must equal the product of the first terms of the divisor and quotient, we divide the first term of the dividend by the first term of the divisor for the first term of the quotient. a is contained in a^2, a times; a times $a+b$ equals a^2+ab. Subtracting and bringing down the next term of the dividend, we have $ab+b^2$.

OPERATION.

$$\begin{array}{l|l} a^2+2ab+b^2 & a+b \\ \hline \underline{a^2+ab} & a+b \\ \quad\ ab+b^2 & \\ \quad\ \underline{ab+b^2} & \end{array}$$

Since the first term of this new dividend must be the product of the first term of the divisor by the second term of the quotient, we divide it by the first term of the divisor. a is contained in ab, b times; b times $a+b$ is is $ab+b^2$. Subtracting, nothing remains. Hence the quotient is $a+b$.

Rule.—I. *Write the divisor at the right of the dividend, arranging both according to the powers of one of the letters.*

II. *Divide the first term of the dividend by the first term of the divisor, and write the result in the quotient; multiply the divisor by it, and subtract the product from the dividend.*

III. *Regard the remainder as a new dividend and proceed as before, and thus continue until the first term of the divisor is not contained in the first term of the dividend.*

NOTES.—1. When the first term of the arranged dividend is not divisible by the first term of the divisor, the division will not be exact.

2. Bring down no more terms of the dividend each time than are needed for use.

EXAMPLES.

2. Divide $a^3+2a^2b+2ab^2+b^3$ by $a+b$.

SOLUTION. We first arrange the dividend with reference to the powers of a, and then proceed as before. a is contained in a^3, a^2 times; a^2 times $a+b$ is a^3+a^2b. Subtracting and bringing down the next term, we have a^2b+2ab^2 for the next dividend, etc.

OPERATION

$$\begin{array}{l|l} a^3+2a^2b+2ab^2+b^3 & a+b \\ \underline{a^3+\ \ a^2b} & a^2+ab+b^2 \\ \quad a^2b+2ab^2 & \\ \quad \underline{a^2b+\ \ ab^2} & \\ \qquad ab^2+b^3 & \\ \qquad \underline{ab^2+b^3} & \end{array}$$

(3.)

$$\begin{array}{l|l} a^3+x^3 & a+x \\ \underline{a^3+a^2x} & a^2-ax+x^2 \\ \ -a^2x+x^3 & \\ \ \underline{-a^2-ax^2} & \\ \qquad ax^2+x^3 & \\ \qquad \underline{ax^2+x^3} & \end{array}$$

(4.)

$$\begin{array}{l|l} x^3-y^3 & x-y \\ \underline{x^3-x^2y} & x^2+xy+y^2 \\ \quad x^2y-y^3 & \\ \quad \underline{x^2y-xy^2} & \\ \qquad xy^2-y^3 & \\ \qquad \underline{xy^2-y^3} & \end{array}$$

5. Divide $a^2-2ax+x^2$ by $a-x$. *Ans.* $a-x$.
6. Divide $a^2-ax-6x^2$ by $a+2x$. *Ans.* $a-3x$.
7. Divide $a^3-ax^2+ax+x^2$ by $a+x$. *Ans.* a^2-ax+x.
8. Divide $m^3+2m^2n-mn^2-2n^3$ by $m+2n$. *Ans.* m^2-n^2.
9. Divide $a^3-3a^2b+3ab^2-b^3$ by $a-b$. *Ans.* $a^2-2ab+b^2$.
10. Divide a^3-b^3 by $a-b$. *Ans.* a^2+ab+b^2.
11. Divide $x^3-9x^2+27x-27$ by $x-3$. *Ans.* x^2-6x+9.

12. Divide m^3-n^3 by m^2+mn+n^2. *Ans.* $m-n$.
13. Divide a^3-1 by $a-1$. *Ans.* a^2+a+1.
14. Divide $8x^3-27y^3$ by $2x-3y$. *Ans.* $4x^2+6xy+9y^2$.
15. Divide a^4-x^4 by $a-x$. *Ans.* $a^3+a^2x+ax^2+x^3$.
16. Divide $a^4+2a^2b^2+9b^4$ by $a^2-2ab+3b^2$. *Ans.* $a^2+2ab+3b^2$.
17. Divide $a^4+a^2c^2+c^4$ by a^2-ac+c^2. *Ans.* a^2+ac+c^2.
18. Divide $x^2+2xy+y^2-z^2$ by $x+y-z$. *Ans.* $x+y+z$.
19. Divide m^4-n^4 by m^2+n^2. *Ans.* m^2-n^2.
20. Divide a^4-1 by $a-1$. *Ans.* a^3+a^2+a+1.
21. Divide $a^{2n}-b^{2n}$ by a^n-b^n. *Ans.* a^n+b^n.
22. Divide m^5-n^5 by $m-n$. *Ans.* m^4+m^3n+, etc.
23. Divide s^4-t^4 by $s^3+s^2t+st^2+t^3$. *Ans.* $s-t$.
24. Divide $a^{2n}+2a^nb^n+b^{2n}$ by a^n+b^n. *Ans.* a^n+b^n.
25. Divide $27x^3-64y^3$ by $3x-4y$. *Ans.* $9x^2+12xy+16y^2$.
26. Divide x^5+1 by $x+1$. *Ans.* $x^4-x^3+x^2-x+1$.
27. Divide $1-z^5$ by $1-z$. *Ans.* $1+z+z^2+z^3+z^4$.
28. Divide $a^{3n}-b^{3n}$ by a^n-b^n. *Ans.* $a^{2n}+a^nb^n+b^{2n}$.
29. Divide a^6-b^6 by $a^3+2a^2b+2ab^2+b^3$. *Ans.* $a^3-2a^2b+2ab^2-b^3$.
30. Divide $(a-x)^2-(x-y)^2$ by $(a-x)-(x-y)$. *Ans.* $(a-x)+(x-y)$, or $a-y$.

PRINCIPLES OF DIVISION.

91. The **Principles** of Division are the truths which relate to the process. They are of three kinds—*Changes of Terms, Zero Exponents* and *Negative Exponents.*

CHANGES OF TERMS.

92 The **Terms** may be changed both in *value* and in *sign.*

PRIN. 1. *Multiplying the dividend or dividing the divisor multiplies the quotient.*

Thus, $abcd \div ab = cd$. Multiplying the dividend by e, we have $abcde \div ab = cde$, which is the quotient, cd, multiplied by e. Dividing the divisor by b, we have $abcd \div a = bcd$, which is cd multiplied by b. Therefore, etc.

PRIN. 2. *Dividing the dividend or multiplying the divisor divides the quotient.*

Thus, $abcd \div ab = cd$. Dividing the dividend by d, we have $abc \div ab = c$, which equals the quotient, cd, divided by d. Multiplying the divisor by c, we have $abcd \div abc = d$, which also equals cd divided by c. Therefore, etc.

PRIN. 3. *Multiplying or dividing both dividend and divisor by the same quantity does not change the quotient.*

For, multiplying the dividend multiplies the quotient, and multiplying the divisor divides the quotient; hence, multiplying both dividend and divisor by the same quantity both multiplies and divides the quotient by that quantity, and hence does not change its value. Therefore, etc.

PRIN. 4. *Changing the sign of either dividend or divisor changes the sign of the quotient.*

If two terms have like signs, the quotient is *positive;* and if the sign of either term be changed, they will have *unlike* signs; hence the quotient will be changed from *plus* to *minus.*

If two terms have unlike signs, the quotient is *negative;* and if the sign of either term be changed, they will have *like* signs; hence the quotient will be changed from *minus* to *plus.* Therefore, etc.

PRIN. 5. *Changing the sign of both dividend and divisor does not change the sign of the quotient.*

For, if the signs of the terms are alike, when both are changed they will still be alike, and the quotient will remain *plus.* If the signs are unlike, when both are changed they will still be unlike, and the quotient will remain *minus.*

ZERO AND NEGATIVE EXPONENTS.

93. A **Zero Exponent** originates in dividing a quantity with any exponent by the same quantity with the same exponent. Thus, a^4 divided by a^4 equals a^{4-4}, or a^0.

PRIN. 1. *Any quantity whose exponent is zero is equal to unity.*

For, $a^4 \div a^4 = a^0$, by subtracting the exponents; but $a^4 \div a^4 = 1$, since any quantity divided by itself equals unity; hence, since a^0 and 1 are both equal to $a^4 \div a^4$, they are equal to each other. Therefore, $a^0 = 1$.

OPERATION.

$$a^4 \div a^4 = a^0$$
$$a^4 \div a^4 = 1$$
$$\therefore a^0 = 1$$

NOTE.—When a quantity whose exponent is zero is a factor of an algebraic expression, it may be omitted without changing the value of the expression, since its value is 1. It is sometimes retained to indicate the process by which the result was obtained.

94. A **Negative Exponent** originates in subtracting exponents when the exponent of the divisor is greater than the exponent of the dividend. Thus, $a^4 \div a^6 = a^{4-6}$, or a^{-2}.

PRIN. 2. *Any quantity with a negative exponent is equal to the reciprocal of the quantity with the sign of its exponent changed.*

For $a^4 \div a^6 = a^{-2}$; but $a^4 \div a^6 = \frac{a^4}{a^6}$, or, dividing both terms by a^4, is equal to $\frac{1}{a^2}$; and since a^{-2} and $\frac{1}{a^2}$ are both equal to $a^4 \div a^6$, they are equal to each other. Therefore, $a^{-2} = \frac{1}{a^2}$.

OPERATION.

$$a^4 \div a^6 = a^{-2}$$
$$a^4 \div a^6 = \frac{a^4}{a^6} = \frac{1}{a^2}$$
$$\therefore a^{-2} = \frac{1}{a^2}$$

PRIN. 3. *Any quantity is equal to the reciprocal of itself with the sign of its exponent changed.*

To prove this we must show that $a^n = \frac{1}{a^{-n}}$. By Prin. 2 we have $a^{-n} = \frac{1}{a^n}$; multiplying by a^n, we have $a^n \times a^{-n} = 1$; dividing by a^{-n}, we have $a^n = \frac{1}{a^{-n}}$.

OPERATION.

$$a^{-n} = \frac{1}{a^n}$$
$$a^n \times a^{-n} = 1$$
$$a^n = \frac{1}{a^{-n}}$$

NOTE.—It has already been seen that a *positive quantity* signifies *addition*, and a *negative* quantity *subtraction.* From the above principles it is also seen that a *positive exponent* implies *multiplication*, and a *negative exponent* implies *division.* Hence, a *negative* sign always denotes the *opposite* of a *positive* sign.

EXAMPLES.

1. Show by dividing a^5 by a^5 that $a^0 = 1$.
2. Show by dividing a^n by a^n that $a^0 = 1$.
3. Prove that $a^{-3} = \frac{1}{a^3}$.
4. Prove that $\frac{1}{a^5} = a^{-5}$.

5. Prove that $a^{-n}=\frac{1}{a^n}$

6. Find the value of a^3c^{-3}. *Ans.* $\frac{a^3}{c^3}$

7. Find the value of $\frac{ab}{cx^{-4}}$. *Ans.* $\frac{abx^4}{c}$

8. Reduce to an integer $\frac{1}{a^{n-1}}$. *Ans.* a^{1-n}.

9. Reduce to an integer $\frac{1}{a^{n-m}}$. *Ans.* a^{m-n}.

10. Multiply $a^{-3}b^4c^{-5}$ by $a^2b^{-2}c^3$. *Ans.* $\frac{b^2}{ac^2}$.

11. Divide $18a^{-6}b^3c^2$ by $3a^{-4}b^{-2}$. *Ans.* $6\frac{b^5c^2}{a^2}$.

12. Divide $a^{-3n}-b^{6n}$ by $a^{-n}-b^{2n}$. *Ans.* $a^{-2n}+a^{-n}b^{2n}+b^{4n}$.

NOTE.—The pupil will readily infer that *negative exponents* may be used in operations in the *same way as positive exponents*; a generalization which is rigidly demonstrated in the *Theory of Exponents.*

REVIEW QUESTIONS.

NOTE.—These REVIEW QUESTIONS are simply suggestive to the teacher, who can extend them as fully as is deemed desirable.

Define Addition. Sum. State the principles. The cases. The rule for each. Is the sum of two quantities ever less than the greater? When?

Define Subtraction. Minuend. Subtrahend. Remainder. State the principles of Subtraction. The cases. The rule for each. The principles of the parenthesis. Is the difference ever greater than the minuend? When?

Define Multiplication. Multiplicand. Multiplier. Product. State the principles of Multiplication. The cases. The rule for each. What is meant by *expanding expressions?* Why do we add exponents in multiplying? Why does plus into minus give minus? Why does minus into minus give plus?

Define Division. Dividend. Divisor. Remainder. State the principles. The cases. The rules. Why do we subtract exponents? Why does plus divided by minus give minus? Why does minus divided by minus give plus?

State the principles of the *Changes of Terms.* Of Zero exponent. Of Negative exponent. Origin of a Zero exponent. Of a Negative exponent.

SECTION III.

COMPOSITION AND FACTORING.

95. Composition is the process of forming *composite quantities.*

96. A **Composite Quantity** is one that is formed by the product of two or more quantities.

97. The **Square** of a quantity is the product obtained by using the quantity twice as a factor.

Composite quantities may be formed by actual multiplication, or, in several cases, by means of the following theorems.

NOTE.—In the fundamental operations each synthetic process has its corresponding analytic process. Thus, Addition is synthetic; Subtraction is analytic; Multiplication is synthetic; Division is analytic. It follows, therefore, that there should be a synthetic process corresponding to the analytic process of Factoring. This process I have called *Composition.* This new generalization, and the term applied to it, will, I trust, meet the approval of teachers and mathematicians.

THEOREM I.

The square of the sum of two quantities equals the square of the first, plus twice the product of the first and second, plus the square of the second.

Let a represent one of the quantities, and b the other; then $a+b$ will represent their sum. Now, $(a+b)^2$ equals $(a+b)(a+b)$, which we find by multiplying equals $a^2+2ab+b^2$. a^2 is the square of the first; $2ab$ is twice the product of the first and second; and b^2 is the square of the third. Therefore, etc.

OPERATION.

$$\begin{array}{r} a+b \\ a+b \\ \hline a^2+ab \\ ab+b^2 \\ \hline a^2+2ab+b^2 \end{array}$$

EXAMPLES.

1. Square $x+y$. *Ans.* $x^2+2xy+y^2$.
2. Square $m+n$. *Ans.* $m^2+2mn+n^2$.
3. Square $2a+3b$. *Ans.* $4a^2+12ab+9b^2$.

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5. Prove that $a^{-n}=\frac{1}{a^n}$

6. Find the value of a^2c^{-3}. *Ans.* $\frac{a^2}{c^3}$

7. Find the value of $\frac{ab}{cx^{-4}}$ *Ans.* $\frac{abx^4}{c}$

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9. Reduce to an integer $\frac{1}{a^{n-m}}$ *Ans.* a^{m-n}.

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OPERATION.

$$\begin{array}{r} a+b \\ a+b \\ \hline a^2+ab \\ ab+b^2 \\ \hline a^2+2ab+b^2 \end{array}$$

EXAMPLES.

1. Square $x+y$. *Ans.* $x^2+2xy+y^2$.
2. Square $m+n$. *Ans.* $m^2+2mn+n^2$.
3. Square $2a+3b$. *Ans.* $4a^2+12ab+9b^2$.

4. Square $3x+4y$. *Ans.* $9x^2+24xy+16y^2$.
5. Square $4p+6q$. *Ans.* $16p^2+48pq+36q^2$.
6. Square $A+B$. *Ans.* $A^2+2AB+B^2$.
7. Square x^2+y^2. *Ans.* $x^4+2x^2y^2+y^4$.
8. Square a^m+b^m. *Ans.* $a^{2m}+2a^mb^m+b^{2m}$.

THEOREM II.

The square of the difference of two quantities equals the square of the first, minus twice the product of the first and second, plus the square of the second.

Let $a-b$ represent the difference of two quantities; then $(a-b)^2$ will equal $(a-b)(a-b)$, which, by multiplying, we find is equal to $a^2-2ab+b^2$. a^2 is the square of the first; $2ab$ is twice the product of the first and second; and b^2 is the square of the second. Therefore, etc.

OPERATION.

$$\begin{array}{r} a-b \\ a-b \\ \hline a^2-ab \\ -ab+b^2 \\ \hline a^2-2ab+b^2 \end{array}$$

EXAMPLES.

1. Square $a-x$. *Ans.* $a^2-2ax+x^2$.
2. Square $c-d$. *Ans.* $c^2-2cd+d^2$.
3. Square $1-c$. *Ans.* $1-2c+c^2$.
4. Square a^2-3c. *Ans.* $a^4-6a^2c+9c^2$.
5. Square $3b^2-5d^3$. *Ans.* $9b^4-30b^2d^3+25d^6$.
6. Square $A-B$. *Ans.* $A^2-2AB+B^2$.
7. Square $4ax-2b^2$. *Ans.* $16a^2x^2-16ab^2x+4b^4$.
8. Square $7a^n-9b^n$. *Ans.* $49a^{2n}-126a^nb^n+81b^{2n}$.

THEOREM III.

The product of the sum and difference of two quantities equal the difference of their squares.

Let $a+b$ represent the sum and $a-b$ the difference of two quantities; then by multiplying we find their product to be a^2-b^2. a^2 is the square of the first, and b^2 is the square of the second. Therefore, etc.

OPERATION

$$\begin{array}{r} a+b \\ a-b \\ \hline a^2+ab \\ -ab-b^2 \\ \hline a^2 \qquad -b^2 \end{array}$$

EXAMPLES.

1. Expand $(c+d)(c-d)$. *Ans.* c^2-d^2.
2. Expand $(m+n)(m-n)$. *Ans.* m^2-n^2.
3. Expand $(3a+2b)(3a-2b)$. *Ans.* $9a^2-4b^2$.
4. Expand $(5x-6y)(5x+6y)$. *Ans.* $25x^2-36y^2$.
5. Expand $(A+B)(A-B)$. *Ans.* A^2-B^2.
6. Expand $(1+3x)(1-3x)$. *Ans.* $1-9x^2$.
7. Expand $(4a^2-\frac{1}{3}c^3)(4a^2+\frac{1}{3}c^3)$. *Ans.* $16a^4-\frac{1}{9}c^6$.
8. Expand $(a^m+b^n)(a^m-b^n)$. *Ans.* $a^{2m}-b^{2n}$.
9. Expand $(a^{2m}+b^{3n})(a^{2m}-b^{3n})$. *Ans.* $a^{4m}-b^{6n}$.

THEOREM IV.

The product of two binomials having a common term equals the square of the common term, plus the algebraic sum of the other two terms into the common term, and the product of the unlike terms.

Let $x+a$ and $x+b$ represent two such quantities; their product is $x^2+(a+b)x+ab$. x^2 is the square of the common term, $(a+b)x$ is the sum of the other two terms into the common term, and ab is the product of the unlike terms.

OPERATION.

$$\begin{array}{r} x+a \\ x+b \\ \hline x^2+ax \\ bx+ab \\ \hline x^2+(a+b)x+ab \end{array}$$

EXAMPLES.

1. Expand $(x+4)(x+3)$. *Ans.* $x^2+7x+12$.
2. Expand $(x+5)(x-9)$. *Ans.* $x^2-4x-45$.
3. Expand $(a+m)(a+n)$. *Ans.* $a^2+a(m+n)+mn$.
4. Expand $(a-c)(a-b)$. *Ans.* $a^2-a(b+c)+bc$.
5. Expand $(2x-7)(2x-1)$. *Ans.* $4x^2-16x+7$.
6. Expand $(5ac+a)(5ac-c)$. *Ans.* $25a^2c^2+(a-c)5ac-ac$.
7. Expand $(3a^n-5)(3a^n+12)$. *Ans.* $9a^{2n}+21a^n-60$.
8. Expand $(2x^n+3a)(2x^n-5a)$. *Ans.* $4x^{2n}-4ax^n-15a^2$.
9. Expand $(2a^{\frac{n}{2}}-3a)(2a^{\frac{n}{2}}+7a)$. *Ans.* $4a^n+8a^{\frac{n+2}{2}}-21a^2$.

APPLICATIONS.

98. These principles may be applied with great convenience in finding the product of three or more binomials.

1. What is the value of $(a+2)(a+2)(a+2)$?

OPERATION.

$$\begin{array}{r} a^2+4a+4 \\ a+2 \\ \hline a^3+4a^2+4a \\ 2a^2+8a+8 \\ \hline a^3+6a^2+12a+8 \end{array}$$

SOLUTION. By Theorem I., $(a+2)(a+2)=a^2+4a+4$, which we can write without multiplying. Then, multiplying by $a+2$, we have $a^3+6a^2+12a+8$.

EXAMPLES.

2. Expand $(a+3)(a+3)(a+3)$. *Ans.* $a^3+9a^2+27a+27$.
3. Expand $(a+c)(a-c)(a+c)$. *Ans.* $a^3+a^2c-ac^2-c^3$.
4. Expand $(x-3)(x-3)(x-4)$. *Ans.* $x^3-10x^2+33x-36$.
5. Expand $(1-x)(1-x)(1+x)$. *Ans.* $1-x-x^2+x^3$.
6. Expand $(1+x)(1-x)(1-x^2)$. *Ans.* $1-2x^2+x^4$.
7. Expand $(x+y)(x-y)(x^2+y^2)$. *Ans.* x^4-y^4.
8. Expand $(3ac-4a^2b)(3ac+4a^2b)$. *Ans.* $9a^2c^2-16a^4b^2$.
9. Expand $2^3(a^2-x^2)(a^2+x^2)$. *Ans.* $8(a^4-x^4)$.
10. Expand $(3+\overline{m-n})(3-\overline{m-n})$. *Ans.* $9-(m-n)^2$.
11. Expand $(a+\overline{b-c})(a-\overline{b-c})$. *Ans.* $a^2-b^2+2bc-c^2$.
12. Expand $(5xz-6yz)(5xz+6yz)$. *Ans.* $25x^2z^2-36y^2z^2$.
13. Expand $(a+c)(a-c)(a^2-c^2)$. *Ans.* $a^4-2a^2c^2+c^4$.
14. Expand $(x-3)(x-4)(x+5)$. *Ans.* $x^3-2x^2-23x+60$.
15. Expand $(a+1)(a-1)(a-2)(a+2)$. *Ans.* a^4-5a^2+4.
16. Expand $(a^n-b^m)(a^n+b^m)(a^{2n}+b^{2m})$. *Ans.* $a^{4n}-b^{4m}$.

FACTORING.

99. Factoring is the process of resolving a composite quantity into its factors.

100. The **Factors** of a composite quantity are the quantities which multiplied together will produce it.

101. A **Prime Quantity** is one that cannot be produced by the multiplication of other quantities; as, 11, a^2+b^2.

102. Quantities are *prime to each other* when they have no common factor; as 5 and 9, $6a^2b$ and $11cd^2$.

103. The **Square Root** of a quantity is *one* of its *two equal* factors; thus, $2ab$ is the square root of $4a^2b^2$.

NOTE.—*Composition* and *Factoring* are the converse of each other; composition is *synthetic;* factoring is *analytic;* one is the putting together of the factors to make the number, and the other is the separating of the number into its factors.

CASE I.

104. To resolve a monomial into its prime factors.

1. Find the prime factors of $6a^2b^3$.

OPERATION.

$$\begin{array}{l} 6 = 2\times 3 \\ a^2 = a\times a \\ b^3 = b\times b\times b \\ \hline 6a^2b^3 = 2\times 3aabbb \end{array}$$

SOLUTION. The factors of 6 are 2 and 3; $a^2=aa$ and $b^3=bbb$; hence, $6a^2b^3$ equals $2\times 3aabbb$.

Hence we have the following rule:

Rule.—*Resolve the numerical coefficient into its prime factors, and annex to it each letter written as many times as there are units in the exponent.*

EXAMPLES.

What are the prime factors—

2. Of $12a^3b$. *Ans.* $2\times 2\times 3aaab$.
3. Of $15a^2b^2c$. *Ans.* $3\times 5aabbc$.
4. Of $18x^3y^2z$. *Ans.* $2\times 3\times 3xxxyyz$.
5. Of $27a^{2n}b^{3n}c$. *Ans.* $3\times 3\times 3a^na^nb^nb^nb^nc$.
6. Of $105a^{n+2}b^{n-2}c$. *Ans.* $3\times 5\times 7a^naab^nb^{-1}b^{-1}c$.
7. Of $72a^nb^{2n}c^{3n}$, when $n=1$. *Ans.* $2\times 2\times 2\times 3\times 3abbccc$.
8. Of $84a^{2n}c^{3n}d^{4n}$, when $n=2$.
Ans. $2\times 2\times 3\times 7aaaaccccccdddddddd$.

CASE II.

105. To find one of the two equal factors of a monomial.

1. Find one of the two equal factors of $25a^2b^4$.

SOLUTION. Resolving $25a^2b^4$ into its factors, we have $5\times 5aabbbb$. Since there are *two* 5's, one of the equal factors will contain *one* 5; since there are *two* a's, the equal factor will contain *one* a; since there are *four* b's, the equal factor will contain *two* b's; hence the factor is $5abb$, or $5ab^2$. This, we see, is equivalent to extracting the square root of the coefficient, and dividing the exponents of the letters by 2.

OPERATION.
$$25a^2b^4 =$$
$$5\times 5aabbbb$$
$$5abb = 5ab^2$$

Rule.—*Extract the square root of the coefficient and divide the exponents of the letters by 2.*

EXAMPLES.

Find one of the two equal factors—

2. Of $4a^4b^2$. *Ans.* $2a^2b$.
3. Of $9a^4b^6$. *Ans.* $3a^2b^3$.
4. Of $16x^6y^2$. *Ans.* $4x^3y$.
5. Of $36c^8d^4$. *Ans.* $6c^4d^2$.
6. Of $\frac{1}{16}x^6y^8z^{10}$. *Ans.* $\frac{1}{4}x^3y^4z^5$.
7. Of $25a^{4n}b^{6n}$. *Ans.* $5a^{2n}b^{3n}$.
8. Of $\frac{1}{64}x^{8a}z^{12c}$. *Ans.* $\frac{1}{8}x^{4a}z^{6c}$.
9. Of $81(a+b)^4$. *Ans.* $9(a+b)^2$.
10. Of $1764(a-x)^6$. *Ans.* $42(a-x)^3$.

CASE III.

106. To resolve a polynomial into two factors one of which is a monomial.

1. Find the factors of $2ac-4ab$.

SOLUTION. We see that $2a$ is a factor common to all the terms; hence, dividing $2ac-4ab$ by $2a$ we find the other factor to be $c-2b$; hence, $2ac-4ab=2a(c-2b)$.

OPERATION.
$$2a)\underline{2ac-4ab}$$
$$c-2b$$
$$2a(c-2b)$$

Rule.—*Divide the polynomial by the greatest factor common to all the terms, enclose the quotient in a parenthesis, and prefix the divisor as a coefficient.*

EXAMPLES.

Find the factors—

2. Of $6a^2b+9a^3c$. *Ans.* $3a^2(2b+3ac)$.
3. Of $8a^3b^2-12ab^4$. *Ans.* $4ab^2(2a^2-3b^2)$.
4. Of $14ax^2z+56abx^2$. *Ans.* $7ax^2(2z+8b)$.
5. Of $ac^2-ba^2c^3+adc^4$. *Ans.* $ac^2(1-abc+dc^2)$.
6. Of $ax^2y-a^2xy^3+dx^3y$. *Ans.* $xy(ax-a^2y^2+dx^2)$.

CASE IV.

107. To resolve a trinomial into two equal binomial factors.

1. Factor $a^2+2ab+b^2$.

SOLUTION. a^2 is the square of a, and b^2 is the square of b, and since $2ab$ is twice the product of a and b, the trinomial is the square of $(a+b)$, (Theo. I.); hence, $a+b$ is one of the two equal factors of $a^2+2ab+b^2$.

OPERATION.
$$a^2+2ab+b^2=$$
$$(a+b)(a+b)$$

Rule.—*Extract the square root of the terms which are squares, and if twice the product of these roots equals the other term, these roots, connected by the sign of this other term, will be one of the equal factors.*

EXAMPLES.

Find one of the two equal factors—

2. Of $a^2-2ab+b^2$. *Ans.* $a-b$.
3. Of $x^2+2xy+y^2$. *Ans.* $x+y$.
4. Of $A^2-2AB+B^2$. *Ans.* $A-B$.
5. Of $4a^2-12ac+9c^2$. *Ans.* $2a-3c$.
6. Of $9m^2+12mn+4n^2$. *Ans.* $3m+2n$.
7. Of $1-2c^2+c^4$. *Ans.* $1-c^2$.
8. Of $16a^{2n}+40a^nc^n+25c^{2n}$. *Ans.* $4a^n+5c^n$.

CASE V.

108. To resolve a binomial consisting of the difference of two squares into its binomial factors.

1. Find the factors of a^2-b^2.

SOLUTION. The difference of the squares of two quantities equals the product of their sum and difference, (Theo. III.); hence, $a^2-b^2=(a+b)$ multiplied by $(a-b)$.

OPERATION.
$a^2-b^2=$
$(a+b)(a-b)$

Rule.—*Take the square root of each term, and make their sum one factor and their difference the other factor.*

EXAMPLES.

2. Factor a^2-c^2. Ans. $(a+c)(a-c)$.
3. Factor c^2-4d^2. Ans. $(c+2d)(c-2d)$.
4. Factor $4x^2-9y^2$. Ans. $(2x+3y)(2x-3y)$.
5. Factor $a^2x^2-b^2z^2$. Ans. $(ax+bz)(ax-bz)$.
6. Factor $\frac{1}{4}x^2-\frac{1}{9}y^2z^4$. Ans. $(\frac{1}{2}x+\frac{1}{3}yz^2)(\frac{1}{2}x-\frac{1}{3}yz^2)$
7. Factor $9a^{2n}-16c^{4n}$. Ans. $(3a^n+4c^{2n})(3a^n-4c^{2n})$.
8. Factor a^4-c^4. Ans. $(a^2+c^2)(a+c)(a-c)$.
9. Factor $x^2y^2-y^2$. Ans. $y^2(x+1)(x-1)$.
10. Factor a^8-b^8. Ans. $(a^4+b^4)(a^2+b^2)(a+b)(a-b)$.

CASE VI.

109. To resolve a trinomial into two unequal binomial factors.

1. Resolve $a^2+5ac+6c^2$ into its factors.

SOLUTION. The first term of each factor is evidently a; the second terms must be $2c$ and $3c$, since their product will be $6c^2$ and their sum $5c$. (Theo. IV.)

OPERATION.
$a^2+5ac+6c^2=$
$(a+2c)(a+3c)$

Rule.—*Take the square root of one term for the first term of each factor, and for the second term take such quantities that their product will equal the third term of the trinomial, and their sum into the first term of the factor will equal the second term of the trinomial.*

EXAMPLES.

2. Factor x^2+3x+2. Ans. $(x+1)(x+2)$.
3. Factor a^2+5a+6. Ans. $(a+2)(a+3)$.
4. Factor x^2-x-2. Ans. $(x+1)(x-2)$.
5. Factor a^2-a-2. Ans. $(a-2)(a+1)$.
6. Factor $x^2+7x+12$. Ans. $(x+3)(x+4)$.
7. Factor $a^2-3a-10$. Ans. $(a-5)(a+2)$.
8. Factor x^4-9x^2-36. Ans. $(x^2+3)(x^2-12)$.
9. Factor $4x^2-6x-40$. Ans. $(2x+5)(2x-8)$.
10. Factor $a^2+4ac-21c^2$. Ans. $(a-3c)(a+7c)$.
11. Factor $a^{2n}+5a^n-84$. Ans. $(a^n-7)(a^n+12)$.

CASE VII.

110. To resolve a quadrinomial into two binomial factors.

111. When two binomials contain a common term their product will be a trinomial; when the terms are dissimilar the product will be a quadrinomial.

1. Factor $ac+bc+ad+bd$.

SOLUTION. $ac+bc$ equals $(a+b)c$, and $ad+bd$ equals $(a+b)d$; now, c times $(a+b)$ plus d times $(a+b)$ equals $(c+d)$ times $(a+b)$, or $(a+b)(c+d)$.

OPERATION.
$ac+bc+ad+bd=$
$(a+b)c+(a+b)d=$
$(a+b)(c+d)$

Rule.—*Factor each two terms which will give a common binomial factor, and then enclose the sum of the monomial factors in a parenthesis, and write it as the coefficient of the common binomial factor.*

EXAMPLES.

2. Factor $ab+ay+bx+xy$. Ans. $(a+x)(b+y)$.
3. Factor $ac-bc+ad-bd$. Ans. $(a-b)(c+d)$.
4. Factor $ax+bx-ay-by$ Ans. $(a+b)(x-y)$.
5. Factor $ab+b-2a-2$. Ans. $(a+1)(b-2)$.

6. Factor $ac-2bc-3ad+6bd$. *Ans.* $(a-2b)(c-3d)$.
7. Factor $a^2c^2-b^2c^2+a^2d^2-b^2d^2$. *Ans.* $(a^2-b^2)(c^2+d^2)$.
8. Factor $a^nx^n-b^nx^n+a^ny^n-b^ny^n$. *Ans.* $(a^n-b^n)(x^n+y^n)$.

CASE VIII.

112. To factor any binomial consisting of two equal powers of two quantities.

This case relates to binomials of the form of a^2-b^2, a^3-b^3, a^4-b^4, etc.; also a^3+b^3, a^5+b^5, etc.

THEOREM I.

The difference of the same powers of two quantities is divisible by the difference of the two quantities.

Let a and b be any quantities, then $a-b$ will be their difference, and a^n-b^n will be the difference of the same power of the quantities; then will a^n-b^n be divisible by $a-b$. Divide a^n-b^n by $a-b$ until we obtain two remainders.

OPERATION.

$$\begin{array}{rl|l} & a^n-b^n & a-b \\ & a^n-a^{n-1}b & a^{n-1}+a^{n-2}b \\ \hline \text{1st Rem.} = & a^{n-1}b-b^n & \\ & a^{n-1}b-a^{n-2}b^2 & \\ \hline \text{2d Rem.} = & a^{n-2}b^2-b^n & \\ n\text{th Rem.} = & a^{n-n}b^n-b^n & \\ = & a^0b^n-b^n & \\ = & 1b^n-b^n=0 & \end{array}$$

Now, if the division terminates, some remainder will reduce to zero; and if we obtain an expression for the nth remainder, we will find that its value is 0. Let us then find the nth remainder.

The *last term* of the nth remainder is evidently $-b^n$. In the 1st remainder the *first term* is $a^{n-1}b$; in the 2d remainder it is $a^{n-2}b^2$; in the 3d remainder it is $a^{n-3}b^3$; hence, in the nth remainder it is $a^{n-n}b^n$; hence, the nth remainder is $a^{n-n}b^n-b^n$, which equals $a^0b^n-b^n$, or $1b^n-b^n$, or 0 Hence, the nth remainder is 0, and the division terminates.

NOTE.—The latter part may be given as follows: If $n=3$, the 3d remainder becomes $a^{3-3}b^3-b^3$, or $a^0b^3-b^3$, or $1b^3-b^3=0$. If $n=4$, the 4th remainder will also reduce to 0; hence, the nth remainder is always equal to 0, and since the remainder is zero, the division is exact.

THEOREM II.

The difference of the same even powers of two quantities is divisible by the sum of the quantities.

Let a^n-b^n be the difference of the same even powers of a and b, n being *even*. Now, a^n-b^n is divisible by $(a-b)$ (Theo. I.). Let $b=-c$; then, substituting, we shall have $[a^n-(-c)^n]\div[a-(-c)]$. But $(-c)^n=c^n$, since n is even, and $[a-(-c)]=a+c$; hence we have a^n-c^n is divisible by $a+c$. Therefore, etc.

OPERATION

$$(a^n-b^n)\div(a-b)$$
$$[a^n-(-c)^n]\div[a-(-c)]$$
$$(a^n-c^n)\div(a+c)$$

NOTE.—This theorem may also be demonstrated independently in a manner similar to Theo. I.

THEOREM III.

The sum of the same odd powers of two quantities is divisible by the sum of the quantities.

Let a^n-b^n be the difference of the same odd powers of a and b; n being odd. Now, a^n-b^n is divisible by $a-b$ (Theo. I.). Let $b=-c$; then, substituting, we shall have $a^n-b^n=a^n-(-c)^n$ and $a-b=a-(-c)$. Now, $(-c)^n$ equals $-c^n$, since n is odd; hence, $a^n-(-c)^n=a^n+c^n$; and $a-(-c)=a+c$; hence, a^n+c^n is divisible by $a+c$.

OPERATION.

$$(a^n-b^n)\div(a-b)=$$
$$[a^n-(-c^n)]\div[a-(-c)]$$
$$(a^n+c^n)\div(a+c)$$

NOTE.—This may also be demonstrated independently in a manner similar to Theo. I.

EXAMPLES.

1. Factor a^3-b^3. *Ans.* $(a-b)(a^2+ab+b^2)$.
2. Factor a^4-c^4. *Ans.* $(a^2+c^2)(a+c)(a-c)$.
3. Factor a^5-x^5. *Ans.* $(a-x)(a^4+a^3x+a^2x^2+ax^3+x^4)$.
4. Factor a^3+b^3. *Ans.* $(a+b)(a^2-ab+b^2)$.
5. Factor a^5+b^5. *Ans.* $(a+b)(a^4-a^3b$, etc.).
6. Factor a^3-8x^3. *Ans.* $(a-2x)(a^2+2ax+4x^2)$.
7. Factor $8a^3+27c^3$. *Ans.* $(2a+3c)(4a^2-6ac+9c^2)$.
8. Factor a^6-b^6. *Ans.* $(a^3+b^3)(a^3-b^3)=$
$(a+b)(a^2-ab+b^2)(a-b)(a^2+ab+b^2)$.

NOTE.—Other factors of the 8th problem may be found; thus, uniting the 1st and 3d, and the 2d and 4th of the last expression, we have $(a^2-b^2)(a^4+a^2b^2+b^4)$.

GREATEST COMMON DIVISOR.

113. A **Common Divisor** of two or more quantities is a quantity that will exactly divide each of them.

114. The **Greatest Common Divisor** of two or more quantities is the greatest quantity that will exactly divide each of them.

PRINCIPLE.—*The greatest common divisor of two or more quantities equals the product of all their common prime factors.*

CASE I.

115. To find the greatest common divisor of quantities by factoring.

1. Find the greatest common divisor of $12a^3b^2c$ and $18a^2bd$.

SOLUTION. Resolving the quantities into their factors, we perceive that 2, 3, a^2, and b are all of the common factors; hence the greatest common divisor is $2\times3\times a^2\times b=6a^2b$.

OPERATION.

$$12a^3b^2c=2\times2\times3a^3b^2c$$
$$18a^2bd=2\times3\times3a^2bd$$
$$\text{G. C. D.}=2\times3a^2b=6a^2b$$

Rule.—*Resolve the quantities into their prime factors, and take the product of all the common factors.*

EXAMPLES.

Find the greatest common divisor—

2. Of $18a^3x^2$ and $24a^4x^3c$. *Ans.* $6a^3x^2$.
3. Of $16a^5b^3z$ and $24a^2b^5z^4$. *Ans.* $8a^2b^3z$.
4. Of $15a^6b^7x^3z^2$ and $25a^2b^5xz^3$. *Ans.* $5a^2b^5xz^2$.
5. Of $28a^nc^{2m}z^5$, $35a^{3n}c^mx^2$, and $14a^{2n}c^{3m}x^2z^4$. *Ans.* $7a^nc^m$.
6. Of a^2-b^2 and $a^2-2ab+b^2$. *Ans.* $a-b$.
7. Of $a^2+2ab+b^2$ and a^2-b^2. *Ans.* $a+b$.
8. Of a^3-b^3 and $a^2-2ab+b^2$. *Ans.* $a-b$.
9. Of a^4-b^4 and a^6-b^6. *Ans.* a^2-b^2.
10. Of x^2-y^2 and $ax+ay+bx+by$. *Ans.* $x+y$.
11. Of $ac+bc+ad+bd$ and $ac-bd-ad+bc$. *Ans.* $a+b$

NOTE.—Young pupils may omit the next case of Greatest Common Divisor until review.

CASE II.

116. To find the greatest common divisor of polynomials when the common factors are not readily seen.

PRIN. 1. *A divisor of a quantity is a divisor of any number of times that quantity.*

Thus, c^2, which is a divisor of bc^3, is a divisor of a times bc^3, or abc^3.

PRIN. 2. *A common divisor of two quantities is a divisor of their sum and their difference.*

For, take the two quantities ac^2 and bc^2, in which the common divisor is c^2; their sum is ac^2+bc^2, or $(a+b)c^2$, which is divisible by c^2; their difference is ac^2-bc^2, or $(a-b)c^2$, which is also divisible by c^2. Therefore, etc.

PRIN. 3. *Either of two quantities may be multiplied or divided by a factor not found in the other, without changing their greatest common divisor.*

For, it neither introduces nor omits a common factor, and hence cannot affect the greatest common divisor. Thus, the greatest common divisor of $2a^3c^2$ and $3bc^2$ is c^2. Multiply the first by a, or divide the second by b, and the greatest common divisor is still c^2.

1. Find the greatest common divisor of any two quantities A and B.

SOLUTION. Divide A by B, indicating the quotient by Q, and the remainder by R; divide B by R, indicating the quotient by Q', and the remainder by R'; divide R by R', indicating the quotient by Q'', and thus continue. Then any remainder which exactly divides the preceding divisor will be the greatest common divisor of A and B. For,

OPERATION.

$$A \,|\, B \quad (\text{quotient } Q), \qquad A-QB=R$$
$$B \,|\, R \quad (\text{quotient } Q'), \qquad B-Q'R=R'$$
$$R \,|\, R' \quad (\text{quotient } Q'')$$
$$A=QB+R$$
$$B=Q'R+R'$$

FIRST. *Each remainder is a number of times the greatest common divisor.* A and B are each a number of times the G. C. D.; hence, $A-QB$ or R is a number of times the G. C. D. (Prin. 2); and since B and R are each a number of times the G. C. D., for the same reason $B-Q'R$ or R' is a number of times the G. C. D. Hence, each remainder is a number of times the G. C. D.

SECOND. *The last divisor will divide A and B.* Suppose that R' divides R, then it will divide $Q'R$ (Prin. 1), and also $Q'R+R'$ or B (Prin. 2);

and since it divides B, it will also divide QB (Prin. 1), and $QB+R$, or A (Prin. 2). Hence, the last divisor will divide A and B.

Therefore, since the last divisor divides A and B, and is a *number of times* the G. C. D., it must be *once* the G. C. D. Hence, the remainder which exactly divides the previous divisor is the G. C. D. of A and B.

NOTE.—The latter part of the solution may be given as follows: Since the last divisor divides A and B, it cannot be *greater* than the G. C. D., and since it is a number of times the G. C. D., it cannot be *less* than the G. C. D.; hence, since it is neither *greater* nor *less* than the G.C.D., it must be the G.C.D.

Rule.—*Divide the greater quantity by the less, the divisor by the remainder, and thus continue to divide the last divisor by the last remainder until there is no remainder; the last divisor will be the greatest common divisor.*

NOTES.—1. When the highest power of the leading letter is the same in both quantities, either quantity may be made the dividend.

2. If both quantities contain a common factor, it may be set aside, and afterward inserted in the greatest common divisor of the other parts.

3. If either quantity contains a factor not found in the other, it may be canceled before beginning the operation. (Prin. 3.)

4. When necessary, the dividend may be multiplied by any quantity not a factor of the divisor, which will render the first term divisible by the first term of the divisor. (Prin. 3.)

5. If we obtain a remainder which does not contain the leading letter, there is no common divisor.

6. When there are more than two quantities, find the G.C.D. of two, then of that divisor and the third quantity, etc. The last divisor will be the greatest common divisor.

2. Find the greatest common divisor of a^2-b^2 and $a^2-2ab+b^2$.

SOLUTION. Dividing $a^2-2ab+b^2$ by a^2-b^2, we have a quotient of 1, and a remainder of $-2ab+2b^2$. Rejecting the factor $-2b$, which does not affect the greatest common divisor (Prin. 3), we have $a-b$. Dividing a^2-b^2 by $a-b$, we have a quotient $a+b$, with no remainder. Hence the last divisor, $a-b$, is the greatest common divisor of a^2-b^2 and $a^2-2ab+b^2$.

OPERATION.

$$
\begin{array}{l}
a^2-b^2)\,a^2-2ab+b^2\,(1 \\
\quad\quad\quad\;\; \underline{a^2-b^2} \\
\quad\quad\quad\;\; -2ab+2b^2 \text{ Rejecting the factor } -2b \\
a-b)\,a^2-b^2\,(a+b \\
\quad\quad\; \underline{a^2-ab} \\
\quad\quad\quad\;\; ab-b^2 \\
\quad\quad\quad\;\; \underline{ab-b^2}
\end{array}
$$

EXAMPLES.

3. Find the greatest common divisor of a^2-b^2 and $a^2+2ab+b^2$. *Ans.* $a+b$.

4. Find the greatest common divisor of $2x^3-5x^2+3x$ and $4x^2-2x-2$. *Ans.* $x-1$.

5. Find the greatest common divisor of a^3-x^3 and a^2-x^2. *Ans.* $a-x$.

6. Find the greatest common divisor of $ab+bc+ad+dc$ and a^2-c^2. *Ans.* $a+c$.

7. Find the greatest common divisor of a^3+x^3 and a^2-x^2. *Ans.* $a+x$.

8. Find the greatest common divisor of $2ax^2-2ay^2$ and $4ax^3+4ay^3$. *Ans.* $2a(x+y)$.

9. Find the greatest common divisor of a^4-b^4 and $a^3+a^2b-ab^2-b^3$. *Ans.* a^2-b^2.

10. Find the greatest common divisor of x^3-x^2-12x and $x^2-4x-21$. *Ans.* $x+3$.

THE LEAST COMMON MULTIPLE.

117. A **Multiple** of a quantity is any quantity of which it is a factor.

118. A **Common Multiple** of two or more quantities is a quantity which is a multiple of each of them.

119. The **Least Common Multiple** of two or more quantities is the least quantity which is a multiple of each of them.

NOTE.—The primary idea of a multiple is that of a number of times a quantity. The above definition is based upon a derivative truth. Another derivative definition is, a multiple of a quantity is any quantity which it will exactly divide.

PRIN. 1. *A multiple of a quantity contains all the factors of the quantity.*

Thus, it is evident that a multiple of ab, as $4a^2b$, must contain all the factors of ab.

PRIN. 2. *A common multiple of two or more quantities contains all the factors of each quantity.*

Thus, it is evident that $12a^2bc$, which is a common multiple of $2ab$ and $3ac$, contains all the factors of $2ab$ and $3ac$.

PRIN. 3. *The least common multiple of two or more quantities contains all the factors of each quantity, and no other factors.*

Thus, it is evident that the least common multiple of $2ab$ and $3ac$, which is $6abc$, must contain all the factors of $2ab$ and $3ac$, or it would not contain the quantities; and it must contain no other factors, or else it would not be the least common multiple.

CASE I.

120. To find the least common multiple by factoring.

1. Find the least common multiple of $12a^3bc^2$ and $18a^2bd^2$.

SOLUTION. We first resolve the quantities into their prime factors.

The least common multiple must contain all the different prime factors, and no others (Prin. 3). All the different prime factors are 2, 2, 3, 3, a^3, b, c^2, d^2, whose product equals $36a^3bc^2d^2$. Hence, the least common multiple of the given quantities is $36a^3bc^2d^2$.

OPERATION.

$$12a^3bc^2 = 2\times2\times3a^3.\,b\,.\,c^2$$
$$18a^2bd^2 = 2\times3\times3a^2.\,b\,.\,d^2$$
$$\text{L. C. M} = 2\times2\times3\times3a^3.\,b\,.\,c^2.\,d^2 = 36a^3bc^2d^2$$

Rule.—*Resolve the quantities into their prime factors, and take the product of all the different factors, using each factor the greatest number of times it appears in either quantity.*

EXAMPLES.

Find the least common multiple—

2. Of $18a^5b^3$ and $24a^3b^4c^3$. *Ans.* $72a^5b^4c^3$.
3. Of $24ax^3z^4$ and $42a^2x^2y^2z$. *Ans.* $168a^2x^3y^2z^4$.
4. Of $18a^2b$, $24ab^3c$ and $27a^2c^3z$. *Ans.* $216a^2b^3c^3z$.
5. Of $2(a+x)$ and (a^2-x^2). *Ans.* $2(a^2-x^2)$.
6. Of $a(b-c)$ and $b(b^2-c^2)$. *Ans.* $ab(b^2-c^2)$.
7. Of (a^2-b^2) and $a^2-2ab+b^2$. *Ans.* $a^3-a^2b-ab^2+b^3$.

8. Of $a^2(a-z)$ and $x^2(a^2-z^2)$. *Ans.* $a^2x^2(a^2-z^2)$.
9. Of $3x^2(2a-1)$ and $4xy(4a^2-1)$. *Ans.* $12x^2y(4a^2-1)$.
10. Of x^2-y^2 and x^3-y^3. *Ans.* $x^4-xy^3+x^3y-y^4$.
11. Of $3a(a-b)$, $4ac(a^2-b^2)$ and $6c^2x(a+b)$. *Ans.* $12ac^2x(a^2-b^2)$.
12. Of $m^2+2mn+n^2$ and m^3+n^3. *Ans.* $(m^3+n^3)(m+n)$.

NOTE.—Young pupils can omit the next case of Least Common Multiple until review.

CASE II.

121. To find the least common multiple when the quantities are not readily factored.

The method will be readily understood from the following principle:

PRINCIPLE.—*The least common multiple of two quantities equals either quantity multiplied by the quotient of the other quantity divided by their greatest common divisor.*

For, let A and B be any two quantities, and let their greatest common divisor be represented by c, and the other factors by a and b, respectively; then we shall have the L. C. M. $= a\times b\times c$, Case I.; but $b\times c = B$, and $a = \frac{A}{c}$; hence, L. C. M. $= \frac{A}{c}\times B$. Therefore, etc.

OPERATION.

$$A = a\times c$$
$$B = b\times c$$
$$\text{L. C. M.} = a\times b\times c = \frac{A}{c}\times B$$

NOTE.—The least common multiple of two quantities equals their product divided by their greatest common divisor, for $\frac{A}{c}\times B = \frac{A\times B}{c}$.

1. Find the least common multiple of a^2-b^2 and $a^2-3ab+2b^2$.

SOLUTION. We first find the greatest common divisor to be $a-b$. Then the L. C. M. equals a^2-b^2 multiplied by the quotient of $a^2-3ab+2b^2$ divided by $a-b$, or $(a^2-b^2)(a-2b)$, which equals $a^3-2a^2b-ab^2+2b^3$.

OPERATION.

$$\text{G. C. D.} = a-b$$
$$\text{L. C. M.} = (a^2-b^2)\times\frac{a^2-3ab+2b^2}{a-b}$$
$$= (a^2-b^2)(a-2b)$$
$$= a^3-2a^2b-ab^2+2b^3$$

Rule.—I. *Find the greatest common divisor of the two quantities; divide one quantity by it, and multiply the other quantity by the quotient.*

II. *When there are more than two quantities, find the least common multiple of two of them, then of this multiple and the third quantity,* etc.

EXAMPLES.

2. Find the least common multiple of x^2-x-12 and $x^2-4x-21$. *Ans.* $x^3-8x^2-5x+84$.

3. Find the least common multiple of x^2+5x+6 and x^2+6x+8. *Ans.* $x^3+9x^2+26x+24$.

4. Find the least common multiple of $a^2+4ab+3b^2$ and a^2-b^2. *Ans.* $a^3+3a^2b-ab^2-3b^3$.

5. Find the least common multiple of $a^2+3ab+2b^2$ and $a^2-ab-6b^2$. *Ans.* $a^3-7ab^2-6b^3$.

6. Find the least common multiple of $x^2-ax+3x-3a$ and $x^2-3x-ax+3a$. *Ans.* $x^3-ax^2-9x+9a$.

7. Find the least common multiple of x^2-x-2, x^2+3x+2 and x^2+5x+4. *Ans.* $x^4+5x^3-20x-16$.

REVIEW QUESTIONS.

Define Composition. A Composite Quantity. State the relation of Composition to Factoring. State the four theorems of Composition.

Define Factoring. Factors. A Prime Quantity. Quantities prime to each other. State each case. Give the rule for each case. State the three theorems of Factoring.

Define Common Divisor. Greatest Common Divisor. State the cases. The principles. The rules. Define a Multiple. A Common Multiple. The Least Common Multiple. State the cases. The principles. The rules.

SECTION IV.

FRACTIONS.

122. A **Fraction** is a number of the equal parts of a unit.

123. A **Fraction** in Algebra is expressed by two quantities, one above the other, with a straight line between them; as, $\frac{a}{c}$ or $\frac{a-x}{a-z}$.

124. The **Denominator** of a fraction denotes the number of equal parts into which the unit is divided. It is written below the line.

125. The **Numerator** of a fraction denotes the number of equal parts taken. It is written above the line.

126. An **Entire Quantity** is one that has no fractional part; as, $2a^2$, $a+b$, etc.

127. A **Mixed Quantity** is one that has both an entire and a fractional part; as, $a+\frac{b}{c}$, $ax-\frac{2ac}{m+n}$.

128. An **Algebraic Fraction** is usually regarded as the expression of one quantity divided by another; thus, $\frac{a}{c}$ means a divided by c, etc.

PRINCIPLES OF FRACTIONS.

129. The **Principles of Fractions** are general laws showing the relation of the value of the fraction to its numerator and denominator.

PRIN. 1. *Multiplying the numerator or dividing the denominator of a fraction by any quantity multiplies the value of the fraction by that quantity.*

Rule.—I. *Find the greatest common divisor of the two quantities; divide one quantity by it, and multiply the other quantity by the quotient.*

II. *When there are more than two quantities, find the least common multiple of two of them, then of this multiple and the third quantity*, etc.

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2. Find the least common multiple of x^2-x-12 and $x^2-4x-21$. *Ans.* $x^3-8x^2-5x+84$.

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4. Find the least common multiple of $a^2+4ab+3b^2$ and a^2-b^2. *Ans.* $a^3+3a^2b-ab^2-3b^3$.

5. Find the least common multiple of $a^2+3ab+2b^2$ and $a^2-ab-6b^2$. *Ans.* $a^3-7ab^2-6b^3$.

6. Find the least common multiple of $x^2-ax+3x-3a$ and $x^2-3x-ax+3a$. *Ans.* $x^3-ax^2-9x+9a$.

7. Find the least common multiple of x^2-x-2, x^2+3x+2 and x^2+5x+4. *Ans.* $x^4+5x^3-20x-16$.

REVIEW QUESTIONS.

Define Composition. A Composite Quantity. State the relation of Composition to Factoring. State the four theorems of Composition.

Define Factoring. Factors. A Prime Quantity. Quantities prime to each other. State each case. Give the rule for each case. State the three theorems of Factoring.

Define Common Divisor. Greatest Common Divisor. State the cases. The principles. The rules. Define a Multiple. A Common Multiple. The Least Common Multiple. State the cases. The principles. The rules.

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PRINCIPLES OF FRACTIONS.

129. The **Principles of Fractions** are general laws showing the relation of the value of the fraction to its numerator and denominator.

PRIN. 1. *Multiplying the numerator or dividing the denominator of a fraction by any quantity multiplies the value of the fraction by that quantity.*

If we multiply the numerator of a fraction by n, there will be n times as many parts taken, each of them the same size as before; hence, the value will be n times as great.

If we divide the denominator by n, the unit will be divided into 1-nth as many parts; hence, each part will be n times as great; and the same number of parts being taken, the value of the fraction will be n times a great. Therefore, etc.

PRIN. 2. *Dividing the numerator or multiplying the denominator of a fraction by any quantity divides the value of the fraction by that quantity.*

If we divide the numerator of a fraction by n, there will be only 1-nth as many parts taken, each of the same size as before; hence, the value of the fraction will be 1-nth as great.

If we multiply the denominator of a fraction by n, the unit will be divided into n times as many parts; hence, each part will be 1-nth as great as before; and the same number of parts being taken, the value of the fraction will be 1-nth as great. Therefore, etc.

PRIN. 3. *Multiplying or dividing both numerator and denominator of a fraction by the same quantity does not change the value of the fraction.*

For, since multiplying the numerator multiplies the fraction, and multiplying the denominator divides the fraction, multiplying both numerator and denominator by the same number both multiplies and divides the value of the fraction by that number, and hence does not change the value. In the same way it may be shown that dividing both terms does not change the value of the fraction. Therefore, etc.

PRIN. 4. *The value of a fraction is equal to the quotient of its numerator divided by its denominator.*

SIGNS OF THE FRACTION.

130. The **Sign** of a fraction is the sign written before the dividing-line, showing whether the fraction is to be added or subtracted.

NOTE.—This sign is sometimes called the *apparent sign* of the fraction and the sign of its value, the *real* sign.

PRIN. 1. *Changing the sign of the numerator or denominator changes the sign of the fraction.*

For, this is the same as multiplying or dividing the fraction by -1, which will change the sign of the quantity (Prin. 1, Art. 129). Thus, $+\frac{a-x}{c+z}$ equals $-\frac{-a+x}{c+z}$, or $-\frac{a-x}{-c-z}$, etc.

PRIN. 2. *Changing the sign of both numerator and denominator does not change the sign of the fraction.*

For, this is the same as multiplying both numerator and denominator by -1, which will not change the value of the fraction. (Prin. 3, Art. 129.)

REDUCTION.

131. Reduction of Fractions is the process of changing their form without changing their value.

CASE I.

132. To reduce a fraction to its lowest terms.

A **Fraction** is in its lowest terms when the numerator and denominator are prime to each other.

1. Reduce $\frac{8a^3b^2c}{12a^2b^3c^2}$ to its lowest terms.

OPERATION.

$$\frac{8a^3b^2c}{12a^2b^3c^2}=\frac{2a}{3c}$$

SOLUTION. Dividing both terms of the fraction by the common factors, 4, a^2, b^2 and c, we have it equal to $\frac{2a}{3c}$ (Prin. 3, Art. 129); and this is the lowest term of the fraction, since the terms are prime to each other.

Rule.—*Divide both numerator and denominator by their common factors;*

Or, *Divide both terms by their greatest common divisor.*

EXAMPLES.

2. Reduce $\frac{15a^4x^2}{25a^7x}$ to its lowest terms. *Ans.* $\frac{3x}{5a^3}$.

3. Reduce $\frac{54x^3z^6}{36x^5z^2}$ to its lowest terms. *Ans.* $\frac{3z^4}{2x^2}$.

4. Reduce $\frac{24a^5x^6z^3}{56a^4x^9z^7}$ to its lowest terms. *Ans.* $\frac{3a}{7x^3z^4}$.

5. Reduce $\frac{a^2-b^2}{a^4-b^4}$ to its lowest terms. *Ans.* $\frac{1}{a^2+b^2}$.

6 Reduce $\frac{2a+2b}{a^2-b^2}$ to its lowest terms. *Ans.* $\frac{2}{a-b}$.

7. Reduce $\frac{a^2-1}{2(ab-b)}$ to its lowest terms. *Ans.* $\frac{a+1}{2b}$.

8. Reduce $\frac{2a^{m+4}}{3a^{m+2}}$ to its lowest terms. *Ans.* $\frac{2a^2}{3}$.

9. Reduce $\frac{4a^2c^{n+1}}{6a^3c^{n-1}}$ to its lowest terms. *Ans.* $\frac{2c^2}{3a}$.

10. Reduce $\frac{x^2-9}{2x^2+10x+12}$ to its lowest terms. *Ans.* $\frac{x-3}{2x+4}$.

11. Reduce $\frac{a^3-ab^2}{a^2+2ab+b^2}$ to its lowest terms. *Ans.* $\frac{a(a-b)}{a+b}$.

12. Reduce $\frac{x^2-4a^2}{x^2+2ax-8a^2}$ to its lowest terms. *Ans.* $\frac{x+2a}{x+4a}$.

13. Reduce $\frac{x^{2n}-9b^{2n}}{x^{2n}-6b^nx^n+9b^{2n}}$ to its lowest terms.

Ans. $\frac{x^n+3b^n}{x^n-3b^n}$.

CASE II.

133. To reduce a fraction to an entire or mixed quantity.

1. Reduce $\frac{ac+b}{c}$ to a mixed quantity.

OPERATION.

$$\frac{ac+b}{c}=a+\frac{b}{c}$$

SOLUTION. The value of a fraction is equal to the quotient of the numerator divided by the denominator. Dividing $ac+b$ by c, we have a for the entire part, and $\frac{b}{c}$ for the fractional part. Therefore, etc.

Rule.—I. *Divide the numerator by the denominator for the entire part, continuing the division as far as necessary.*

II. *Write the denominator under the remainder, and annex the result to the entire part with the proper sign.*

EXAMPLES.

2. Reduce $\frac{ac^2+b}{c}$ to a mixed quantity. *Ans.* $ac+\frac{b}{c}$.

3. Reduce $\frac{2ax+x^2}{a+x}$ to a mixed quantity. *Ans.* $2x-\frac{x^2}{a+x}$.

4. Reduce $\frac{a^2-4c^2}{a-c}$ to a mixed quantity. *Ans.* $a+c-\frac{3c^2}{a-c}$.

5. Reduce $\frac{3a^3-3x^3}{a-x}$ to an entire quantity.

Ans. $3(a^2+ax+x^2)$.

6. Reduce $\frac{x^3-z^3}{(x-z)^2}$ to a mixed quantity. *Ans.* $x+2z+\frac{3z^2}{x-z}$.

7. Reduce $\frac{a^3-b^3}{a+b}$ to a mixed quantity.

Ans. $a^2-ab+b^2-\frac{2b^3}{a+b}$.

8. Reduce $\frac{x^4-z^4}{(x^2-z^2)^2}$ to a mixed quantity. *Ans.* $1+\frac{2z^2}{x^2-z^2}$.

9. Reduce $\frac{x^3+z^3}{(x+z)^3}$ to a mixed quantity. *Ans.* $1-\frac{3xz}{(x+z)^2}$.

CASE III.

134. To reduce a mixed quantity to a fraction.

1. Reduce $a+\frac{c}{x}$ to a fraction.

OPERATION.

$$a+\frac{c}{x}=$$
$$\frac{ax}{x}+\frac{c}{x}=\frac{ax+c}{x}$$

SOLUTION. It is evident that $1=\frac{x}{x}$; hence a equals a times $\frac{x}{x}$, or $\frac{ax}{x}$, which, added to $\frac{c}{x}$, equals $\frac{ax}{x}+\frac{c}{x}$, or $\frac{ax+c}{x}$.

Rule.—*Multiply the entire part by the denominator of the fraction; add the numerator when the sign of the fraction is plus, and subtract it when the sign is minus, and write the denominator under the result.*

NOTE.—When the sign of the fraction is minus, remember to change the signs of all the terms.

EXAMPLES.

2. Reduce $3a+\frac{a-3}{2}$ to a fraction. *Ans.* $\frac{7a-3}{2}$

3. Reduce $2z+\frac{3-2z}{3}$ to a fraction. *Ans.* $\frac{4z+3}{3}$

4. Reduce $x+\frac{ax}{a+x}$ to a fraction. *Ans.* $\frac{2ax+x^2}{a+x}$

5. Reduce $a+c-\frac{3c^2}{a-c}$ to a fraction. *Ans.* $\frac{a^2-4c^2}{a-c}$

6. Reduce $4x-\frac{3-5x}{4}$ to a fraction. *Ans.* $\frac{21x-3}{4}$.

7. Reduce $a-\frac{2ac-c^2}{a}$ to a fraction. *Ans.* $\frac{(a-c)^2}{a}$.

8. Reduce $4a+x+\frac{3ax+x^2}{a-x}$ to a fraction. *Ans.* $\frac{4a^2}{a-x}$.

9. Reduce $2x-5-\frac{2x^2+4}{x-3}$ to a fraction. *Ans.* $\frac{11(1-x)}{x-3}$.

10. Reduce $a+x-\frac{c^2-x^2}{a-x}$ to a fraction. *Ans.* $\frac{a^2-c^2}{a-x}$.

CASE IV.

135. To reduce fractions by changing factors from one term to another.

PRINCIPLE.—*Any factor may be transferred from one term of a fraction to the other if the sign of the exponent be changed.*

This principle is readily proved by applying the principles of Article 94.

1. Change the terms in the fraction $\frac{a^2}{c^3}$.

SOLUTION. First, $\frac{a^2}{c^3}=a^2\times\frac{1}{c^3}$; but $a^2=\frac{1}{a^{-2}}$ (Prin. 3, Art. 94); hence, $a^2\times\frac{1}{c^3}=\frac{1}{a^{-2}}\times\frac{1}{c^3}=\frac{1}{a^{-2}c^3}$.

OPERATION.

1st. $\frac{a^2}{c^3}=a^2\times\frac{1}{c^3}=\frac{1}{a^{-2}}\times\frac{1}{c^3}=\frac{1}{a^{-2}c^3}$

2d. $\frac{a^2}{c^3}=a^2\times\frac{1}{c^3}=a^2\times c^{-3}=a^2c^{-3}$

Second, $\frac{a^2}{c^3}=a^2\times\frac{1}{c^3}$; but $\frac{1}{c^3}=c^{-3}$; hence, $a^2\times\frac{1}{c^3}=a^2\times c^{-3}$, or a^2c^{-3}.

Rule.—*In changing a factor from one term of a fraction to the other, change the sign of its exponent.*

EXAMPLES.

2. Change the terms in $\frac{a^5}{c^2x^3}$. *Ans.* $a^5c^{-2}x^{-3}$.

3. Change the terms in $\frac{x^3y^4}{a^{-3}z^5}$. *Ans.* $a^3x^3y^4z^{-5}$.

4. Change $\frac{3c^3}{ab^2}$ to an entire quantity. *Ans.* $3a^{-1}b^{-2}c^3$.

5. Change $\frac{a-x}{a+x}$ to an entire quantity. *Ans.* $(a-x)(a+x)^{-1}$.

6. Change $\frac{3(a+c)}{5c^{-2}}$ to an entire quantity. *Ans.* $3\times5^{-1}(a+c)c^2$.

7. Change $\frac{4(c-z^3)}{6c^{-2}}$ to a simpler form. *Ans.* $\frac{2}{3}(c^3-c^2z^3)$.

8. Change $\frac{x^2-2xy+y^2}{x^2-y^2}$ to an entire quantity. *Ans.* $(x-y)(x+y)^{-1}$.

9. Change $\frac{a(b-c)}{(b+c)^{-1}}$ to an entire quantity. *Ans.* $a(b^2-c^2)$.

10. Change $\frac{4a(c-z)^{-1}}{c+z}$ to positive exponents. *Ans.* $\frac{4a}{c^2-z^2}$.

11. Change $\frac{(a-b)^2(x-y)^{-1}}{(a-b)^{-1}(x-y)}$ to positive exponents. *Ans.* $\frac{(a-b)^3}{(x-y)^2}$.

CASE V.

136. To reduce fractions to a common denominator.

137. Fractions have a *common denominator* when they have the same expression for a denominator.

138. The **Least Common Denominator** of several fractions is the least denominator to which they may all be reduced.

PRIN. 1. *A common denominator of several fractions is a common multiple of their denominators.*

For, when a fraction is reduced to higher terms, its denominator is multiplied by some number, and the common denominator to which several fractions are reduced must therefore be a multiple of each given denominator.

PRIN. 2. *The least common denominator of several fractions is the least common multiple of their denominators.*

For, since a *common* denominator is a *common* multiple of the denominators, it is evident that a *least common denominator* is a *least common* multiple of the denominators.

1. Reduce $\frac{a}{bc}$ and $\frac{c}{b^2d}$ to their least common denominator.

SOLUTION. The least common multiple of bc and b^2d is b^2cd, which is the least common denominator. Dividing b^2cd by bc, the denominator of the first fraction, we find we must multiply both terms of $\frac{a}{bc}$ by bd to reduce it to the common denominator. Multiplying both terms of $\frac{a}{bc}$ by bd, we have $\frac{abd}{b^2cd}$. Dividing b^2cd by b^2d, we have c. Multiplying both terms of $\frac{c}{b^2d}$ by c, we have $\frac{c^2}{b^2cd}$.

OPERATION.

$$\frac{a}{bc}, \frac{c}{b^2d}; \quad \text{L.C.M.} = b^2cd$$

$$b^2cd \div bc = bd; \quad b^2cd \div b^2d = c$$

$$\frac{a}{bc} = \frac{a \times bd}{bc \times bd} = \frac{abd}{b^2cd}$$

$$\frac{c}{b^2d} = \frac{c \times c}{b^2d \times c} = \frac{c^2}{b^2cd}$$

Rule.—*Find the least common multiple of the denominators; divide this by the denominator of each fraction, and multiply the numerator by the quotient;*

Or, *Multiply both terms of each fraction by the denominators of the other fractions.*

NOTE.—Fractions may be reduced to a common denominator by multiplying both terms of one or more fractions by such quantities as will make the denominators equal.

EXAMPLES.

Reduce to a common denominator—

2. $\frac{a}{mn}, \frac{b}{m^2}$ and $\frac{c}{mn^2}$. *Ans.* $\frac{amn}{m^2n^2}, \frac{bn^2}{m^2n^2}, \frac{cm}{m^2n^2}$.

3. $\frac{2a}{3x}, \frac{3b}{4z}$ and $\frac{c}{ax^2}$. *Ans.* $\frac{8a^2xz}{12ax^2z}, \frac{9abx^2}{12ax^2z}, \frac{12cz}{12ax^2z}$.

4. $\frac{c}{4a}, \frac{5}{6c}$ and $\frac{2ac}{3n^2}$. *Ans.* $\frac{3c^2n^2}{12acn^2}, \frac{10an^2}{12acn^2}, \frac{8a^2c^2}{12acn^2}$.

5. $\frac{3}{2a^2c}, \frac{4b^2}{3ac^2}$ and $\frac{2ac}{4c^2z}$. *Ans.* $\frac{18cz}{12a^2c^2z}, \frac{16ab^2z}{12a^2c^2z}, \frac{6a^3c}{12a^2c^2z}$.

6. $\frac{a-b}{a^2c}, \frac{a+b}{3ac^2}$ and $5\frac{1}{2}$. *Ans.* $\frac{6c(a-b)}{6a^2c^2}, \frac{2a(a+b)}{6a^2c^2}, \frac{33a^2c^2}{6a^2c^2}$.

7. $\frac{ab}{a-b}, \frac{bc}{a+b}$ and $\frac{cd}{a^2-b^2}$. *Ans.* $\frac{ab(a+b)}{a^2-b^2}, \frac{bc(a-b)}{a^2-b^2}, \frac{cd}{a^2-b^2}$.

8. $\frac{2ax}{x-1}, \frac{3ax}{x+1}$ and $\frac{4ax}{x^2-1}$.
Ans. $\frac{2ax^2+2ax}{x^2-1}, \frac{3ax^2-3ax}{x^2-1}, \frac{4ax}{x^2-1}$.

9. $a, \frac{a}{c}$ and $\frac{a-b}{a-c}$. *Ans.* $\frac{a^2c-ac^2}{c(a-c)}, \frac{a^2-ac}{c(a-c)}, \frac{c(a-b)}{c(a-c)}$.

10. $\frac{a-c}{(a+c)^2}$ and $\frac{a+c}{(a-c)^2}$. *Ans.* $\frac{(a-c)^3}{(a^2-c^2)^2}, \frac{(a+c)^3}{(a^2-c^2)^2}$.

11. $\frac{a+c}{a-c}, \frac{a-c}{a+c}$ and $\frac{a^2+c^2}{a^2-c^2}$. *Ans.* $\frac{(a+c)^2}{a^2-c^2}, \frac{(a-c)^2}{a^2-c^2}, \frac{a^2+c^2}{a^2-c^2}$.

12. $\frac{a}{a^2+1}, \frac{a^2}{a^2-1}$ and $\frac{a^4}{a^4-1}$.
Ans. $\frac{a(a^2-1)}{a^4-1}, \frac{a^2(a^2+1)}{a^4-1}, \frac{a^4}{a^4-1}$.

13. $\frac{a}{(a+b)(b-c)}$ and $\frac{b}{(a+b)(c-b)}$.
Ans. $\frac{a}{(a+b)(b-c)}, \frac{-b}{(a+b)(b-c)}$.

ADDITION.

139. Addition of Fractions is the process of finding the simplest expression for the *sum* of two or more fractions.

PRINCIPLE.—*To be added, fractions must have a common denominator.*

For, they then express similar fractional units, and only similar units can be united into one sum.

1. Find the sum of $\frac{a}{n}$ and $\frac{b}{n}$.

SOLUTION. a divided by n, plus b divided by n, equals $(a+b)$ divided by n.

OPERATION.
$$\frac{a}{n}+\frac{b}{n}=\frac{a+b}{n}$$

2. Find the sum of $\frac{a}{n}$ and $\frac{b}{m}$.

SOLUTION. Since the denominators are not alike, we must first reduce the fractions to a common denominator. The common denominator is mn. $\frac{a}{n}$ equals $\frac{am}{mn}$, and $\frac{b}{m}$ equals $\frac{bn}{mn}$; am divided by mn, plus bn divided by mn, equals $(am+bn)$ divided by mn.

OPERATION.
$$\frac{a}{n}+\frac{b}{m}=\frac{am}{mn}+\frac{bn}{mn}$$
$$=\frac{am+bn}{mn}$$

Rule.—I. *Reduce the fractions, when necessary, to their least common denominator.*

II. *Add the numerators, and write the common denominator under their sum.*

NOTES.—1. Reduce each fraction to its lowest terms before adding, and also the result after addition.

2. Mixed quantities may be added by adding the integral parts and uniting the sum with the sum of the fractions.

EXAMPLES.

Find the sum—

3. Of $\frac{a}{b}+\frac{c}{d}$. *Ans.* $\frac{ad+bc}{bd}$.

4. Of $\frac{a}{b}+\frac{b}{d}+\frac{d}{c}$. *Ans.* $\frac{adc+b^2c+bd^2}{bdc}$.

5. Of $\frac{2a}{n}+\frac{3c}{mn}+\frac{4b}{m^2n}$. *Ans.* $\frac{2am^2+3cm+4b}{m^2n}$.

6. Of $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$. *Ans.* $\frac{yz+xz+xy}{xyz}$.

7. Of $\frac{2a}{3}+a-\frac{2x^2}{3ac}$. *Ans.* $\frac{5a^2c-2x^2}{3ac}$.

8. Of $\frac{a+b}{2}+\frac{a-b}{2}$. *Ans.* a.

9. Of $\frac{1}{m+n}+\frac{1}{m-n}$. *Ans.* $\frac{2m}{m^2-n^2}$.

10. Of $\frac{a}{a+b}+\frac{b}{a-b}$. *Ans.* $\frac{a^2+b^2}{a^2-b^2}$.

11. Of $\frac{a}{2a-2b}+\frac{b}{2b-2a}$. *Ans.* $\frac{1}{2}$.

12. Of $\frac{a}{2c}$, $\frac{a-c}{ac}$ and $\frac{c-a}{ac}$. *Ans.* $\frac{a}{2c}$.

13. Of $\frac{a^n}{2x}$, $\frac{x-a^{2n}}{a^nx}$ and $\frac{a^{2n}x-2z^2}{2a^nx^2}$. *Ans.* $\frac{x^2-z^2}{a^nx^2}$.

14. Of $\frac{1+a}{1-a}$ and $\frac{1-a}{1+a}$. *Ans.* $\frac{2(1+a^2)}{1-a^2}$.

15. Of $\frac{x-y}{xy}$, $\frac{y-z}{yz}$ and $\frac{z-x}{xz}$. *Ans.* 0.

16. Of $3a^2+\frac{x-3}{3}$ and $4a^2+\frac{2a-z}{2a}$. *Ans.* $7a^2+\frac{2ax-3z}{6a}$.

17. Of $\frac{2}{z+1}$ and $\frac{1+z^2}{z^2+z}$. *Ans.* $\frac{1+z}{z}$.

18. Of $\frac{1+x^2}{1-x^2}$ and $\frac{1-x^2}{1+x^2}$. *Ans.* $\frac{2(1+x^4)}{1-x^4}$.

19. Of $\frac{a}{(a-b)(b-c)}$ and $\frac{b}{(a-b)(c-b)}$. *Ans.* $\frac{1}{b-c}$.

20. Of $\frac{a+b}{(b-c)(c-a)}+\frac{b+c}{(c-a)(a-b)}+\frac{c+a}{(a-b)(b-c)}$. *Ans.* 0.

SUBTRACTION.

140. Subtraction of Fractions is the process of finding the simplest expression for the *difference* of two fractions.

PRINCIPLE.—*To be subtracted, fractions must have a common denominator.*

For, they then express similar fractional units, and only similar units can be subtracted.

1. Subtract $\frac{b}{n}$ from $\frac{a}{n}$.

SOLUTION. a divided by n, minus b divided by n, equals $(a-b)$ divided by n.

OPERATION.

$$\frac{a}{n}-\frac{b}{n}=\frac{a-b}{n}$$

2. Subtract $\frac{b}{n}$ from $\frac{a}{m}$.

SOLUTION. Since the denominators are not alike, we must first reduce the fractions to a common denominator. The common denominator is mn. $\frac{a}{m}=\frac{an}{mn}$ and $\frac{b}{n}=\frac{bm}{mn}$; an divided by mn, minus bm divided by mn, equals $(an-bm)$ divided by mn.

OPERATION.

$$\frac{a}{m}-\frac{b}{n}=\frac{an}{mn}-\frac{bm}{mn}=\frac{an-bm}{mn}$$

Rule.—I. *Reduce the fractions, when necessary, to their least common denominator.*

II. *Subtract the numerator of the subtrahend from the numerator of the minuend, and write the common denominator under the result.*

NOTES.—1. Reduce each fraction to its lowest terms before subtracting, and also the result after subtraction.

2. Mixed quantities may be subtracted by subtracting the integral parts, and uniting the difference with the difference of the fractions.

EXAMPLES.

3. From $\frac{a}{b}$ take $\frac{c}{d}$. *Ans.* $\frac{ad-bc}{bd}$.

4. From $\frac{2a}{n}$ take $\frac{2c}{mn}$. *Ans.* $\frac{2(am-c)}{mn}$.

5. From $\frac{3ax}{2c^2}$ take $\frac{4x}{3ac}$. *Ans.* $\frac{9a^2x-8cx}{6ac^2}$.

6. From $\frac{a+b}{2}$ take $\frac{a-b}{2}$. *Ans.* b.

7. From $\frac{a+b}{a}$ take $\frac{a-b}{a}$. *Ans.* $\frac{2b}{a}$.

8. From $5a^2+\frac{3x}{ac}$ take $3a^2+\frac{2a}{ax}$. *Ans.* $2a^2+\frac{3x^2-2ac}{acx}$.

9. From $4c-\frac{a}{a-3}$ take $2c-\frac{a+3}{a}$. *Ans.* $2c-\frac{9}{a^2-3a}$.

10. From $\frac{a+b}{a}$ take $\frac{b-a}{b}$. *Ans.* $\frac{a^2+b^2}{ab}$.

11. From $\frac{a}{a-b}$ take $\frac{b}{a+b}$. *Ans.* $\frac{a^2+b^2}{a^2-b^2}$.

12. From $\frac{1}{a-b}$ take $\frac{b}{a^2-b^2}$. *Ans.* $\frac{a}{a^2-b^2}$.

13. From $\frac{1}{1-a}$ take $\frac{1}{1+a}$. *Ans.* $\frac{2a}{1-a^2}$.

14. From $\frac{x+1}{x-1}$ take $\frac{x-1}{x+1}$. *Ans.* $\frac{4x}{x^2-1}$.

15. From $\frac{1+z^2}{1-z^2}$ take $\frac{1-z^2}{1+z^2}$. *Ans.* $\frac{4z^2}{1-z^4}$.

16. Find value of $\frac{a}{a-x}+\frac{3a}{a+x}-\frac{2ax}{a^2-x^2}$. *Ans.* $\frac{4a}{a+x}$.

17. Find value of $\frac{1}{a+b}+\frac{b}{a^2-b^2}-\frac{a}{a^2+b^2}$. *Ans.* $\frac{2ab^2}{a^4-b^4}$.

18. Find value of $\frac{3}{1-2x}-\frac{7}{1+2x}+\frac{4-20x}{1-4x^2}$. *Ans.* 0

19. Find value of $\frac{a}{(a+b)(b-c)}-\frac{b}{(a+b)(c-b)}$. *Ans.* $\frac{1}{b-c}$

20. Find value of $\left(\frac{1}{m}+\frac{1}{n}\right)(a+b)-\left(\frac{a+b}{m}-\frac{a-b}{n}\right)$. *Ans.* $\frac{2a}{n}$

MULTIPLICATION.

141. Multiplication of Fractions is the process of finding a *product* when one or both factors are fractions.

CASE I.

142. To multiply a fraction by an entire quantity.

1. Multiply $\frac{a}{b}$ by c.

SOLUTION. Since multiplying the numerator of a fraction multiplies the value of the fraction (Prin. 1, Art. 129), c times $\frac{a}{b}$ equals $\frac{ac}{b}$.

OPERATION.
$$\frac{a}{b}\times c=\frac{ac}{b}$$

2. Multiply $\frac{a}{b^2}$ by b.

SOLUTION. Since dividing the denominator of a fraction multiplies the fraction (Prin. 1, Art. 129), b times $\frac{a}{b^2}$ equals $\frac{a}{b}$.

OPERATION
$$\frac{a}{b^2}\times b=\frac{a}{b}$$

Rule.—*Multiply the numerator or divide the denominator of the fraction by the multiplier.*

NOTES.—1. The second method is preferred when the denominator is divisible by the multiplier.

2. It is often convenient to indicate the multiplication, and cancel equal factors in numerators and denominators.

EXAMPLES.

3. Multiply $\frac{a}{c}$ by n. *Ans.* $\frac{an}{c}$

4. Multiply $\frac{a^2b}{c^2d}$ by cd. *Ans.* $\frac{a^2b}{c}$.

5. Multiply $\frac{5ax^2}{12cz^3}$ by $4z^2$. *Ans.* $\frac{5ax^2}{3cz}$

6. Multiply $\frac{5an}{c^3(a-x)}$ by $3ac^2$. *Ans.* $\frac{15a^2n}{c(a-x)}$.

7. Multiply $\frac{mx}{(m-x)^2}$ by $2(m-x)$. *Ans.* $\frac{2mx}{m-x}$.

8. Multiply $\frac{3a^2z}{(a-x)^2}$ by a^2-x^2. *Ans.* $\frac{3a^2z(a+x)}{a-x}$.

9. Multiply $\frac{x+y}{x^2-2xy+y^2}$ by x^2-y^2. *Ans.* $\frac{(x+y)^2}{x-y}$.

10. Multiply $\frac{5a^2x}{a-1}$ by a^2-1. *Ans.* $5a^3x+5a^2x$.

11. Multiply $\frac{3a^2z^3}{x^3-x}$ by $2a(x-1)$. *Ans.* $\frac{6a^3z^3}{x(x+1)}$.

CASE II.

143. To multiply an entire or fractional quantity by a fraction.

1. Multiply a by $\frac{b}{c}$.

SOLUTION. a multiplied by b is ab; hence, a multiplied by b divided by c is ab divided by c, or $\frac{ab}{c}$.

OPERATION.
$$a\times\frac{b}{c}=\frac{ab}{c}$$

2. Multiply $\frac{a}{b}$ by $\frac{c}{d}$.

SOLUTION. $\frac{a}{b}$ multiplied by c is $\frac{ac}{b}$; hence, $\frac{a}{b}$ multiplied by c divided by d must be $\frac{ac}{b}$ divided by d, which is $\frac{ac}{bd}$. (Prin. 2, Art. 129.)

OPERATION.
$$\frac{a}{b}\times\frac{c}{d}=\frac{ac}{bd}$$

SOLUTION 2D. From Art. 135, Prin., $\frac{a}{b}$ equals ab^{-1} and $\frac{c}{d}$ equals cd^{-1}, and ab^{-1} multiplied by cd^{-1} equals $acb^{-1}d^{-1}$, which, by Art. 135, Prin., is equal to $\frac{ac}{bd}$.

OPERATION.
$$\frac{a}{b}\times\frac{c}{d}=ab^{-1}\times cd^{-1}=acb^{-1}d^{-1}=\frac{ac}{bd}$$

Rule.—*Multiply the numerators together for the numerator, and the denominators together for the denominator, canceling common factors.*

NOTES.—1. When there are common factors in the numerators and denominators, indicate the multiplication and then cancel the common factors.

2. If either factor is a mixed quantity, reduce it to a fraction before multiplying.

EXAMPLES.

3. Multiply $\frac{a}{m}$ by $\frac{c}{n}$. *Ans.* $\frac{ac}{mn}$.

4. Multiply $\frac{2ax}{3c}$ by $\frac{3x^2}{2a}$. *Ans.* $\frac{x^3}{c}$.

5. Multiply $\frac{3a^2c}{2n^2}$ by $\frac{4cn}{5a^3}$. *Ans.* $\frac{6c^2}{5an}$.

6. Multiply $\frac{ab^{2n}}{x^5}$ by $\frac{ax^3}{mb^n}$. *Ans.* $\frac{a^2b^n}{mx^2}$.

7. Multiply $\frac{a-x}{a^2}$ by $\frac{a^3x^2}{3b}$. *Ans.* $\frac{ax^2(a-x)}{3b}$.

8. Multiply $\frac{a+c}{c^2}$ by $\frac{a+x}{a+c}$. *Ans.* $\frac{a+x}{c^2}$.

9. Multiply $\frac{a+x}{4ax}$ by $\frac{a-x}{a+x}$. *Ans.* $\frac{a-x}{4ax}$.

10. Multiply $\frac{1-a^2}{6a^3}$ by $\frac{4ab^3}{1-a}$. *Ans.* $\frac{2b^3(1+a)}{3a^2}$.

11. Multiply $\frac{(a-b)^2}{a+b}$ by $\frac{(a+b)^2}{a-b}$. *Ans.* a^2-b^2.

12. Multiply $a-\frac{a}{c}$ by $\frac{2bc}{3a}$. *Ans.* $\frac{2b(c-1)}{3}$.

13. Multiply $a+\frac{a}{x}$ by $a-\frac{a}{x}$. *Ans.* $a^2-\frac{a^2}{x^2}$.

14. Multiply $\frac{n^2-z^2}{3m^2}$ by $\frac{6n^2}{n+z}$. *Ans.* $\frac{2n^2(n-z)}{m^2}$

15. Multiply $\frac{a^2+ab}{(1+b)^2}$ by $\frac{c+bc}{a+b}$. *Ans.* $\frac{ac}{1+b}$.

16. Multiply $\frac{ac+bc}{(a-b)^2}$ by $\frac{a^2-ab}{c^2}$. *Ans.* $\frac{a(a+b)}{c(a-b)}$.

17. Multiply $\frac{a}{b}+\frac{c}{d}$ by $\frac{b}{a}+\frac{d}{c}$. *Ans.* $2+\frac{bc}{ad}+\frac{ad}{bc}$.

18. Multiply $\frac{n}{m+n}$, $\frac{m^2-n^2}{m^2}$ and $\frac{m}{m-n}$ together. *Ans.* $\frac{n}{m}$.

19. Multiply $\frac{1-x^2}{1-c}$ by $\frac{1-c^2}{x+x^2}$. *Ans.* $\frac{(1-x)(1+c)}{x}$.

20. Multiply $\frac{a(a-b)}{a^2+2ab+b^2}$ by $\frac{a(a+b)}{a^2-2ab+b^2}$. *Ans.* $\frac{a^2}{a^2-b^2}$.

21. Multiply $\frac{a^4-b^4}{a^2-2ab+b^2}$ by $\frac{a-b}{a^2+ab}$. *Ans.* $\frac{a^2+b^2}{a}$.

22. Required the value of $\left(\frac{a^2}{b-c}-\frac{b^2}{b-c}\right)\times\left(\frac{b}{a^2-b^2}-\frac{c}{a^2-b^2}\right)$. *Ans.* 1.

23. Required the value of $\left(\frac{a+b}{b-c}\right)(a-b)\times\left(\frac{b-c}{a+b}\right)\left(\frac{1}{a-b}\right)$. *Ans.* 1.

DIVISION.

144. Division of Fractions is the process of dividing when one or both terms are fractional.

CASE I.

145. To divide a fraction by an entire quantity.

1. Divide $\frac{ab}{c}$ by b.

SOLUTION. Since dividing the numerator of a fraction divides the fraction (Prin. 2, Art. 129), to divide $\frac{ab}{c}$ by b we divide the numerator by b, and have $\frac{a}{c}$.

OPERATION.
$\frac{ab}{c}\div b=\frac{a}{c}$

2. Divide $\frac{a}{c}$ by b.

Solution. Since multiplying the denominator of a fraction divides the fraction, to divide $\frac{a}{c}$ by b we multiply the denominator by b, and have $\frac{a}{bc}$.

OPERATION

$$\frac{a}{c}\div b=\frac{a}{bc}$$

Rule.—*Divide the numerator or multiply the denominator of the fraction, by the divisor.*

Note.—It is often convenient to indicate the division and then cancel common factors.

EXAMPLES.

3. Divide $\frac{6ax^3}{bc}$ by $2ax$. *Ans.* $\frac{3x^2}{bc}$.
4. Divide $\frac{12b^2c^4}{ad}$ by $4bc^3$. *Ans.* $\frac{3bc}{ad}$.
5. Divide $\frac{abcd}{mn}$ by $2x^2$. *Ans.* $\frac{abcd}{2mnx^2}$.
6. Divide $\frac{15x^2z^5}{a^3b^2}$ by $5z^4c$. *Ans.* $\frac{3x^2z}{a^3b^2c}$.
7. Divide $\frac{a^2(x-z)}{3c}$ by $2a^4c$. *Ans.* $\frac{x-z}{6a^2c^2}$.
8. Divide $\frac{x^2-1}{ab^2}$ by $a(x+1)$. *Ans.* $\frac{x-1}{a^2b^2}$.
9. Divide $\frac{a^3-ab^2}{a-c}$ by $c^2(a-b)$. *Ans.* $\frac{a(a+b)}{c^2(a-c)}$.
10. Divide $\frac{a^3x-ax^3}{c^n}$ by ac^n+c^nx. *Ans.* $\frac{ax(a-x)}{c^{2n}}$.

CASE II.

146. To divide an entire or a fractional quantity by a fraction.

1. Divide a by $\frac{b}{c}$.

Solution. a divided by b equals $\frac{a}{b}$; but since the divisor is b divided by c, the quotient must be c times as great, or $\frac{a}{b}\times c=\frac{ac}{b}$.

OPERATION.

$$a\div\frac{b}{c}=\frac{ac}{b}$$

2. Divide $\frac{a}{b}$ by $\frac{c}{d}$.

Solution. $\frac{a}{b}$ divided by c equals $\frac{a}{b\times c}$ (Prin. 2, Art. 129); but since the divisor is c divided by d, the quotient must be d times as great, or $\frac{a}{b\times c}\times d$, which equals $\frac{a\times d}{b\times c}$, or $\frac{ad}{bc}$.

OPERATION.

$$\frac{a}{b}\div\frac{c}{d}=\frac{a\times d}{b\times c}=\frac{ad}{bc}$$

Solution 2d. From Art. 135, Prin., $\frac{a}{b}=ab^{-1}$, and also $\frac{c}{d}=cd^{-1}$; ab^{-1} divided by cd^{-1} equals $\frac{ab^{-1}}{cd^{-1}}$, which equals $\frac{ad}{bc}$.

OPERATION.

$$\frac{a}{b}\div\frac{c}{d}=ab^{-1}\div cd^{-1}=\frac{ab^{-1}}{cd^{-1}}=\frac{ad}{bc}$$

By inspection, we see that the same result can be obtained by inverting the terms of the divisor and multiplying; hence the following

Rule.—*Invert the terms of the divisor and proceed as in multiplication.*

Notes.—1. When there are common factors in numerators and denominators, indicate the operation and then cancel common factors.

2. If either term is a mixed quantity, reduce it to a fraction before dividing.

EXAMPLES.

3. Divide $\frac{a^2}{c}$ by $\frac{b^2}{d}$. *Ans.* $\frac{a^2d}{b^2c}$.
4. Divide $\frac{ax}{cy}$ by $\frac{a^2}{xy^3}$. *Ans.* $\frac{x^2y^2}{ac}$.
5. Divide $\frac{4c^3x}{5ac^2}$ by $\frac{3cx^2}{5a^2b}$. *Ans.* $\frac{4ab}{3x}$.
6. Divide $(a-x)$ by $\frac{(a-x)c}{2a^2}$. *Ans.* $\frac{2a^2}{c}$.
7. Divide $\frac{(a-x)^2}{3c}$ by $\frac{a-x}{4a^2}$. *Ans.* $\frac{4a^2(a-x)}{3c}$.
8. Divide $3a+\frac{c}{d}$ by $\frac{a}{x}$. *Ans.* $\frac{3adx+cx}{ad}$.
9. Divide $a+\frac{b}{c}$ by $a+\frac{d}{c}$. *Ans.* $\frac{ac+b}{ac+d}$.

10. Divide $\frac{a+1}{2a}$ by $\frac{a-1}{4a^2}$. *Ans.* $\frac{2a(a+1)}{a-1}$

11. Divide $\frac{a^2-x^2}{a-1}$ by $\frac{a+x}{a(a-1)}$. *Ans.* $a(a-x)$.

12. Divide $\frac{ax^2-bx^3}{3a}$ by $\frac{5cx^2}{6ab}$. *Ans.* $\frac{2b(a-bx)}{5c}$

13. Divide $\frac{4a^3n}{a^2-b^2}$ by $\frac{an^3}{a+b}$. *Ans.* $\frac{4a^2}{n^2(a-b)}$.

14. Divide $1+\frac{1}{x}$ by $1-\frac{1}{x^2}$. *Ans.* $\frac{x}{x-1}$

15. Divide $1+\frac{a^n}{x^n}$ by $1+\frac{x^n}{a^n}$. *Ans.* $\frac{a^n}{x^n}$

16. Divide $\frac{(x-1)^2}{a^2-1}$ by $\frac{x^2+x-2}{a-1}$. *Ans.* $\frac{x-1}{(a+1)(x+2)}$.

17. Divide $\frac{a^2-b^2}{a^2+ax}$ by $\frac{(a-b)^2}{a+x}$. *Ans.* $\frac{a+b}{a(a-b)}$.

18. Divide $\frac{x^2+3x+2}{x+3}$ by $\frac{x^2+x}{x+3}$. *Ans.* $1+\frac{2}{x}$.

19. Divide $\frac{x^2-5x+6}{x+4}$ by $\frac{x-2}{x^2+x-12}$. *Ans.* $(x-3)^2$.

20. Divide $1-\frac{a^{2n}}{x^{2n}}$ by $1+\frac{a^n}{x^n}$. *Ans.* $1-\frac{a^n}{x^n}$.

COMPLEX FRACTIONS.

147. A Complex Fraction is one in which the numerator or denominator, or both, contain a fraction.

1. Reduce $\frac{\frac{a}{b}}{\frac{c}{d}}$ to a simple fraction.

OPERATION.

$$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b} \div \frac{c}{d} =$$
$$\frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}$$

SOLUTION. This complex fraction may be regarded as an expression of $\frac{a}{b}$ divided by $\frac{c}{d}$, which equals $\frac{a}{b} \times \frac{d}{c}$ or $\frac{ad}{bc}$

OPERATION.

$$\frac{\frac{a}{b} \times bd}{\frac{c}{d} \times bd} = \frac{ad}{bc}$$

SOLUTION 2D. Since multiplying both terms of a fraction by the same quantity does not change its value (Prin. 3, Art. 129), if we multiply both terms of the complex fraction by the least common multiple of their denominators, b and d, we will have the complex fraction equal to $\frac{ad}{bc}$.

Rule.—*Divide the numerator by the denominator, as in division;*

Or, *Multiply both terms of the complex fraction by the least common multiple of their denominators.*

EXAMPLES.

2. Reduce $\frac{\frac{a}{c}}{\frac{x}{z}}$ to a simple fraction. *Ans.* $\frac{az}{cx}$.

3. Reduce $\frac{\frac{2a^2}{c^3}}{\frac{4ax}{bc}}$ to a simple fraction. *Ans.* $\frac{ab}{2c^2x}$.

4. Reduce $\frac{\frac{a}{2}-\frac{b}{a}}{1+\frac{1}{a}}$ to a simple fraction. *Ans.* $\frac{a^2-2b}{2a+2}$

5. Reduce $\frac{1+\frac{1}{c}}{a+\frac{1}{a}}$ to a simple fraction. *Ans.* $\frac{a(c+1)}{c(a^2+1)}$.

6. Reduce $\frac{\frac{c}{c-1}-1}{1-\frac{c}{c+1}}$ to a simple fraction. *Ans.* $\frac{c+1}{c-1}$.

7. Reduce $\frac{\frac{a+b}{x+y}}{\frac{a^2-b^2}{x^2-y^2}}$ to a simple fraction. *Ans.* $\frac{x-y}{a-b}$

8. Reduce $1-\dfrac{1}{1+\dfrac{1}{a}}$ to a simple fraction. *Ans.* $\dfrac{1}{a+1}$

9. Reduce $\dfrac{1}{1-\dfrac{1}{1+\dfrac{1}{n}}}$ to a simple fraction. *Ans.* $n+1$.

10. Reduce $\dfrac{a-1+\dfrac{6}{a-6}}{a-2+\dfrac{3}{a-6}}$ to a simple fraction. *Ans.* $\dfrac{a-4}{a-5}$.

VANISHING FRACTIONS.

148. A **Vanishing Fraction** is one which reduces to the form $\frac{0}{0}$ when certain suppositions are made.

Thus, $\dfrac{x^2-1}{x-1}$, when $x=1$, becomes $\dfrac{1-1}{1-1}$, or $\dfrac{0}{0}$. So, also, $\dfrac{a^2-x^2}{a-x}$, when $a=x$, becomes equal to $\frac{0}{0}$.

PRINCIPLE.—*Vanishing fractions contain a common factor in the numerator and denominator, which reduces to zero when a special supposition is made.*

Thus, $\dfrac{x^2-1}{x-1}$ by factoring becomes $\dfrac{(x-1)(x+1)}{x-1}$, in which the factor, $x-1$, is common to both terms, and is equal to 0 when $x=1$.

1. Find the value of $\dfrac{a^2-x^2}{a-x}$ when $a=x$.

SOLUTION. If we substitute a for x, we will have $\frac{0}{0}$; but factoring the numerator and dividing by the denominator, we have the fraction equal to $a+x$. Substituting the value of x, we have $a+a$, or $2a$. Hence, the value of the given fraction when $a=x$ is $2a$.

OPERATION.

$$\frac{a^2-x^2}{a-x}=\frac{(a+x)(a-x)}{a-x}=a+x=2a$$

Rule.—*Cancel the common factor which reduces to zero, and then make the supposition which reduced the fraction to* $\frac{0}{0}$.

EXAMPLES.

2. Find the value of $\dfrac{x^2-1}{x-1}$ when $x=1$. *Ans.* 2.
3. Find the value of $\dfrac{x^3-1}{x-1}$ when $x=1$. *Ans.* 3.
4. Find the value of $\dfrac{x^3-a^3}{x-a}$ when $x=a$. *Ans.* $3a^2$.
5. Find the value of $\dfrac{x^3-a^3}{x^2-a^2}$ when $x=a$. *Ans.* $\dfrac{3a}{2}$.
6. Find the value of $\dfrac{x^4-a^4}{x-a}$ when $x=a$. *Ans.* $4a^3$.
7. Find the value of $\dfrac{(x-a)^2}{x^3-a^3}$ when $x=a$. *Ans.* 0.
8. Find the value of $\dfrac{x-x^6}{1-x}$ when $x=1$. *Ans.* 5.
9. Find the value of $\dfrac{x^2+2x-15}{x^2+4x-21}$ when $x=3$. *Ans.* $\dfrac{4}{5}$.
10. Find the value of $\dfrac{1-x^m}{1-x}$ when $x=1$. *Ans.* m.
11. Find the value of $\dfrac{x^n-a^n}{x-a}$ when $x=a$. *Ans.* na^{n-1}.

REVIEW QUESTIONS.

Define a Fraction. The Terms. Numerator. Denominator. A Mixed Quantity. State the principles. What is the sign of a fraction? State the principles of the signs.

Define Reduction of Fractions. State the cases. The rule for each case. Define Addition. Subtraction. Multiplication. Divison. How many cases in each? Give the rule for each case. How is a quantity changed from one term of a fraction to another?

Define a Complex Fraction. Give the rule for the reduction of complex fractions to simple fractions. Define a Vanishing Fraction. When does a fraction become a vanishing fraction? How do we find the value of a vanishing fraction?

SECTION V.

SIMPLE EQUATIONS.

149. An **Equation** is an expression of equality between two equal quantities. Thus, $2x+4=20$ is an equation.

150. The **First Member** of an equation is the quantity on the left of the sign of equality.

151. The **Second Member** of an equation is the quantity on the right of the sign of equality.

152. A **Numerical Equation** is one in which the known quantities are expressed by figures; as, $3x-4=17$.

153. A **Literal Equation** is one in which some or all of the known quantities are expressed by letters; as, $2x-a=b$, or $x-2=b$.

154. An **Identical Equation** is one in which the two members are the same, or will become the same by performing the operations indicated; as, $3x+1=3x+1$; or, $3x+a=2x+2a+x-a$.

155. The **Degree** of an equation containing but one unknown quantity is determined by the highest power of the unknown quantity.

156. A **Simple Equation**, or an *equation of the first degree*, is one in which the first power is the highest power of the unknown quantity; as, $x+a=b$.

157. A **Quadratic Equation**, or an *equation of the second degree*, is one in which the second power is the highest power of the unknown quantity; as, $x^2+ax=b$, or $x^2=a$.

158. A **Cubic Equation**, or an *equation of the third degree*, is one in which the third power is the highest power of the unknown quantity; as, $x^3+4x^2+3x=8$.

NOTE.—The symbol = was introduced by *Robert Recorde*, who gave as his reason for it that "noe 2 thynges can be moare equalle" than two parallel lines.

TRANSFORMATION OF EQUATIONS.

159. The **Transformation** of an equation is the process of changing the terms without affecting the equality of the members.

160. Equations may be transformed by means of the following axiomatic principles:

PRINCIPLES.

1. *The same or equal quantities may be added to both members of an equation.*
2. *The same or equal quantities may be subtracted from both members of an equation.*
3. *Both members of an equation may be multiplied by the same or equal quantities.*
4. *Both members of an equation may be divided by the same or equal quantities.*
5. *Both members of an equation may be raised to the same power.*
6. *Both members of an equation may have the same root extracted.*

161. In the transformation of equations there are two principal cases—

1. Clearing of fractions: 2. Transposition of the terms.

CASE I.

162. To clear an equation of fractions.

1. Clear $\frac{3x}{4}-\frac{2x}{3}=\frac{5}{6}$ of fractions.

SOLUTION. The least common multiple of the denominators is 12; multiplying both members of the equation by 12, we have $9x-8x=10$; hence, multiplying both members of the equation by the least common multiple of the denominators clears it of fractions and does not change the equality of the members. (Prin. 3.)

OPERATION.

$$\frac{3x}{4}-\frac{2x}{3}=\frac{5}{6}$$

$$9x-8x=10$$

Rule.—*Multiply both members of the equation by the least common multiple of the denominators, reducing fractional terms to integers.*

NOTES.—1. An equation may be cleared of fractions by multiplying each term by all the denominators.

2. If a fraction has the minus sign before it, the signs of all the terms of the numerator must be changed when the denominator is removed.

EXAMPLES.

Clear the following equations of fractions:

2. $\frac{x}{2}+\frac{x}{3}=\frac{5}{3}$. *Ans.* $3x+2x=10$.

3. $\frac{2x}{3}+\frac{3x}{4}=\frac{5}{6}$. *Ans.* $8x+9x=10$.

4. $\frac{3x}{4}-\frac{5x}{6}=\frac{7}{8}$. *Ans.* $18x-20x=21$.

5. $\frac{x}{2}+\frac{x}{6}=4-\frac{x}{3}$. *Ans.* $3x+x=24-2x$.

6. $\frac{x}{3}-a+6=\frac{a+x}{6}$. *Ans.* $2x-6a+36=a+x$.

7. $\frac{x}{a-b}-b=\frac{c}{2}$. *Ans.* $2x-2ab+2b^2=ac-bc$.

8. $\frac{ax}{3}-\frac{bx}{4}=\frac{cx-dx}{6}$. *Ans.* $4ax-3bx=2cx-2dx$.

9. $\frac{2x}{a+b}-\frac{3x}{a-b}=4$. *Ans.* $2ax-2bx-3ax-3bx=4a^2-4b^2$.

10. $\frac{3x-4}{a}=2-3a^{-1}$. *Ans.* $3x-4=2a-3$.

11. $\frac{x+1}{x-1}-\frac{x-1}{x+1}=\frac{3a}{x^2-1}$. *Ans.* $(x+1)^2-(x-1)^2=3a$.

12. $\frac{2-\frac{x}{2}}{\frac{x}{3}}=\frac{1-\frac{1}{2}}{\frac{2}{3}}$. *Ans.* $8-2x=2x-x$.

13. $\frac{x+3}{x-3}-\frac{x-3}{x+3}=6\frac{6}{7}$. *Ans.* $7(x+3)^2-7(x-3)^2=48(x^2-9)$.

CASE II.

163. To transpose the terms of an equation.

164. Transposition is the process of changing a term from one member of an equation to the other without affecting the equality of the members.

1. In $x+b=a$, transpose b to the second member.

SOLUTION. Subtracting b from both members, which does not affect the equality of the members (Prin. 2, Art. 160), we have $x=a-b$.

OPERATION.

$$\begin{array}{r} x+b=a \\ b=b \\ \hline x=a-b \end{array}$$

2. In $ax-a=bx+c$, transpose a and bx.

SOLUTION. Adding a to both members and subtracting bx from both members, which, according to Prin. 1 and 2, will not affect the equality of the members, we have $ax-bx=a+c$.

OPERATION.

$$\begin{array}{r} ax-a=bx+c \\ a=a \\ \hline ax=bx+a+c \\ bx=bx \\ \hline ax-bx=a+c \end{array}$$

In both of these examples we see that in changing a quantity from one member to the other, the *sign* of the quantity is changed; hence the following rule.

Rule.—*A term may be transposed from one member of an equation to the other, if, at the same time, the sign be changed.*

EXAMPLES.

In the following examples transpose the known terms to the second member and the unknown terms to the first member:

3. $2x+c=a$. *Ans.* $2x=a-c$.

4. $3x-2=x+4$. *Ans.* $3x-x=4+2$.

5. $5x+b=3x+a$. *Ans.* $5x-3x=a-b$.

6. $3a-2b=6x-ax$. *Ans.* $ax-6x=2b-3a$.

7. $a^2-a^2x-an=3x^2$. *Ans.* $-3x^2-a^2x=an-a^2$.

8. $ax^2+5ac-2a=cx$. *Ans.* $ax^2-cx=2a-5ac$.

9. $2x^3-\frac{a-b}{2}=5a+\frac{6x^2-x}{3}$. *Ans.* $2x^2-\frac{6x^2-x}{3}=5a+\frac{a-b}{2}$.

SOLUTION OF SIMPLE EQUATIONS,

CONTAINING ONE UNKNOWN QUANTITY.

165. The **Solution** of an equation is the process of finding the value of the unknown quantity.

166. The **Root** of an equation is the value of the unknown quantity.

167. To **Verify** the root of an equation, we substitute its value for the unknown quantity and reduce the members to identity. The equation is then said to be *satisfied*.

NOTE.—The solution of an equation is often called the *reduction of the equation*. *To reduce an equation* is therefore to solve it.

CASE I.

NUMERICAL EQUATIONS.

1. Find the value of x in the equation $3x-4=12-x$.

SOLUTION. Transposing the unknown terms to the first member and the known terms to the second member, we have $3x+x=12+4$; uniting the terms, we have $4x=16$; dividing by the coefficient of x, we have $x=4$.

OPERATION.

$$3x-4=12-x$$
$$3x+x=12+4$$
$$4x=16$$
$$x=4$$

VERIFICATION. Substituting for x its value in the given equation, we have $3\times 4-4=12-4$; reducing, we have $8=8$; and since this is an identical equation, the root is *verified*.

VERIFICATION.

$$3\times 4-4=12-4$$
$$8=8$$

2. Find the value of x in the equation $\frac{x}{2}-\frac{5}{6}=\frac{1}{2}+\frac{x}{6}$.

SOLUTION. Clearing the equation of fractions by multiplying by 6, we have $3x-5=3+x$; transposing the terms, we have $3x-x=3+5$; uniting the terms, we have $2x=8$; dividing by the coefficient of x, we have $x=4$.

OPERATION

$$\frac{x}{2}-\frac{5}{6}=\frac{1}{2}+\frac{x}{6}$$
$$3x-5=3+x$$
$$3x-x=3+5$$
$$2x=8$$
$$x=4$$

VERIFICATION.—Substituting the value of x in the equation, we have $\frac{4}{2}-\frac{5}{6}=\frac{1}{2}+\frac{4}{6}$; reducing, we have $\frac{7}{6}=\frac{7}{6}$; and since this is an identical equation, the root is *verified*.

VERIFICATION

$$\frac{4}{2}-\frac{5}{6}=\frac{1}{2}+\frac{4}{6}$$
$$\frac{7}{6}=\frac{7}{6}$$

Rule.—I. *Clear the equation of fractions, if necessary.*

II. *Transpose the unknown terms to the first member of the equation and the known terms to the second member.*

III. *Reduce each member to its simplest form, and divide both members by the coefficient of the unknown quantity.*

To verify the result: *Substitute the value of the unknown quantity in the equation, and if the members are identical the result is correct.*

NOTES.—1. It is sometimes advantageous to transpose and make some reductions before clearing of fractions.

2. When the coefficient of the unknown quantity is negative we may multiply both members by -1, or divide by the negative coefficient.

EXAMPLES.

3. Given $4x+6=2x+10$, to find x. *Ans.* $x=2$.
4. Given $5x+4-2x=10+x$, to find x. *Ans.* $x=3$.
5. Given $18-3x=5x+2$, to find x. *Ans.* $x=2$.
6. Given $7x+6=5x+14$, to find x. *Ans.* $x=4$.
7. Given $4x-17=4x+13-6x$, to find x. *Ans.* $x=5$.
8. Given $2x-12=5x-30$, to find x. *Ans.* $x=6$.
9. Given $6x+16=9x-5$, to find x. *Ans.* $x=7$.
10. Given $\frac{x}{2}+\frac{x}{3}=5\frac{1}{3}+4\frac{2}{3}$, to find x. *Ans.* $x=12$.
11. Given $\frac{x}{3}-\frac{x}{4}+\frac{x}{5}=2$, to find x. *Ans.* $x=7\frac{1}{17}$.
12. Given $\frac{x}{2}-\frac{x}{6}+3=\frac{x}{5}+4\frac{3}{5}$, to find x. *Ans.* $x=12$.
13. Given $\frac{3x}{5}+\frac{5x}{6}=\frac{4x}{5}+1\frac{4}{15}$, to find x. *Ans.* $x=2$.
14. Given $\frac{2x}{7}-\frac{3x}{4}=\frac{7x}{6}-11\frac{5}{12}$, to find x. *Ans.* $x=7$.
15. Given $4(x+1)=3(x+2)$, to find x. *Ans.* $x=2$.
16. Given $\frac{2x}{3}+\frac{x-1}{6}=\frac{3x+1}{2}-10$, to find x. *Ans.* $x=14$.

17. Given $\frac{x+3}{2}+\frac{x}{3}=4-\frac{x-5}{4}$, to find x. *Ans.* $x=3\frac{6}{13}$.

18. Given $x+\frac{2x-4}{3}=12-\frac{3x-5}{2}$, to find x. *Ans.* $x=5$.

19. Given $\frac{x-5}{3}+\frac{x}{2}=12-\frac{x-10}{3}$, to find x. *Ans.* $x=14\frac{4}{7}$.

20. Given $\frac{x+3}{2}-\frac{x-2}{3}=\frac{3x-5}{12}+\frac{1}{4}$, to find x. *Ans.* $x=28$.

CASE II.

LITERAL EQUATIONS.

1. Given $ax+bx=ac+bc$, to find x.

OPERATION.

$$ax+bx=ac+bc \quad (1)$$
$$(a+b)x=(a+b)c \quad (2)$$
$$x=c \quad (3)$$

SOLUTION. Factoring both members of equation (1), we obtain equation (2); dividing by $(a+b)$, the coefficient of x, we obtain $x=c$.

EXAMPLES.

2. Given $ax+bx=an+bn$, to find x. *Ans.* $x=n$.

3. Given $ax+d=c-bx$, to find x. *Ans.* $x=\frac{c-d}{a+b}$.

4. Given $nx-c=nc-x$, to find x. *Ans.* $x=c$.

5. Given $mx-n=nx+m$, to find x. *Ans.* $x=\frac{m+n}{m-n}$.

6. Given $nx+m=mx+n$, to find x. *Ans.* $x=1$.

7. Given $ax+b=\frac{x}{a}+\frac{1}{b}$, to find x. *Ans.* $x=\frac{a(1-b^2)}{b(a^2-1)}$.

8. Given $\frac{x}{a}+\frac{x}{b-a}=\frac{a}{b+a}$, to find x. *Ans.* $x=\frac{a^2(b-a)}{b(b+a)}$.

9. Given $\frac{1-x}{1+x}=1-\frac{1}{c}$, to find x. *Ans.* $x=\frac{1}{2c-1}$.

10. Given $x+a=\frac{x^2}{a+x}$, to find x. *Ans.* $x=-\frac{a}{2}$.

11. Given $(a+x)(b+x)-a(b+c)=\frac{a^2c}{b}+x^2$, to find x. *Ans.* $x=\frac{ac}{b}$.

12. Given $\frac{a+b}{a-x}+\frac{b+c}{c-x}=\frac{a-b}{a-x}$, to find x. *Ans.* $x=\frac{ab+2bc+ac}{c+3b}$

SPECIAL ARTIFICES.

168. The solution of a problem may often be abridged by the use of particular operations, called *Artifices.*

CASE I.

169. Uniting terms before clearing of fractions.

1. Given $\frac{4x}{5}+12=\frac{3x}{4}+15$, to find x.

OPERATION.

$$\frac{4x}{5}+12=\frac{3x}{4}+15 \quad (1)$$
$$\frac{4x}{5}=\frac{3x}{4}+3 \quad (2)$$
$$16x=15x+60 \quad (3)$$
$$x=60$$

SOLUTION. Transposing the 12 and uniting the terms, we have equation (2); clearing of fractions, we have equation (3); transposing and uniting terms, we have $x=60$.

EXAMPLES.

2. Given $2x-4=\frac{x}{2}+2$, to find x. *Ans.* $x=4$.

3. Given $\frac{x}{2}-6+\frac{x}{3}=\frac{x}{6}+2$, to find x. *Ans.* $x=12$.

4. Given $\frac{3x}{4}+\frac{4x}{5}-1\frac{5}{6}=\frac{3x}{5}+17\frac{1}{5}$, to find x. *Ans.* $x=20$.

5. Given $\frac{x}{a}-\frac{x-a}{3}-a=2a$, to find x. *Ans.* $x=\frac{8a^2}{3-a}$.

6. Given $\frac{2x}{3}-3\frac{1}{3}+13=13\frac{11}{12}-\frac{3x}{4}$, to find x. *Ans.* $x=3$.

7. Given $\frac{3x}{4}-4\frac{1}{2}+a=a+\frac{x}{5}+1$, to find x. *Ans.* $x=10$.

8. Given $2ax+3m-\frac{1}{3}cx=ax+2m+\frac{2}{3}cx+n$, to find x. *Ans.* $x=\frac{n-m}{a-c}$.

9. Given $3ax-2bx-\frac{1}{3}c-\frac{1}{4}mx=\frac{2}{3}c+\frac{3}{4}mx-n-bx+2ax$, to find x.

Ans. $x=\dfrac{c-n}{a-b-m}$.

CASE II.

170. Indicating some of the operations.

1. Given $\frac{x}{3}+\frac{x}{4}+\frac{x}{5}=47$, to find x.

SOLUTION. We multiply both members by 60, the least common multiple of the denominators, indicating the operation in the second member, and obtain (2); reducing, we have (3); dividing by 47, we have (4).

OPERATION.

$$\frac{x}{3}+\frac{x}{4}+\frac{x}{5}=47 \quad (1)$$
$$20x+15x+12x=47\times 60 \quad (2)$$
$$47x=47\times 60 \quad (3)$$
$$x=60 \quad (4)$$

EXAMPLES.

2. Given $\frac{x}{2}+\frac{x}{3}+\frac{x}{4}=26$, to find x. Ans. $x=24$.

3. Given $\frac{x}{2}+\frac{x}{6}+\frac{x}{12}=45$, to find x. Ans. $x=60$.

4. Given $\frac{1}{3}x+\frac{1}{5}x+\frac{1}{6}x=42$, to find x. Ans. $x=60$.

5. Given $3x+\frac{2x}{3}+\frac{5x}{6}=54$, to find x. Ans. $x=12$.

CASE III.

171. Substituting some other unknown quantity for a common expression.

1. Given $x+2+3(x+2)=\frac{x+2}{4}+15$, to find x.

SOLUTION. Let y represent $x+2$; substituting y for $x+2$, we have equation (2); uniting terms, clearing of fractions, etc., we have the equation $y=4$; but $y=x+2$; hence $x+2=4$, from which we have $x=2$.

OPERATION.

$$x+2+3(x+2)=\frac{x+2}{4}+15 \quad (1)$$
$$y+3y=\frac{y}{4}+15 \quad (2)$$
$$4y=\frac{y}{4}+15 \quad (3)$$
$$y=4 \quad (4)$$
$$\therefore x+2=4 \quad (5)$$
$$x=2 \text{ Ans.} \quad (6)$$

2. Given $x+3+2(x+3)=18$, to find x. Ans. $x=3$.

3. Given $\frac{x+4}{2}+\frac{3(x+4)}{4}=\frac{4(x+4)}{5}+4\frac{1}{20}$, to fin(d) x. Ans. $x=5$.

4. Given $\frac{x-5}{3}+\frac{2(x-5)}{4}=\frac{3}{5}(x-5)+14$, to find x. Ans. $x=65$.

5. Given $\frac{2x+4}{3}-\frac{x-3}{4}=\frac{x+2}{3}+3\frac{1}{3}$, to find x. Ans. $x=23$.

6. Given $\frac{x+c}{3}-\frac{3(x+c)}{4}=\frac{1}{3}(x+c)-c$, to find x. Ans. $x=\frac{c}{3}$.

CASE IV.

172. Separating and uniting terms before clear-ing of fractions.

1. Given $\frac{5x+4}{4}-\frac{6x-8}{5x-20}=\frac{10x-5}{8}-\frac{3}{8}$, to find x.

SOLUTION.—Separating the terms, we have equation (2); transposing and reducing, we have equation (3); clearing of fractions, we have equation (4); transposing, uniting terms and dividing by 4, we have equation (6).

OPERATION.

$$\frac{5x+4}{4}-\frac{6x-8}{5x-20}=\frac{10x-5}{8}-\frac{3}{8} \quad (1)$$
$$\frac{5x}{4}+1-\frac{6x-8}{5x-20}=\frac{10x}{8}-\frac{5}{8}-\frac{3}{8} \quad (2)$$
$$2=\frac{6x-8}{5x-20} \quad (3)$$
$$10x-40=6x-8 \quad (4)$$
$$4x=32 \quad (5)$$
$$x=8 \quad (6)$$

NOTE.—Considerable labor is saved by this artifice. Let the pupil solve the problem by the ordinary method and observe the difference.

2. Given $\frac{7-9x}{12}-\frac{12-4x}{5-3x}=\frac{15-6x}{8}-\frac{7}{24}$, to find x. Ans. $x=2\frac{3}{7}$.

3 Given $\frac{6x-15}{9}-\frac{10x-17}{15}=\frac{4x-15}{3-2x}+\frac{2}{15}$, to find x. Ans. $x=4\frac{7}{8}$.

4. Given $\frac{x-16}{18}-\frac{17-4x}{9}=\frac{5x}{7}-\frac{4-26x}{32-17x}-\frac{3x}{14}$, to find x. Ans. $x=4$.

PROBLEMS IN SIMPLE EQUATIONS.

173. A **Problem** is a question requiring some unknown result from things which are known.

174. The **Solution** of a problem is the process of finding the required unknown result.

175. The solution of a problem in Algebra consists of two distinct parts—

1st. The formation of the equation;

2d. The solution of the equation.

176. The **Method of Solving** a problem cannot be stated by any general or precise rule. The following directions may be of some value:

1. *Represent the unknown quantity by one of the final letters of the alphabet.*

2. *Form an equation by indicating the operations necessary to verify the result were it known.*

3. *Solve the equation thus derived.*

NOTE.—The formation of the equation is called the *concrete* part of the solution; the reduction of the equation the *abstract* part. The first part is also called the *statement* of the problem. It is merely a translation of the problem from *common* into *algebraic* language.

PROBLEMS.

CASE I.

1. A farmer bought a cow and a horse for $375, paying 4 times as much for the horse as for the cow; required the cost of each.

SOLUTION. Let x represent the cost of the cow; then, since he paid 4 times as much for the horse as for the cow, $4x$ will represent the cost of the horse; and since both cost $375, we have the equation $x+4x=375$; uniting the terms, we have $5x=375$; dividing by 5, we have $x=75$; and multiplying by 4, we have $4x=300$. Hence, the cow cost $75, and the horse $300.

OPERATION.

Let $x=$ the cost of the cow.
Then $4x=$ the cost of the horse.

$$x+4x=375 \quad (1)$$
$$5x=375 \quad (2)$$
$$x=75, \text{ cost of cow.} \quad (3)$$
$$4x=300, \text{ cost of horse} \quad (4)$$

2. The income of A and B for one month was $1728, and B's income was 3 times A's; required the income of each.

Ans. A's, $432; B's, $1296.

3. A tree, 96 feet high, in falling broke into three unequal parts; the longest piece was 5 times the shortest, and the other was twice the shortest; required the length of each piece.

Ans. 1st, 12 ft.; 2d, 24 ft.; 3d, 60 ft.

4. Divide the number represented by a into 3 parts, such that m times the first part shall equal the second part, and n times the second part shall equal the third part.

Ans. $\frac{a}{1+m+mn}$; $\frac{ma}{1+m+mn}$; $\frac{mna}{1+m+mn}$.

CASE II.

1. A boy bought a book and a toy for $2.25, and the toy cost $\frac{2}{3}$ as much as the book; required the cost of each.

SOLUTION. Let x represent the cost of the book; then will $\frac{2}{3}x$ represent the cost of the toy; and since both cost $2.25, we have the equation $x+\frac{2}{3}x=225$. Clearing of fractions, we have $3x+2x=675$; uniting the terms, we have $5x=675$, from which $x=135$, and $\frac{2}{3}x=90$. Hence, etc.

OPERATION.

Let $x=$ cost of book.
Then $\frac{2}{3}x=$ cost of toy.

$$x+\tfrac{2}{3}x=225 \quad (1)$$
$$3x+2x=675 \quad (2)$$
$$5x=675 \quad (3)$$
$$x=135, \text{ cost of book.}$$
$$\tfrac{2}{3}x=90, \text{ cost of toy.}$$

2. A watch and chain cost $350; what was the cost of each if the chain cost $\frac{3}{4}$ as much as the watch?

Ans. Watch, $200; chain, $150.

3. Divide $2782 among Harry, Harvey and Hinkley, so that Harry may have $\frac{2}{3}$ as much as Hinkley, and Harvey $\frac{3}{4}$ as much as Harry.

Ans. Harry, $856; Harvey, $642; Hinkley, $1284.

4. Divide the number a into 2 parts, so that $\frac{n}{m}$ of the first part shall equal the second part. *Ans.* $\frac{ma}{m+n}$; $\frac{na}{m+n}$.

CASE III.

1. A man weighs 27 lbs. more than his wife, and the sum of their weights is 313 lbs.; required the weight of each.

Ans. Man, 170 lbs.; wife, 143 lbs.

2. An angler's pole and line measure 28 feet, and $\frac{2}{3}$ of the sum obtained by increasing the length of the pole by 12 feet equals the length of the line; required the length of each.

Ans. Pole, 12 ft.; line, 16 ft.

3. The sum of 3 numbers is 215; the first equals twice the second, increased by 15, and the second equals $\frac{2}{3}$ of the remainder of the third diminished by 20; required the numbers.

Ans. 1st, 95; 2d, 40; 3d, 80.

4. Divide the number a into 2 parts, such that the second part shall equal m times the first part, plus n.

Ans. 1st, $\frac{a-n}{1+m}$; 2d, $\frac{ma+n}{1+m}$.

CASE IV.

1. A gentleman gave 6 cents each to some poor children had he given them 9 cents each, it would have taken 48 cents more; how many children were there?

SOLUTION. Let x equal the number of children; then $6x$ will equal what he gave them, and $9x$ will equal what he would have given them by giving them 9 cents each. Then, by the conditions of the problem, we have the equation $9x-6x=48$; uniting the terms, we have $3x=48$, or $x=16$. Hence, there were 16 children.

OPERATION.

Let x = number of children;
then $6x$ = what he gave them;
and $9x$ = what he would have given

$$9x-6x=48 \quad (1)$$
$$3x=48 \quad (2)$$
$$x=16 \quad (3)$$

2. A man gave some beggars 10 cents each, and had 75 cents remaining; had he given them 15 cents each, it would have taken all his money; required the number of beggars.

Ans. 15 beggars.

3. A lady gave 60 cents to some poor children; to each boy she gave 2 cents, and to each girl 4 cents; how many were there of each, provided there were 3 times as many boys as girls?

Ans. 6 girls; 18 boys.

4. A and B had equal sums of money; A bought sheep at $12 each, and had $40 remaining; B bought twice as many lambs at $8 each, and wanted $40 to pay for them; how much did each invest?

Ans. A, $240; B, $320.

5. A man gave a number of beggars m cents each, and had a cents remaining; had he given them n cents each, he would have had b cents remaining; how many beggars were there, and what was his money? *Ans.* Number, $\frac{b-a}{m-n}$; money, $\frac{bm-an}{m-n}$.

CASE V.

1. A can do a piece of work in 6 days, and B in 8 days; in what time can they together do it?

SOLUTION. Let x equal the time in which they together can do the work; then $\frac{1}{x}$ will equal what they both do in a day; but A does $\frac{1}{6}$ of it, and B $\frac{1}{8}$ of it, in a day; hence, $\frac{1}{6}+\frac{1}{8}=\frac{1}{x}$; from which we find $x=3\frac{3}{7}$ days.

OPERATION.

Let x = the time;
$\frac{1}{x}$ = what both do in 1 day;
$\frac{1}{6}$ = what A does in 1 day;
$\frac{1}{8}$ = what B does in 1 day;

then, $\frac{1}{6}+\frac{1}{8}=\frac{1}{x}$;
$$8x+6x=48;$$
$$x=3\tfrac{3}{7} \text{ days.}$$

2. A pound of tea lasted a man and wife 3 months, and the wife alone 4 months; how long will it last the man alone?

Ans. 12 months.

3. A can build a wall in 20 days, B in 30 days, and C in 40 days; in what time can they together build it. *Ans.* $9\frac{3}{13}$ days.

4. A can paper a room in $\frac{1}{2}$ of a day, B in $\frac{1}{3}$ of a day, and C in $\frac{1}{4}$ of a day; in what time will they do it working together?

Ans. $\frac{1}{9}$ of a day.

5. A can do a piece of work in a days, B in b days, C in c days; in what time can they perform it if all work together?

Ans. $\frac{abc}{ab+ac+bc}$ days.

CASE VI.

1. A man receives \$4 a day for his labor, and forfeits \$2 each day he is idle, and at the end of 30 days receives \$60; how many days has he worked?

SOLUTION.

Let x = number of working days,
and $30-x$ = number of idle days;
then, $4x$ = sum earned,
and $(30-x)2$ = sum forfeited.
Then, $4x-2(30-x)=60$
whence, $x=20$, number of days he worked,
and $30-x=10$, number of days he was idle.

2. Fannie James agreed to carry 12 dozen eggs to a store for $\frac{1}{4}$ cent each, on condition that she should forfeit $2\frac{1}{4}$ cents for each one she broke; she received 26 cents; how many were broken? *Ans.* 4.

3. Francis receives \$2.50 a day for his labor, and pays 50 cents a day for his board, and at the end of 40 days has saved \$50; how many days was he idle? *Ans.* 12.

4. A man receives \$$a$ a day for his labor, on condition that he forfeits \$$b$ each day he is idle; at the expiration of n days he has received \$$c$; required the number of working and idle days. *Ans.* Working days, $\frac{c+nb}{a+b}$; idle days, $\frac{an-c}{a+b}$.

CASE VII.

1. The head of a fish is 10 inches long, the tail is as long as the head, plus $\frac{1}{2}$ of the body, and the body is as long as the head and tail both; required the length of the fish.

SOLUTION.

Let x = the length of the body,
and $10+\frac{x}{2}$ = the length of the tail.
Then, $x=10+\frac{x}{2}+10$, by the last condition;
whence, $x=40$, the length of the body, etc.

2. The head of a whale is $\frac{1}{2}$ as long as the tail, plus 3 feet; the tail is $\frac{1}{4}$ as long as the body, plus 4 feet; and the body is twice as long as the head and tail; what is the length of the whale? *Ans.* 108 ft.

3. The artillery of an army corps consisted of 60 men less than $\frac{1}{8}$ of the cavalry; the cavalry consisted of 2040 men more than the artillery; and $\frac{1}{6}$ of the infantry was 370 men less than the cavalry; how many men were in the corps? *Ans.* 19,500.

4. The head of a fish is a inches long; the tail is as long as the head plus 1-nth of the length of the body; and the body is as long as the head and tail; what is the length of the fish? *Ans.* $\frac{4an}{n-1}$.

CASE VIII.

1. How far may a person ride in a coach, going at the rate of 10 miles an hour, and walking back at the rate of 6 miles an hour, provided he is gone 8 hours?

SOLUTION.

Let x = the distance he goes;
then $\frac{x}{10}$ = time in going,
and $\frac{x}{6}$ = time in returning.
Then $\frac{x}{10}+\frac{x}{6}=8$, etc.

2. A steamboat, whose propelling rate in still water is 15 miles an hour, descends a river whose current is 3 miles an hour; how far may it go that it may be gone but 10 hours? *Ans.* 72 miles.

3. An equestrian rides 24 miles, going at a certain rate. He walks back at the rate of 3 miles an hour, and is gone 11 hours. At what rate does he ride? *Ans.* 8 miles an hour.

4. How far may a person ride in a stage-coach going at the rate of a miles an hour, provided he returns immediately by railroad at the rate of c miles an hour, and is gone n hours? *Ans.* $\frac{acn}{a+c}$.

5. A steam-packet, whose propelling rate in still water is a miles an hour, descends a river whose current is c miles an hour; how far may it go that it may be gone n hours?

Ans. $\frac{(a^2-c^2)n}{2a}$.

CASE IX.

1. Eight men hire a coach to ride to Lancaster, but by taking in 4 more persons the expense of each is diminished by $\$\frac{3}{4}$; what do they pay for the coach?

SOLUTION.

Let x = the sum to be paid;

then $\frac{x}{8}$ = share of each by 1st condition,

and $\frac{x}{12}$ = share of each by 2d condition.

Hence, $\frac{x}{8}-\frac{x}{12}=\frac{3}{4}$.

2. Fifteen persons engage a yacht, but, before sailing, 3 of the company decline going, by which the expense of each is increased $\$2\frac{1}{2}$; what do they pay for the yacht?

Ans. \$150.

3. A company of 15 persons engage a dinner at a hotel for \$15, but, before paying the bill, a number of them withdraw, by which each person's bill is augmented $\$\frac{1}{2}$; how many withdraw? *Ans.* 5 persons.

4. A number of persons, n, hire a coach to ride, but, by taking in m more persons, the expense of each is diminished a dollars; what do they pay for the coach? *Ans.* $\frac{(n^2+nm)a}{m}$.

5. A number of persons, n, chartered a steamboat for an excursion, for which they were to pay $\$a$; but, before starting, several of the company declined going, by which each person's share of the expense was increased $\$b$; how many persons went and how many remained?

Ans. Went, $\frac{an}{a+bn}$; remained, $\frac{bn^2}{a+bn}$.

CASE X.

1. A, at a game of chess, won \$120, and then lost $\frac{1}{4}$ of what he then had, and then found he had 3 times as much as at first; how much had he at first?

SOLUTION.

Let x = the sum at first;

then $\frac{3}{4}(x+120)=3x$, etc.

2. A person being asked his age, said that if his age were increased by its $\frac{2}{5}$ and $2\frac{3}{5}$ years, the sum would equal 4 times his age 13 years ago; what was his age? *Ans.* 21 yrs.

3. A merchant lost \$1400 of his stock, and the next year gained $\frac{4}{5}$ as much as remained of his stock, and then had $\frac{3}{4}$ as much as at first; what was his original stock? *Ans.* \$2400.

4. A, having a certain sum of money, found a dollars, and then lost 1-nth of what he then had, and then found he had m times as much as he had at first; how much had he at first?

Ans. $\frac{a(n-1)}{mn-n+1}$.

CASE XI.

1. If 80 lbs. of sea water contain 2 lbs. of salt, how much fresh water must be added to these 80 lbs., so that 10 lbs. of the new mixture may contain $\frac{1}{6}$ of a pound of salt?

SOLUTION.

Let x = number of pounds to be added.

Then, $\frac{2}{80+x}=\frac{1}{60}$,

or $x=40$. *Ans.*

2. In a mixture of silver and copper consisting of 60 oz. there are 4 oz. of copper; how much silver must be added that there may be $\frac{2}{3}$ oz. of copper in 12 oz. of the mixture? *Ans.* 12 oz.

3. In a mixture of gold and silver there are 6 oz. of silver; and if 56 oz. of gold be added, there will be 10 oz. of gold to $\frac{2}{5}$ oz. of silver; how much gold was there at first? *Ans.* 94 oz.

NOTE.—In No. 2, let x = no. of oz. to be added; then $4\div(60+x)=\frac{2}{3}\div12$. In No. 3, let x = no. of oz. of gold at first; then $6\div(x+56)=\frac{2}{5}\div10$.

4. In a mixture of silver and copper there are 4 oz. of copper; and if 12 oz. of silver be added to the mixture, there will be 12 oz. of the mixture to $\frac{2}{3}$ oz. of copper; how many ounces in the mixture? *Ans.* 60 oz.

NOTE.—Let x = no. of ounces of the mixture; then $4 \div (x+12) = \frac{2}{3} \div 12$.

5. If a lbs. of sea water contain b lbs. of salt, how much salt must be added so that m lbs. of the mixture may contain n lbs. of salt? *Ans.* $\frac{an-bm}{m-n}$.

CASE XII.

1. Two men, A and B, in partnership gain \$300. A owns $\frac{2}{3}$ of the stock, lacking \$40, and gains \$180; required the whole stock and share of each.

SOLUTION.

Let x = the stock;

then, $\frac{2x}{3} - 40$ = A's stock,

and $\frac{x}{3} + 40$ = B's stock.

Then $300 \div x = \frac{300}{x}$ = gain on \$1,

and $120 \div \left(\frac{x}{3}+40\right) = \frac{120}{\frac{x}{3}+40}$ = gain on \$1.

Hence, $\frac{300}{x} = \frac{120}{\frac{x}{3}+40}$

from which we find $x = 600$, the entire stock.

2. C and D in partnership gain \$820; C owns \$12,750 of the stock, and D's gain is \$565; required the amount of stock that D owns. *Ans.* \$28,250.

3. Two men engage to build a boat for \$84; the first labors 6 days more than $\frac{1}{3}$ as many as the second, and receives \$48; how many days does each labor? *Ans.* 1st, 8 days; 2d, 6 days.

4. Two men, A and B, in partnership gain \$$a$; A owns 1-$n$th of the stock, lacking \$$b$, and gains \$$c$; required the entire stock and share of each. *Ans.* Stock, $\frac{anb}{a-cn}$.

CASE XIII.

1. What time of day is it, provided $\frac{1}{3}$ of the time past midnight equals the time to noon?

SOLUTION.

Let x = the time past midnight,

and $\frac{x}{3}$ = the time to noon.

Then, $x + \frac{x}{3} = 12$, etc.

2. What is the time of day, provided $\frac{3}{7}$ of the time past midnight equals the time past noon? *Ans.* 9 P. M.

3. What is the hour of day when $\frac{3}{5}$ of the time to noon equals the time past midnight? *Ans.* $4\frac{1}{2}$ A. M.

4. Required the hour of day if $\frac{2}{3}$ of the time past 10 o'clock A. M. equals $\frac{1}{2}$ of the time to midnight. *Ans.* 4 P. M.

5. What time of day is it if $\frac{3}{5}$ of the time past 4 o'clock A. M. equals $\frac{3}{4}$ of the time to 10 o'clock P. M. *Ans.* 2 P. M.

6. What time of day is it if 1-nth of the time past midnight equals the time to noon? *Ans.* $\frac{12n}{n+1}$ A. M.

CASE XIV.

1. A man being asked the time of day said, "It is between 2 and 3 o'clock, and the hour- and minute-hands are together;" what was the time?

SOLUTION.

Let x = the distance the minute-hand goes;

then, $\frac{x}{12}$ = the distance the hour-hand goes.

Then, $x - \frac{x}{12} = 10$, the number of minute-spaces they are apart at 2 o'clock;

whence, $\frac{11x}{12} = 10$, and $x = 10\frac{10}{11}$; $\therefore$ it is $10\frac{10}{11}$ minutes past 2.

2. A man being asked the hour of the day replied, "It is between 3 and 4 o'clock, and the hour- and minute-hands are together;" what was the time? *Ans.* $16\frac{4}{11}$ min. past 3 o'clock.

3. A lady being asked the time of day replied, "It is between 4 and 5 o'clock, and the hands of my watch are 5 minute-spaces apart;" what was the time?

Ans. $16\frac{4}{11}$ min. past 4 o'clock.

4. A companion of the lady also said, "By my watch it is between 4 and 5 o'clock, and the hour- and minute-hands are 5 minutes of time apart;" what was the time by her watch?

Ans. $16\frac{9}{11}$ min. past 4.

5. What is the time of day if it is between m and $m+1$ o'clock, and the hands of the clock are together?

Ans. $5\frac{5}{11}\,m$ min. past m o'clock.

6. What is the time of day if it is between m and $m+1$ o'clock, and the two hands are n minute-spaces apart?

Ans. $\frac{12(5m-n)}{11}$ min. past m o'clock.

CASE XV.

1. A is 6 years old, and B is 5 times as old; in how many years will B be only 4 times as old as A?

SOLUTION.

Let x = the number of years;
then, $6+x$ = A's age at that time,
and $30+x$ = B's age at that time.
Then, $4(6+x)=30+x$, etc.

2. Jones is 10 years old, and Smith is 3 times as old; how long since Smith was 5 times as old as Jones? *Ans.* 5 yrs.

3. Mary is $\frac{1}{4}$ as old as her aunt, but in 20 years she will be $\frac{1}{2}$ as old; what is the age of each? *Ans.* Mary, 10; aunt, 40.

4. Six years ago B's house was 4 times as old as his barn, but 2 years hence it will be only twice as old; how long has each been built? *Ans.* House, 22 yrs.; barn, 10 yrs.

5. A is a years old, and B is b years old; in what time will A be n times as old as B? *Ans.* $\frac{nb-a}{1-n}$ yr.

6. A is m times as old as B, but in c years he will be n times as old as B; required the age of each at present.

Ans. A, $\frac{mc(n-1)}{m-n}$ yr.; B, $\frac{c(n-1)}{m-n}$ yr.

MISCELLANEOUS PROBLEMS.

1. How many roses and pinks in my garden if there are 70 of both, and the number of roses plus $\frac{1}{2}$ of the number of pinks equals 3 times the number of pinks?

Ans. 50 roses; 20 pinks.

2. Five times a certain number, plus 60, equals 3 times the sum obtained by increasing the number by 60; what is the number? *Ans.* 60.

3. Divide the number 130 into 4 parts, so that each part is greater than the immediately preceding one by its $\frac{1}{3}$.

Ans. 16; 24; 36; 54.

4. In an orchard $\frac{1}{2}$ of the trees bear apples, $\frac{1}{3}$ bear peaches, and the remainder, 24, bear plums; how many trees are there in the orchard? *Ans.* 144.

5. Find a number such that, if we add to it its $\frac{1}{2}$, the sum exceeds 60 by as much as the number itself is less than 65.

Ans. 50.

6. A lady has 2 purses; if she puts \$12 in the first, the whole is worth 5 times as much as the second purse; what is the value of each if the first is worth twice as much as the second?

Ans. 1st, \$8; 2d, \$4.

7. In a mixture of copper and zinc, the copper comprised 6 oz. more than $\frac{1}{2}$ of the mixture, and the zinc 4 oz. more than $\frac{2}{3}$ of the copper; how much was there of each?

Ans. Copper, 48 oz.; zinc, 36 oz.

8. A young man received a fortune from England, and spent $\frac{2}{5}$ of it the first year, and $\frac{3}{5}$ of the remainder the following year, and then had only \$6000 remaining; what was the fortune? *Ans.* \$25,000.

9. A lady gave \$2.10 to her pupils: to each boy she gave 3 cents, and to each girl 5 cents; how many were there of each, provided there were 3 times as many boys as girls?

Ans. 15 girls; 45 boys.

10. A cistern has two supply-pipes, which will singly fill it in 4 and 6 hours respectively, and it has also a leak by which

3. A lady being asked the time of day replied, "It is between 4 and 5 o'clock, and the hands of my watch are 5 minute-spaces apart;" what was the time?

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Ans. Copper, 48 oz.; zinc, 36 oz.

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Ans. 15 girls; 45 boys.

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it would be emptied in 8 hours; in what time will it be filled if all flow together? *Ans.* $3\frac{3}{7}$ hours.

11. An Englishman having bought some nutmegs, said that 3 of them cost as much more than a penny as 4 cost him more than twopence half-penny; required the price of the nutmegs. *Ans.* $1\frac{1}{2}$d. each.

12. Two persons, A and B, having received equal sums of money, A spent $25 and B $60, and then it appeared that A had twice as much as B; required the sum each received. *Ans.* $95.

13. How many cows must a person buy at $24 each, that, after paying for their keeping at the rate of $1 for 12, he may gain $142 by selling them at $30 each? *Ans.* 24 cows.

14. Find a number which being doubled, and 16 subtracted from the result, the remainder shall exceed 100 as much as the required number is less than 100. *Ans.* 72.

15. There is a number such that the sum of its $\frac{1}{4}$ and $\frac{1}{5}$ exceeds the sum of its $\frac{1}{6}$ and $\frac{1}{8}$ by 19; required the number. *Ans.* 120.

16. Out of a cask of wine, from which $\frac{1}{4}$ had already been taken away, 24 gallons were afterward drawn, and then, being gauged, it was found to be half full; how much did it hold? *Ans.* 96 gals.

17. A and B, traveling with $500 each, are met by robbers, who take twice as much from A as from B, and leave B with three times as much money as A; how much was taken from each? *Ans.* A, $400; B, $200.

18. In a mixture of copper, tin and lead, $\frac{1}{2}$ of the whole, minus 16lbs., was copper; $\frac{1}{3}$ of the whole, minus 12 lbs., tin; $\frac{1}{4}$ of the whole, plus 4 lbs., lead; what quantity of each was there in the composition? *Ans.* 128 lbs.; 84 lbs.; 76 lbs.

19. A person agreed to do a piece of work on condition that he received $4 for each day he worked, and forfeited $1 each day he was idle; he worked twice as many days as he was idle, and received $140; how many days was he idle? *Ans.* 20 days.

20. The sum of $750 was raised by 4 persons, A, B, C and D; B contributing twice as much as A, C twice as much as A and B, and D twice as much as B and C; what did each contribute? *Ans.* A, $30; B, $60; C, $180; D, $480.

21. A person borrowed a certain sum of money on interest at 6%: in 12 years the interest received amounted to $140 less than the sum loaned; what was the sum loaned? *Ans.* $500.

22. A farmer has 128 animals, consisting of horses, sheep and cows; required the number of each, provided $\frac{1}{2}$ of the number of sheep, plus 12, equals the number of cows, and $\frac{1}{2}$ of the number of cows, plus 12, equals the number of horses. *Ans.* Sheep, 56; cows, 40; horses, 32.

23. Divide the number 90 into 4 such parts that the first increased by 2, the second diminished by 2, the third multiplied by 2, and the fourth divided by 2, may all be equal. *Ans.* 18; 22; 10; 40.

24. A general drawing up his army in the form of a solid square, finds he has 44 men over; then increasing the side of the square by 1 man, he finds he lacks 225 men to complete the square; what was the number of men in the army? *Ans.* 18,000 men.

25. Said Mary to William, "Our purses contain the same sum of money, but if you give me $60 and I give you $20, I shall have 3 times as much as you;" how much had each? *Ans.* $80.

26. A lady bought a number of eggs, half of them at 2 for a penny, and half of them at 3 for a penny; she sold them at the rate of 5 for twopence, and lost a penny by the transaction; what was the number of eggs? *Ans.* 60.

27. There are three sisters, the sum of whose ages is $43\frac{1}{2}$ years, and their birth-days are $2\frac{1}{2}$ years apart, respectively; what is the age of each? *Ans.* 12 yrs.; $14\frac{1}{2}$ yrs.; 17 yrs.

28. At an election $\frac{1}{2}$ of the votes were cast for A, $\frac{1}{3}$ for B, and the remainder for C, and A's majority over C was 800; how many voted for each? *Ans.* A, 1200; B, 800; C, 400.

29. A had twice as much money as B; but after each had spent $\frac{1}{5}$ of his money, and A had paid B $600, B had twice as much as A; how much had each? *Ans.* A, $1500; B, $750.

30. A, B and C can do a piece of work in 24 hours; how long will it take each to do it if A does $\frac{1}{2}$ as much as B, and B $\frac{1}{2}$ as much as C?

Ans. A, 168 hrs.; B, 84 hrs.; C, 42 hrs.

31. If 10 men, 20 women and 30 children receive $424 for a week's work, and 2 men receive as much as 3 women or 5 children, what does each child receive for a day's work?

Ans. 80 cts.

32. Said E to F, My age is 9 years greater than yours; but 12 years ago my age was $\frac{3}{5}$ of what yours will be 7 years hence; what was the age of each? *Ans.* E's, 27 yrs.; F's, 18 yrs.

33. From one end of a line I cut off 3 feet more than $\frac{1}{4}$ of it, and from the other end 6 feet less than $\frac{1}{5}$ of it, and then there remained 25 feet; how long was the line? *Ans.* 40 ft.

34. In a bag containing eagles and dollars, there are 4 times as many eagles as dollars; but if 6 eagles and as many dollars be taken away, there will be left 6 times as many eagles as dollars; how many were there of each?

Ans. Eagles, 60; dollars, 15.

35. A bright young lady being asked her age by a gentleman who was "not very smart at figures," said, "Twice my age $2\frac{1}{2}$ years ago will equal 3 times $\frac{1}{2}$ of my age $2\frac{1}{2}$ years hence;" what was her age? *Ans.* $17\frac{1}{2}$ yrs.

36. An English lady distributed 20 shillings among 20 persons, giving 6 pence each to some, and 16 pence each to the rest; how many persons received 6 pence each?

Ans. 8 persons.

37. A mason receives $450 for building a wall; if he had received $$1\frac{1}{5}$ more a rod he would have received for the entire work $540; how many rods of wall did he build?

Ans. 75 rods.

38. A man loans $8250, part at 5% and part at 6%; how much did he loan at each rate if he receives equal sums of interest for each part? *Ans.* $4500; $3750.

39. An army lost $\frac{1}{8}$ of its number in killed and wounded and 5000 prisoners; it was then reinforced by 10,000 men, but retreating, it lost $\frac{1}{5}$ of its number on the march, when there remained 60,000 men; what was the original force?

Ans. 80,000.

40. Find two consecutive numbers, such that the fifth and the seventh of the first taken together shall equal the sum of the fourth and the twelfth of the second taken together.

Ans. 35 and 36.

41. A person being asked the time of day, replied, "It is between 3 and 4 o'clock, and the hour- and minute-hands of my watch are exactly opposite each other;" what was the time?

Ans. $49\frac{1}{11}$ min. past 3.

42. A colonel forms his regiment into a solid square, and then sending out a picket-guard of 295 men and re-forming the square, finds there were 5 men less on a side; what was the number of men in the regiment at first? *Ans.* 1024.

43. A person being asked the time of day, replied, "The number of minutes it lacks of being 4 o'clock is equal to $\frac{1}{2}$ of the number of minutes it was past 2 o'clock $\frac{3}{4}$ of an hour ago;" what was the time? *Ans.* 25 min. of 4.

44. A regiment consisting of 1296 men can be formed into a hollow square 12 men deep; required the number of men in the outer rank of a side. *Ans.* 39.

45. A boatman who can row at the rate of 12 miles an hour finds that it takes twice as long to run his boat a mile up the river as to run the same distance down the river; what is the rate of the current? *Ans.* 4 miles an hour.

SIMPLE EQUATIONS,

CONTAINING TWO UNKNOWN QUANTITIES.

177. Independent Equations are such as cannot be derived from one another, or be reduced to the same form.

178. The equations $4x+2y=6$ and $6x+3y=9$ are not independent, since both can be reduced to the form $2x+y=3$.

179. Simultaneous equations are those in which the unknown quantities have respectively the same values.

180. To find the value of any unknown quantity in two equations of two unknown quantities, we must derive from them a single equation containing this unknown quantity. The process of doing this is called *Elimination*.

181. Elimination is the process of deducing from two or more simultaneous equations a less number of equations containing a less number of unknown quantities.

182. There are three principal methods of elimination:

1. By Substitution; 2. By Comparison;
3. By Addition and Subtraction.

NOTE.—There is a fourth method of elimination, called the method of *Indeterminate Multipliers*, due to the French mathematician *Bezout*. The three methods given above, however, are all that are generally used in the solution of equations.

CASE I.

ELIMINATION BY SUBSTITUTION.

183. Elimination by Substitution consists in finding an expression for the value of an unknown quantity in one equation, and substituting it in another.

1. Given the equations $2x+3y=12$ and $3x+y=11$, to find the values of x and y.

OPERATION.

$$2x+3y=12 \qquad (1)$$
$$3x+y=11 \qquad (2)$$
$$x=\frac{12-3y}{2} \qquad (3)$$
$$\frac{36-9y}{2}+y=11 \qquad (4)$$
$$36-9y+2y=22 \qquad (5)$$
$$-7y=-14 \qquad (6)$$
$$y=2 \qquad (7)$$
$$x=\frac{12-6}{2}=3 \qquad (8)$$

SOLUTION. From equation (1), by transposing $3y$ and dividing by 2, we have eq. (3). Substituting this value of x in eq. (2), we have eq. (4). Clearing of fractions, we have eq. (5). Transposing and uniting terms, we have eq. (6). Dividing by -7, we have $y=2$. Substituting this value of y in eq. (3), we have eq. (8). Hence, in the given equations, $x=3$ and $y=2$.

NOTE.—In explaining, the pupil may read the equation instead of giving its number, as above.

Rule.—I. *Find an expression for the value of one of the unknown quantities in either equation.*

II. *Substitute this value for the same unknown quantity in the other equation, and reduce.*

NOTES.—1. Use first the equation which will give the simplest expression for the value of the unknown quantity.

2. This method is especially appropriate when the coefficient of one of the unknown quantities is 1.

EXAMPLES.

2. Given $\begin{cases} x+2y=7 \\ 2x+3y=12 \end{cases}$ to find x and y. *Ans.* $x=3$; $y=2$.

3. Given $\begin{cases} 3x-y=10 \\ x+4y=12 \end{cases}$ to find x and y. *Ans.* $x=4$; $y=2$.

4. Given $\begin{cases} 6x-2y=2 \\ 2x+3y=19 \end{cases}$ to find x and y. *Ans.* $x=2$; $y=5$.

5. Given $\begin{cases} 3y-x=7 \\ y+2x=14 \end{cases}$ to find x and y. *Ans.* $x=5$; $y=4$.

6. Given $\begin{cases} 5x+2y=41 \\ 3x-4y=9 \end{cases}$ to find x and y. *Ans.* $x=7$; $y=3$.

7. Given $\begin{cases} 5x-y=48 \\ x+5y=46 \end{cases}$ to find x and y. *Ans.* $x=11$; $y=7$.

8. Given $\left\{\begin{array}{l}\frac{1}{2}x+\frac{1}{3}y=8\\ \frac{1}{3}x+\frac{1}{2}y=7\end{array}\right\}$ to find x and y. *Ans.* $x=12$; $y=6$.

9. Given $\left\{\begin{array}{l}\frac{3x}{4}-\frac{5y}{2}=-9\\ \frac{x}{2}+\frac{y}{3}=6\end{array}\right\}$ to find x and y. *Ans.* $x=8$; $y=6$.

10. Given $\left\{\begin{array}{l}\frac{4x}{5}+\frac{3z}{4}=21\\ \frac{x}{3}-\frac{2z}{3}=-3\end{array}\right\}$ to find x and z. *Ans.* $x=15$; $z=12$.

CASE II.

ELIMINATION BY COMPARISON.

184. Elimination by Comparison consists in finding an expression for the value of the unknown quantity in each equation, and placing these values equal to each other.

1. Given $\left\{\begin{array}{l}2x+3y=12\\ 3x+2y=13\end{array}\right\}$ to find x and y.

OPERATION.

$$2x+3y=12 \quad (1)$$
$$3x+2y=13 \quad (2)$$
$$x=\frac{12-3y}{2} \quad (3)$$
$$x=\frac{13-2y}{3} \quad (4)$$
$$\frac{12-3y}{2}=\frac{13-2y}{3} \quad (5)$$
$$36-9y=26-4y \quad (6)$$
$$-9y+4y=26-36 \quad (7)$$
$$-5y=-10 \quad (8)$$
$$y=2. \quad (9)$$
$$x=\frac{12-6}{2} \quad (10)$$
$$x=3. \quad (11)$$

SOLUTION. Transposing $3y$ in equation (1), and dividing by 2, we obtain (3). Transposing $2y$ in (2), and dividing by 3, we obtain (4). Quantities which are equal to the same thing are equal to each other; hence, we have (5). Clearing of fractions, we have (6). Transposing terms, we have (7). Uniting terms, we have (8). Dividing by -5, we have $y=2$. Substituting the value of y in (3), we have (10). Reducing, we have $x=3$.

Rule.—I. *Find an expression for the value of the same unknown quantity in each equation.*

II. *Place these values equal to each other, and reduce the resulting equation.*

EXAMPLES.

2. Given $\left\{\begin{array}{l}2x+5y=18\\ 3x+4y=20\end{array}\right\}$ to find x and y. *Ans.* $x=4$; $y=2$.

3. Given $\left\{\begin{array}{l}4x-3y=11\\ 5x+y=28\end{array}\right\}$ to find x and y. *Ans.* $x=5$; $y=3$.

4. Given $\left\{\begin{array}{l}x+3y=17\\ 2x+5y=31\end{array}\right\}$ to find x and y. *Ans.* $x=8$; $y=3$.

5. Given $\left\{\begin{array}{l}8x-5y=3\\ 12x-7y=5\end{array}\right\}$ to find x and y. *Ans.* $x=1$; $y=1$.

6. Given $\left\{\begin{array}{l}7x-5y=-3\\ 3x+7y=81\end{array}\right\}$ to find x and y. *Ans.* $x=6$; $y=9$.

7. Given $\left\{\begin{array}{l}\frac{x}{3}+\frac{4z}{5}=6\\ \frac{x}{6}+\frac{3z}{5}=4\end{array}\right\}$ to find x and z. *Ans.* $x=6$; $z=5$.

8. Given $\left\{\begin{array}{l}\frac{2x}{3}-\frac{4y}{5}=-4\\ \frac{3x}{4}-\frac{2y}{3}=-1\end{array}\right\}$ to find x and y. *Ans.* $x=12$; $y=15$

9. Given $\left\{\begin{array}{l}\frac{3x-2y}{5}+3y=16\\ 2x-\frac{2x-3y}{5}=11\end{array}\right\}$ to find x and y. *Ans.* $x=5$; $y=5$

10. Given $\left\{\begin{array}{l}\frac{5x-6y}{6}+\frac{5y}{3}=13\\ \frac{5x}{6}-\frac{7x-4y}{3}=7\end{array}\right\}$ to find x and y. *Ans.* $x=6$; $y=12$.

CASE III.

ELIMINATION BY ADDITION AND SUBTRACTION.

185. Elimination by addition and subtraction consists in adding or subtracting the equations when the coefficients of one of the unknown quantities are alike or are made alike.

1. Given $\left\{\begin{array}{l} 2x+3y=12 \\ 3x+2y=13 \end{array}\right\}$ to find x and y.

OPERATION.

$$\begin{array}{rl} 2x+3y=12 & (1) \\ 3x+2y=13 & (2) \\ \hline 6x+9y=36 & (3) \\ 6x+4y=26 & (4) \\ \hline 5y=10 & (5) \\ y=2 & (6) \\ 2x+6=12 & (7) \\ x=3 & (8) \end{array}$$

SOLUTION. Multiplying equation (1) by 3, and equation (2) by 2, to make the coefficients of x alike, we have equations (3) and (4). Subtracting (4) from (3), we have $5y=10$, from which $y=2$. Substituting the value of y in equation (1), we have (7), from which we find $x=3$.

Rule.—I. *Multiply or divide one or both equations if necessary, so that the coefficients of one unknown quantity shall be the same in both.*

II. *Add these equations when the signs of the equal coefficients are unlike, and subtract the equations when they are alike.*

NOTES.—1. If the coefficients to be made alike are prime to each other, multiply each equation by the coefficient of that quantity in the other equation.

2. This method is generally preferred in finding the value of one quantity, the value of the other quantity being found by substitution.

EXAMPLES.

2. Given $\left\{\begin{array}{l} 3x+4y=7 \\ 4x+2y=6 \end{array}\right\}$ to find x and y. *Ans.* $x=1$; $y=1$.

3. Given $\left\{\begin{array}{l} 5x-2y=14 \\ 2x+3y=17 \end{array}\right\}$ to find x and y. *Ans.* $x=4$; $y=3$.

4. Given $\left\{\begin{array}{l} 3x+5y=25 \\ 2x+4y=18 \end{array}\right\}$ to find x and y. *Ans.* $x=5$; $y=2$.

5. Given $\left\{\begin{array}{l} 3x+5y=46 \\ 6x-8y=2 \end{array}\right\}$ to find x and y. *Ans.* $x=7$; $y=5$.

6. Given $\left\{\begin{array}{l} 15y-6x=87 \\ 9y+3x=105 \end{array}\right\}$ to find x and y. *Ans.* $x=8$; $y=9$.

7. Given $\left\{\begin{array}{l} x+y=a \\ x-y=b \end{array}\right\}$ to find x and y.

Ans. $x=\dfrac{a+b}{2}$; $y=\dfrac{a-b}{2}$.

8. Given $\left\{\begin{array}{l} ax+by=ab \\ 2ax+3by=\dfrac{5ab}{2} \end{array}\right\}$ to find x and y.

Ans. $x=\dfrac{b}{2}$; $y=\dfrac{a}{2}$.

9. Given $\left\{\begin{array}{l} \dfrac{x+8}{4}+6y=21 \\ \dfrac{x+y}{3}=22\frac{1}{3}-5x \end{array}\right\}$ to find x and y.

Ans. $x=4$; $y=3$.

10. Given $\left\{\begin{array}{l} \dfrac{2x-8}{4}-\dfrac{3x-4y}{6}=2 \\ \dfrac{3x-2y}{3}+\dfrac{2y+6}{3}=14 \end{array}\right\}$ to find x and y.

Ans. $x=12$; $y=6$.

MISCELLANEOUS EXAMPLES.

186. In the following examples the pupil will exercise his judgment which method to use:

1. Given $\left\{\begin{array}{l} 3x+4y=24 \\ 4x+3y=25 \end{array}\right\}$. *Ans.* $\left\{\begin{array}{l} x=4, \\ y=3. \end{array}\right.$

2. Given $\left\{\begin{array}{l} 5x-6y=7 \\ 3x+3y=24 \end{array}\right\}$. *Ans.* $\left\{\begin{array}{l} x=5, \\ y=3. \end{array}\right.$

3. Given $\left\{\begin{array}{l} 2x-6y=-18 \\ 6x-7y=1 \end{array}\right\}$. *Ans.* $\left\{\begin{array}{l} x=6, \\ y=5. \end{array}\right.$

4. Given $\left\{\begin{array}{l} 6x-5z=1\frac{3}{4} \\ 4z-5x=-1\frac{1}{2} \end{array}\right\}$. *Ans.* $\left\{\begin{array}{l} x=\frac{1}{2}, \\ z=\frac{1}{4}. \end{array}\right.$

5. Given $\left\{\begin{array}{l} x+y=2a \\ x-y=2b \end{array}\right\}$. *Ans.* $\left\{\begin{array}{l} x=a+b, \\ y=a-b. \end{array}\right.$

6. Given $\left\{\begin{array}{l} x+y=a+b \\ x-y=a-b \end{array}\right\}$. *Ans.* $\left\{\begin{array}{l} x=a, \\ y=b. \end{array}\right.$

7. Given $\left\{\begin{array}{l} \frac{1}{2}x+\frac{1}{3}y=7 \\ \frac{1}{3}x+\frac{1}{4}y=5 \end{array}\right\}$. *Ans.* $\left\{\begin{array}{l} x=6, \\ y=12. \end{array}\right.$

8. Given $\left\{\begin{array}{l} x+\frac{1}{4}y=14 \\ \frac{1}{3}x+y=12 \end{array}\right\}$. *Ans.* $\left\{\begin{array}{l} x=12, \\ y=8. \end{array}\right.$

9. Given $\left\{\begin{array}{l} \dfrac{a}{2}x+\dfrac{b}{3}y=2ab \\ \dfrac{a}{2}x-by=-2ab \end{array}\right\}$. *Ans.* $\left\{\begin{array}{l} x=2b, \\ y=3a. \end{array}\right.$

10. Given $\left\{\begin{array}{l} 6x-7y=42 \\ .7x-.6y=7.5 \end{array}\right\}$. *Ans.* $\left\{\begin{array}{l} x=21, \\ y=12. \end{array}\right.$

11. Given $\left\{\begin{array}{l}\frac{x+1}{y-1}-\frac{x-1}{y}=\frac{6}{y}\\ x-y=1\end{array}\right\}$. *Ans.* $\left\{\begin{array}{l}x=3,\\ y=2.\end{array}\right.$

12. Given $\left\{\begin{array}{l}\frac{x+y}{3}+x=15\\ \frac{x-y}{5}+y=6\end{array}\right\}$. *Ans.* $\left\{\begin{array}{l}x=10,\\ y=5.\end{array}\right.$

13. Given $\left\{\begin{array}{l}\frac{1}{x}+\frac{1}{y}=a\\ \frac{1}{x}-\frac{1}{y}=b\end{array}\right\}$. *Ans.* $\left\{\begin{array}{l}x=\frac{2}{a+b},\\ y=\frac{2}{a-b}.\end{array}\right.$

14. Given $\left\{\begin{array}{l}bx+ay=2ab\\ x+y=a+b\end{array}\right\}$. *Ans.* $\left\{\begin{array}{l}x=a,\\ y=b.\end{array}\right.$

15. Given $\left\{\begin{array}{l}ax+by=2m\\ ax-by=2n\end{array}\right\}$. *Ans.* $\left\{\begin{array}{l}x=\frac{m+n}{a},\\ y=\frac{m-n}{b}.\end{array}\right.$

16. Given $\left\{\begin{array}{l}ax+by=c\\ bx+ay=d\end{array}\right\}$. *Ans.* $\left\{\begin{array}{l}x=\frac{ac-bd}{a^2-b^2},\\ y=\frac{ad-bc}{a^2-b^2}.\end{array}\right.$

17. Given $\left\{\begin{array}{l}\frac{x}{a}+\frac{y}{b}=2\\ bx-ay=0\end{array}\right\}$. *Ans.* $\left\{\begin{array}{l}x=a,\\ y=b.\end{array}\right.$

18. Given $\left\{\begin{array}{l}ax+by=a\\ bx-ay=b\end{array}\right\}$. *Ans.* $\left\{\begin{array}{l}x=1.\\ y=0.\end{array}\right.$

19. Given $\left\{\begin{array}{l}\frac{x}{a}+\frac{y}{b}=1\\ \frac{x}{b}+\frac{y}{a}=1\end{array}\right\}$. *Ans.* $\left\{\begin{array}{l}x=\frac{ab}{a+b},\\ y=\frac{ab}{a+b}.\end{array}\right.$

20. Given $\left\{\begin{array}{l}\frac{a}{x}+\frac{b}{y}=m\\ \frac{c}{x}+\frac{d}{y}=n\end{array}\right\}$. *Ans.* $\left\{\begin{array}{l}x=\frac{bc-ad}{nb-md},\\ y=\frac{bc-ad}{mc-na}.\end{array}\right.$

21. Given $\left\{\begin{array}{l}(a+c)x-by=bc\\ x+y=a+b\end{array}\right\}$. *Ans.* $\left\{\begin{array}{l}x=b\\ y=a\end{array}\right.$

PRACTICAL PROBLEMS.

187. Several of the problems in Art. 176 contained more than one unknown quantity, but the conditions were such that they were readily solved with one unknown quantity.

188. In the following problems the solution is most readily effected by using a separate symbol for each unknown quantity.

189. In such problems the conditions must give as many independent equations as there are unknown quantities.

CASE I.

1. A drover sold 6 lambs and 7 sheep for \$71, and at the same price 4 lambs and 8 sheep for \$64; what was the price of each?

SOLUTION. Let x = the price of the lambs, and y = the price of the sheep; then, by the first condition we have equation (1), and by the second condition we have eq. (2). Multiplying eq. (1) by 2, we have (3). Multiplying (2) by 3, we have (4). Subtracting (3) from (4), we have (5). Dividing by 10, we have (6). From which we find $x=6$.

OPERATION.

Let x = the price of lambs, and y = the price of sheep.

$6x+7y=71$	(1)
$4x+8y=64$	(2)
$12x+14y=142$	(3)
$12x+24y=192$	(4)
$10y=50$	(5)
$y=5$	(6)
$6x+35=71$	(7)
$x=6$	(8)

2. A farmer hired 8 men and 6 boys one day for \$36, and at the same rate next day he hired 6 men and 11 boys for \$40; what did he pay each per day? *Ans.* Men, \$3; boys, \$2.

3. A man paid \$1.14 for 12 oranges and 13 lemons; but the price falling $\frac{1}{3}$, he received only 62 cents for 9 oranges and 11 lemons; what was the price paid for each?

Ans. Oranges, 3 cts.; lemons, 6 cts.

4. A man hired a men and b boys for \$$m$, and at the same price c men and d boys for \$$n$; what price did he pay each?

Ans. Men, \$$\frac{md-nb}{ad-bc}$; boys, \$$\frac{an-mc}{ad-bc}$.

CASE II.

1. There is a fraction such that 1 added to its numerator will make it $\frac{1}{2}$, and 1 added to its denominator will make it $\frac{1}{3}$; what is the fraction?

SOLUTION.

Let $\frac{x}{y}$ = the fraction.

Then, by the 1st condition, $\frac{x+1}{y}=\frac{1}{2}$, (1)

and by the 2d condition, $\frac{x}{y+1}=\frac{1}{3}$. (2)

Clearing (1) of fractions, $2x+2=y$ (3)

clearing (2) of fractions, $3x=y+1$. (4)

Subtracting (3) from (4), $x-2=1$;

transposing, $x=3$;

substituting in (3) and reducing, $y=8$;

hence the fraction is $\frac{x}{y}=\frac{3}{8}$. *Ans.*

2. Find a fraction such that if 2 be subtracted from the numerator the fraction will equal $\frac{1}{4}$, or if 2 be subtracted from the denominator the fraction will equal $\frac{1}{2}$. *Ans.* $\frac{5}{12}$.

3. If 4 be subtracted from both terms of a fraction, the value will be $\frac{1}{3}$, and if 5 be added to both terms the value will be $\frac{5}{6}$; what is the fraction? *Ans.* $\frac{5}{7}$.

4. If 1 be added to both terms of a fraction, its value will be $\frac{1}{2}$; and if the denominator be increased by the numerator, and the numerator be diminished by 1, the value will be $\frac{1}{4}$; required the fraction. *Ans.* $\frac{5}{11}$.

5. Required the fraction such that if the numerator be increased by a the result will equal $\frac{m}{n}$, and if the denominator be increased by a the result will be $\frac{n}{m}$. *Ans.* $\frac{an\,(m-n)}{an\,(m-n)}$.

CASE III.

1. Required the number of two figures which, added to the number obtained by changing the place of the digits, gives 77, and subtracted from it leaves 27.

SOLUTION.

Let x = the tens' digit,

and y = the units' digit;

then $10x+y$ = the number,

and $10y+x$ = the number with digits inverted.

Then, by 1st condition, $10x+y+10y+x=77$, (1)

and by 2d condition, $10y+x-(10x+y)=27$; (2)

uniting terms in (1), $11x+11y=77$; (3)

dividing by 11, $x+y=7$; (4)

uniting terms of (2), $9y-9x=27$; (5)

dividing by 9, $y-x=3$; (6)

from (4) and (6), $x=2$, and $y=5$;

hence the number is 2 tens and 5 units, or 25.

2. Required a number of two figures which, increased by the number obtained by inverting the figures, will equal 132, and which diminished by that number will equal 18. *Ans.* 75.

3. A number consisting of two places being divided by the sum of its digits, the quotient is 4; and if 36 be added to it, the digits will be inverted; required the number. *Ans.* 48.

4. There is a number which equals 5 times the sum of its two digits; and if three times the sum of the digits, plus 9, be subtracted from twice the number, the digits will be inverted; required the number. *Ans.* 45.

CASE IV.

1. The amount of a certain principal at simple interest, for a certain time, at 5%, is \$260; and the amount for the same time, at 8%, is \$296; required the principal and time.

SOLUTION.

Let x = the principal,
and y = the time.

Then, by the 1st condition, $xy \times \frac{5}{100} + x = 260$ (1)

By the 2d condition, $xy \times \frac{8}{100} + x = 296$ (2)

Whence, $x = 200$, the principal, and $y = 6$, the time.

2. A certain sum of money on simple interest amounts in a certain time, at 6%, to \$310, and at 10%, for the same time, to \$350; required the time and principal.

Ans. Prin., \$250; time, 4 yrs.

3. The amount of a certain principal for 7 years, at a certain rate per cent., is \$810, and for 12 years, at the same rate, is \$960; required the principal and rate.

Ans. Prin., \$600; rate, 5%.

4. A certain sum of money, put out at simple interest, amounts in m years to a dollars, and at the same rate, in n years, to b dollars; required the sum and rate per cent.

Ans. Prin., $\frac{an - bm}{n - m}$; rate, $\frac{100(b - a)}{an - bm}$.

CASE V.

1. There are two numbers whose sum equals twice their product, and whose difference equals once their product; required the numbers.

SOLUTION.

Let x = the greater number,
and y = the less number.

Then, by the 1st condition, $x + y = 2xy$ (1)
and by the 2d condition, $x - y = xy$ (2)
Adding (1) and (2), $2x = 3xy$
Dividing by x, $2 = 3y$
Whence, $y = \frac{2}{3}$
and $x = 2$.

2. There are two numbers whose sum equals three times their product, and whose difference equals once their product; what are the numbers? *Ans.* 1 and $\frac{1}{2}$.

3. There are two numbers such that twice their sum equals 3 times their product, and twice their difference equals once their product; what are the numbers? *Ans.* 2 and 1.

4. There are two numbers such that their sum equals 4 times their quotient, and their difference equals twice their quotient; what are the numbers? *Ans.* 9 and 3.

5. There are two numbers whose sum equals a times their product, and whose difference equals b times their product; what are their values? *Ans.* $\frac{2}{a - b}$; $\frac{2}{a + b}$.

MISCELLANEOUS PROBLEMS.

1. A person buys 8 lbs. of tea and 3 lbs. of sugar for \$2.64, and at another time, at the same rates, 5 lbs. of tea and 4 lbs. of sugar for \$1.82; find the price per pound of the tea and sugar. *Ans.* Tea, 30 cts.; sugar, 8 cts.

2. If the greater of two numbers be added to $\frac{1}{3}$ of the less, the sum will be 30, but if the less be divided by $\frac{1}{5}$ of the greater, the quotient will be 3; what are the numbers? *Ans.* 25; 15.

3. A said to B, "Give me \$200 and I shall have 3 times as much as you;" but B replied, "Give me \$200 and I shall have twice as much as you;" how much had each?

Ans. A, \$520; B, \$440.

4. A lady having two watches bought a chain for \$20: if the chain be put on the silver watch, their value will equal $\frac{1}{3}$ of the gold watch, but if it be put on the gold watch, they will be worth 7 times as much as the silver watch; required the value of each watch. *Ans.* Gold, \$120; silver, \$20.

5. Find the fraction which becomes equal to $\frac{1}{3}$ when the numerator is increased by 1, and equal to $\frac{1}{4}$ when the denominator is increased by 1. *Ans.* $\frac{4}{15}$.

6. Mary and Jane have a certain number of plums: if Mary had 4 more she would have 3 times as many as Jane, but if she

had 4 less she would have $\frac{1}{3}$ as many as Jane; how many has each? *Ans.* Mary, 5; Jane, 3.

7. A bill of £120 was paid in guineas (21s.) and moidores (27s.), and the number of pieces of both kinds was just 100; required the number of pieces of each kind. *Ans.* 50.

8. A boy expends 30 cents for apples and pears, buying his apples at 4 and his pears at 5 for a cent, and afterward accommodates his friend with half his apples and one-third of his pears for 13 cents at the price paid; how many did he buy of each? *Ans.* Apples, 72; pears, 60.

9. A person rents 25 acres of land for £7 12s. per annum: for the better part he receives 8s. an acre, and for the poorer part, 5s. an acre; required the number of acres of each sort. *Ans.* 9; 16.

10. If 9 be added to a number consisting of two digits, the two digits will change places, and the sum of the two numbers will be 33; what is the number? *Ans.* 12.

11. Said A to B, "If you will give me $20 of your money, I will have twice as much as you have left;" but says B to A, "If you will give me $20 of your money, I will have thrice as much as you have left;" how much had each? *Ans.* A, $44; B, $52.

12. A and B laid a wager of $20: if A loses, he will have as much as B will then have; if B loses, he will have half of what A will then have; find the money of each. *Ans.* A, $140; B, $100.

13. There is a number consisting of two figures which is double the sum of its digits; and if 9 be subtracted from 5 times the number, the digits will be inverted; what is the number? *Ans.* 18.

14. Several persons engaged a boat for sailing: if there had been 3 more, they would have paid $1 each less than they did; if there had been 2 less, they would have paid $1 each more than they did; required the number of persons and what each paid. *Ans.* 12 persons; $5 each.

15. A and B ran a race which lasted five minutes: B had a start of 20 yards, but A ran 3 yards while B was running 2,

and won by 30 yards; find the length of the course and the speed of each. *Ans.* 150 yds.; A 30 yds. and B 20 yds. a min.

16. If a certain rectangular field were 10 feet longer and 5 feet broader, it would contain 400 square feet more; but if it were 5 feet longer and 10 feet broader, it would contain 450 square feet more; required its length and breadth. *Ans.* 30 ft.; 20 ft.

17. A market-woman bought eggs—some at 3 for 7 cents, and some at two for 5 cents, paying $2.62 for the whole; she afterward sold them at 36 cents a dozen, clearing 62 cents; how many of each kind did she buy? *Ans.* 48; 60.

18. A and B ran a mile, A giving B a start of 20 yards at the first heat, and beating him 30 seconds; at the second heat A gives B a start of 32 seconds, and beats him $9\frac{5}{11}$ yards; at what rate per hour does A run? *Ans.* 12 miles.

SIMPLE EQUATIONS,

CONTAINING THREE OR MORE UNKNOWN QUANTITIES.

190. Equations containing three or more unknown quantities may be solved by either of the methods of elimination already explained.

1. Given $\left\{\begin{aligned} x+y+z&=6 \\ x+2y+3z&=14 \\ 2x+3y+z&=11 \end{aligned}\right\}$ to find x, y and z.

OPERATION.

$$\begin{aligned}
x+y+z&=6 &(1)\\
x+2y+3z&=14 &(2)\\
2x+3y+z&=11 &(3)\\
y+2z&=8 &(4)\\
y-z&=-1 &(5)\\
3z&=9 &(6)\\
z&=3 &(7)\\
y+6&=8 &(8)\\
y&=2 &(9)\\
x+2+3&=6 &(10)\\
x&=1 &(11)
\end{aligned}$$

SOLUTION. Subtracting equation (1) from equation (2), we have equation (4). Multiplying (1) by 2, and subtracting from (3), we have (5); subtracting (5) from (4), we have (6); dividing (6) by 3, we have $z=3$; substituting the value of z in (4), we have 8. Transposing, we have $y=2$; substituting the values of z and y in (1), we have (10): from which we have $x=1$.

had 4 less she would have $\frac{1}{3}$ as many as Jane; how many has each? *Ans.* Mary, 5; Jane, 3.

7. A bill of £120 was paid in guineas (21s.) and moidores (27s.), and the number of pieces of both kinds was just 100; required the number of pieces of each kind. *Ans.* 50.

8. A boy expends 30 cents for apples and pears, buying his apples at 4 and his pears at 5 for a cent, and afterward accommodates his friend with half his apples and one-third of his pears for 13 cents at the price paid; how many did he buy of each? *Ans.* Apples, 72; pears, 60.

9. A person rents 25 acres of land for £7 12s. per annum: for the better part he receives 8s. an acre, and for the poorer part, 5s. an acre; required the number of acres of each sort. *Ans.* 9; 16.

10. If 9 be added to a number consisting of two digits, the two digits will change places, and the sum of the two numbers will be 33; what is the number? *Ans.* 12.

11. Said A to B, "If you will give me $20 of your money, I will have twice as much as you have left;" but says B to A, "If you will give me $20 of your money, I will have thrice as much as you have left;" how much had each? *Ans.* A, $44; B, $52.

12. A and B laid a wager of $20: if A loses, he will have as much as B will then have; if B loses, he will have half of what A will then have; find the money of each. *Ans.* A, $140; B, $100.

13. There is a number consisting of two figures which is double the sum of its digits; and if 9 be subtracted from 5 times the number, the digits will be inverted; what is the number? *Ans.* 18.

14. Several persons engaged a boat for sailing: if there had been 3 more, they would have paid $1 each less than they did: if there had been 2 less, they would have paid $1 each more than they did; required the number of persons and what each paid. *Ans.* 12 persons; $5 each.

15. A and B ran a race which lasted five minutes: B had a start of 20 yards, but A ran 3 yards while B was running 2, and won by 30 yards; find the length of the course and the speed of each. *Ans.* 150 yds.; A 30 yds. and B 20 yds. a min.

16. If a certain rectangular field were 10 feet longer and 5 feet broader, it would contain 400 square feet more; but if it were 5 feet longer and 10 feet broader, it would contain 450 square feet more: required its length and breadth. *Ans.* 30 ft.; 20 ft.

17. A market-woman bought eggs—some at 3 for 7 cents, and some at two for 5 cents, paying $2.62 for the whole; she afterward sold them at 36 cents a dozen, clearing 62 cents; how many of each kind did she buy? *Ans.* 48; 60.

18. A and B ran a mile, A giving B a start of 20 yards at the first heat, and beating him 30 seconds; at the second heat A gives B a start of 32 seconds, and beats him $9\frac{5}{11}$ yards; at what rate per hour does A run? *Ans.* 12 miles.

SIMPLE EQUATIONS,

CONTAINING THREE OR MORE UNKNOWN QUANTITIES.

190. Equations containing three or more unknown quantities may be solved by either of the methods of elimination already explained.

1. Given $\left\{\begin{array}{r} x+y+z=6 \\ x+2y+3z=14 \\ 2x+3y+z=11 \end{array}\right\}$ to find x, y and z.

OPERATION.

$$\begin{array}{rr} x+y+z=6 & (1) \\ x+2y+3z=14 & (2) \\ 2x+3y+z=11 & (3) \\ \hline y+2z=8 & (4) \\ y-z=-1 & (5) \\ \hline 3z=9 & (6) \\ z=3 & (7) \\ y+6=8 & (8) \\ y=2 & (9) \\ x+2+3=6 & (10) \\ x=1 & (11) \end{array}$$

SOLUTION. Subtracting equation (1) from equation (2), we have equation (4). Multiplying (1) by 2, and subtracting from (3), we have (5); subtracting (5) from (4), we have (6); dividing (6) by 3, we have $z=3$; substituting the value of z in (4), we have 8. Transposing, we have $y=2$; substituting the values of z and y in (1), we have (10): from which we have $x=1$.

Rule.—I. *Eliminate successively the same unknown quantity from each of the equations; this will give a number of equations and of unknown quantities one less than the given number.*

II. *In the same way eliminate one of the unknown quantities from each of the derived equations, and thus continue until an equation is found containing one unknown quantity, and then find the value of the unknown quantity in this last equation.*

III. *Substitute this value in one of the equations containing two unknown quantities, and find the value of a second; substitute these values in an equation containing three unknown quantities, and find the value of a third, and thus continue until all are found.*

NOTES.—1. The pupil will exercise his judgment which quantity to eliminate first, and also what method of elimination to use.

2. When one of the equations contains but one or two of the unknown quantities, the method of substitution will often be found the shortest.

EXAMPLES.

2. Given $\begin{cases} x+2y+z=13 \\ x+y+3z=15 \\ 2x+3y+2z=22 \end{cases}$. Ans. $\begin{cases} x=2, \\ y=4, \\ z=3. \end{cases}$

3. Given $\begin{cases} 2x+4y+3z=35 \\ x+3y+2z=23 \\ 3x-5y+4z=17 \end{cases}$. Ans. $\begin{cases} x=4, \\ y=3, \\ z=5. \end{cases}$

4. Given $\begin{cases} 2x+4y-3z=22 \\ 4x-2y+5z=18 \\ 6x+7y-z=63 \end{cases}$. Ans. $\begin{cases} x=3, \\ y=7, \\ z=4. \end{cases}$

5. Given $\begin{cases} 2x+3y-z=27 \\ 3x-4y+3z=12 \\ 4x+2y-5z=15 \end{cases}$. Ans. $\begin{cases} x=7, \\ y=6, \\ z=5. \end{cases}$

6. Given $\begin{cases} x+\frac{1}{2}y+\frac{1}{3}z=32 \\ \frac{1}{3}x+\frac{1}{4}y+\frac{1}{5}z=15 \\ \frac{1}{4}x+\frac{1}{5}y+\frac{1}{6}z=12 \end{cases}$. Ans. $\begin{cases} x=12, \\ y=20, \\ z=30. \end{cases}$

7. Given $\begin{cases} x+y+z=24 \\ x-y+z=8 \\ x+y-z=6 \end{cases}$. Ans. $\begin{cases} x=7, \\ y=8, \\ z=9 \end{cases}$

8. Given $\begin{cases} x+y+z=a \\ x+y-z=b \\ x-y+z=c \end{cases}$ Ans. $\begin{cases} x=\frac{1}{2}(b+c), \\ y=\frac{1}{2}(a-c), \\ z=\frac{1}{2}(a-b). \end{cases}$

9. Given $\begin{cases} x+y=a \\ x+z=b \\ y+z=c \end{cases}$. Ans. $\begin{cases} x=\frac{1}{2}(a+b-c), \\ y=\frac{1}{2}(a+c-b), \\ z=\frac{1}{2}(b+c-a). \end{cases}$

10. Given $\begin{cases} \frac{1}{x}+\frac{1}{y}=5 \\ \frac{1}{x}+\frac{1}{z}=6 \\ \frac{1}{y}+\frac{1}{z}=7 \end{cases}$. Ans. $\begin{cases} x=\frac{1}{2}, \\ y=\frac{1}{3}, \\ z=\frac{1}{4}. \end{cases}$

11. Given $\begin{cases} \frac{x}{a}+\frac{y}{b}=1 \\ \frac{x}{a}+\frac{z}{c}=1 \\ \frac{y}{b}+\frac{z}{c}=1 \end{cases}$. Ans. $\begin{cases} x=\frac{a}{2}, \\ y=\frac{b}{2} \\ z=\frac{c}{2}. \end{cases}$

12. Given $\begin{cases} x+y-z=c \\ x+z-y=b \\ y+z-x=a \end{cases}$. Ans. $\begin{cases} x=\frac{1}{2}(b+c), \\ y=\frac{1}{2}(a+c), \\ z=\frac{1}{2}(a+b). \end{cases}$

13. Given $\begin{cases} x+y+z=12 \\ x+y+u=13 \\ x+z+u=14 \\ y+z+u=15 \end{cases}$. Ans. $\begin{cases} x=3, \\ y=4, \\ z=5, \\ u=6. \end{cases}$

14. Given $\begin{cases} \frac{a}{x}+\frac{b}{y}+\frac{c}{z}=3 \\ \frac{a}{x}-\frac{b}{y}+\frac{c}{z}=1 \\ \frac{a}{x}+\frac{2b}{y}-\frac{c}{z}=2 \end{cases}$. Ans. $\begin{cases} x=a, \\ y=b, \\ z=c. \end{cases}$

NOTE.—There are certain artifices which may be employed to simplify the solution of several of these problems. In the 9th, take the sum of the three equations, and subtract twice each equation from it. In the 10th, eliminate without clearing of fractions.

In the 13th, let the sum of the four quantities be represented by s; then we shall have $s-u=12$; $s-z=13$; $s-y=14$; $s-x=15$; from which we can find s, and then readily find the other quantities.

PROBLEMS

PRODUCING SIMPLE EQUATIONS CONTAINING THREE OR MORE UNKNOWN QUANTITIES.

1. A bin contains 47 bushels of wheat, rye and oats; there are 7 bushels less of oats than of wheat and rye, and 17 bushels less of rye than of wheat and oats; required the quantity of each. *Ans.* Wheat, 12 bu.; rye, 15 bu.; oats, 20 bu.

2. A, B and C have \$1800: the sum of $\frac{1}{2}$ of A's, $\frac{1}{3}$ of B's and $\frac{1}{4}$ of C's equals \$600, and the sum of 4 times A's, 3 times B's and twice C's is \$5000; required the fortune of each.
Ans. A's, \$400; B's, \$600; C's, \$800.

3. A drover bought 230 animals: the number of horses added to $\frac{1}{2}$ of the number of sheep and cows, equals 145; the number of cows, plus $\frac{1}{5}$ of the number of horses and sheep, equals 110; how many were there of each?
Ans. Horses, 60; cows, 80; sheep, 90.

4. Divide the number 150 into 3 such parts that twice the first part increased by 35, 3 times the second part increased by 5, and 4 times the third part divided by 5, may all be equal to each other. *Ans.* $23\frac{1}{10}$; $25\frac{2}{3}$; $101\frac{1}{2}$.

5. Three men, A, B and C, were discussing their ages, when it appeared that the sum of A's and B's was 94 years, the sum of B's and C's was 98 years, and the sum of A's and C's was 96 years; what was the age of each?
Ans. A's, 46 yrs.; B's, 48 yrs.; C's, 50 yrs.

6. A man has \$82 in one, five and ten dollar bills; his ones, plus $\frac{1}{2}$ of his fives and tens, amount to \$47; and his fives, plus $\frac{1}{4}$ of his ones and tens, amount to \$43; how many has he of each? *Ans.* 12 ones; 6 fives; 4 tens.

7. A boy bought at one time 3 apples and 4 peaches for 11 cents; at another time 4 apples and 5 pears for 19 cents; at another, 4 pears and 5 oranges for 32 cents; and at another, 6 peaches and 7 oranges for 34 cents; what was the price of each?
Ans. Apples, 1 ct.; peaches, 2 cts.; pears, 3 cts.; oranges, 4 cts.

8. Divide the number 27 into 4 such parts that if the first part is increased by 2, the second diminished by 2, the third multiplied by 2, and the fourth divided by 2, the results will be equal. *Ans.* 4; 8; 3; 12.

9. A and B can do a piece of work in 12 days, A and C in 15 days, and B and C in 20 days; how many days will it take each person to perform the same work alone?
Ans. A, 20; B, 30; C, 60.

10. The sum of 3 fractions is $2\frac{1}{4}$: the sum of the first and third equals twice the second fraction, and the difference between the first and third is $\frac{1}{3}$ of the third fraction; what are the fractions? *Ans.* $\frac{2}{3}$, $\frac{3}{4}$, $\frac{5}{6}$.

11. In a naval engagement the number of ships captured was 7 more, and the number burned was 2 less, than the number sunk. Fifteen escaped, and the fleet consisted of 8 times the number sunk. Of how many ships did the fleet consist?
Ans. 32.

12. A cistern has 3 pipes opening into it. If the first be closed, the cistern may be filled in 20 minutes; if the second be closed, in 25 minutes; if the third be closed, in 30 minutes. How long would it take each pipe alone to fill it?
Ans. 1st, $85\frac{5}{7}$ min.; 2d, $46\frac{2}{13}$ min.; 3d, $35\frac{5}{17}$ min.

13. I have 3 watches, and a chain which is worth \$60. The first watch and chain are worth $\frac{1}{3}$ as much as the second and third watches; the second watch and chain are worth $\frac{3}{5}$ as much as the first and third watches; and the third watch and chain are worth 3 times as much as the first and second watches. What is the value of each watch?
Ans. 1st, \$20; 2d, \$60; 3d, \$180.

14. There is a number consisting of 3 digits: the sum of the digits is 9; the digit in the tens' place is half the sum of the other 2 digits; and if 198 be added to the number, the result will be expressed by the figures of the number reversed; required the number. *Ans.* 234.

15. A sum of money consists of quarter dollars, dimes and half dimes. It is worth as many dimes as there are pieces of money; it is worth as many quarters as there are dimes; and

the number of half dimes is one more than the number of dimes. What is the number of each?

Ans. 3 quarters; 8 dimes; 9 half dimes.

16. Three boys, A, B and C, were playing marbles. First A loses to B and C as many as each of them has; next, B loses to A and C as many as each of them now has; lastly, C loses to A and B as many as each of them now has; and it is then found that each of them has 16 marbles. How many had each at first? *Ans.* A, 26; B, 14; C, 8.

17. Some smugglers discovered a cave which would exactly hold the cargo of their boat, which consisted of 13 bales of cotton and 33 casks of liquor. While they were unloading, a custom-house cutter hove in sight, and they sailed away with 9 casks and 5 bales, leaving the cave $\frac{2}{3}$ full; how many bales or casks would it hold? *Ans.* 24 bales; 72 casks.

18. A person has 2 horses and 2 saddles; the better saddle cost $50, and the other $15. If he puts the better saddle upon the first horse, and the worse saddle upon the second, then the latter is worth $50 more than the former; but if he puts the worse saddle upon the first, and the better saddle upon the second horse, the latter is worth $1\frac{2}{3}$ times as much as the former; what is the value of each horse? *Ans.* 1st, $165; 2d, $250.

REVIEW QUESTIONS.

Define Independent Equations. Give an example of them. Define Simultaneous Equations. Give an example of them. Define Elimination. What is the use of Elimination? How many methods are there? State the methods of elimination.

Define Elimination by Substitution. State the rule for it. Define Elimination by Comparison. State the rule for it. Define Elimination by Addition and Subtraction. State the rule for it. Is there any other method of elimination? Which method of elimination is preferred?

What effect has transposition upon the signs of terms? Why does transposition change the sign of a term? Why does changing the signs of all the terms of an equation not affect the equality of the members? In solving problems when should a separate symbol be used for each unknown quantity?

SUPPLEMENT

TO SIMPLE EQUATIONS.

191. This **Supplement** to Simple Equations embraces the following subjects: *Zero and Infinity, Generalization, Negative Solutions, Discussion of Problems,* and *Indeterminate Problems.*

NOTE.—Teachers desiring a short course may omit this Supplement to simple equations.

ZERO AND INFINITY.

192. Zero and **Infinity** often occur in algebraic expressions. Such expressions may be interpreted by the following principles:

PRIN. 1. $0 \times A = 0$; that is, *if zero be multiplied by a finite quantity the product is zero.*

For, 0 multiplied by 2 is 0, 0 multiplied by 3 is 0, etc.; hence, 0 multiplied by *any number* is 0, or $0 \times A$ is 0.

PRIN. 2. $\frac{0}{A} = 0$; that is, *if zero be divided by a finite quantity the quotient is zero.*

For, 0 divided by 2 is 0, 0 divided by 3 is 0, etc.; hence, 0 divided by *any number* is 0, or $0 \div A$ is 0.

PRIN. 3 $\frac{0}{0} = A$; that is, *if zero be divided by zero the quotient is any finite quantity, or is indeterminate.*

For, if in Prin. 1 we divide both members by 0, we have $\frac{0}{0} = A$, in which A is any finite quantity.

PRIN. 4. $\frac{A}{0} = \infty$; that is, *if a finite quantity be divided by zero the quotient is infinity.*

To prove this, suppose we have the fraction $\frac{a}{b}$. Now, if a remains constant, the smaller b is, the greater will be the quotient, hence, if b becomes *infinitely small*, the quotient will become *infinitely large;* hence, when b is *zero*, the quotient is *infinity*.

PRIN. 5. $0 \times \infty = A$; that is, *if zero be multiplied by infinity, the product is a finite quantity.*

For, clearing the equation in Prin. 4 of fractions, we have $0 \times \infty = A$ n which A is any finite quantity.

PRIN. 6. $\frac{A}{\infty} = 0$; that is, *if a finite quantity be divided by infinity, the quotient is zero.*

For, dividing both members of the equation in Prin. 5 by ∞, we have 0 equals A divided by infinity, which proves the principle.

GENERALIZATION.

193. Generalization is the process of solving general problems, and interpreting the results.

194. A **General Problem** is one in which the quantities are represented by letters.

195. A **Formula** is a general expression for the solution of a problem. A formula expressed in ordinary language gives a *rule* by which all the problems of a class may be solved.

CASE I.

1. The difference between a times a number and b times the number is c; required the number.

Let $x =$ the number.

Then, $ax - bx = c$;

whence, $x = \frac{c}{a-b}$.

Expressing this formula in ordinary language, we have the following rule:

Rule.—*Divide the difference of the products by the difference of the multipliers.*

Apply this formula to the following problems:

2. The difference between 5 times a number and 3 times the number is 12; required the number.

3. The difference between 9 times a number and 6 times a number is 162; what is the number?

CASE II.

1. Find a number which being divided by two given numbers, a and b, the sum of the quotients may be c.

Let $x =$ the number.

Then, $\frac{x}{a} + \frac{x}{b} = c$;

whence, $x = \frac{abc}{a+b}$.

This formula, expressed in ordinary language, gives the following rule:

Rule.—*Divide the product of the three given quantities by the sum of the divisors when the sum of the quotients is given, and by the difference of the divisors when the difference of the quotients is given.*

Apply this formula to the following problems:

2. Find a number which being divided by 4 and by 6, the difference of the quotients is 4.

3. Find a number which being divided by 3 and by 7, the difference of the quotients is 16.

CASE III.

1. A can do a piece of work in a days, and B in b days; in what time can they both do it?

Let $x =$ the number of days in which both can do it.

Then, $\frac{1}{a} + \frac{1}{b} = \frac{1}{x}$.

whence, $x = \frac{ab}{a+b}$.

This formula, expressed in ordinary language, gives the following rule:

Rule.—*Divide the product of the numbers expressing the time in which each can perform the work by their sum.*

Apply this formula to the following problems:

2. A can do a piece of work in 4 days, and B in 8 days; in what time will they together do it?

3. A can reap a field in 6 days, and B in 9 days; in what time can they together reap it?

4. A pound of tea would last a man 12 months, and his wife 6 months; how long would it last them both?

CASE IV.

1. The sum of two numbers is a, and their difference is b; what are the numbers?

Let x = the greater number,
and y = the smaller number.
Then, $x+y=a$,
and $x-y=b$.
Whence $x=\frac{a+b}{2}$,
and $y=\frac{a-b}{2}$.

These formulas, expressed in ordinary language, will give the following rules:

Rule.—I. *To find the greater number, add half the difference to half the sum.*

II. *To find the less number, subtract half the difference from half the sum.*

Apply the formulas to the following problems;

Required the numbers whose—

2. Sum is 20; difference is 4.
3. Sum is 62; difference is 14.
4. Sum is 221; difference is 29.

MISCELLANEOUS PROBLEMS.

196. The pupils will obtain the formulas in the following problems, and derive rules from them:

1. Divide the number a into two parts, so that one part shall be n times the other part. *Ans.* $\frac{a}{n+1}$; $\frac{na}{n+1}$.

2. The sum of two numbers is a, and n times one number equals m times the other; what are the numbers?
Ans. $\frac{ma}{m+n}$; $\frac{na}{m+n}$.

3. The difference of two numbers is a, and n times one number equals m times the other; what are the numbers?
Ans. $\frac{ma}{m-n}$; $\frac{na}{m-n}$.

4. The sum of two numbers is a, and n times their sum equals m times their difference; what are the numbers?
Ans. $\frac{(m+n)a}{2m}$; $\frac{(m-n)a}{2m}$.

5. Divide the number a into two such parts that one part increased by b shall be equal to m times the other part.
Ans. $\frac{ma-b}{m+1}$; $\frac{a+b}{m+1}$.

6. A is m times as old as B, and in a years he will be n times as old; what is the age of each at present?
Ans. A's, $\frac{am(n-1)}{m-n}$; B's, $\frac{a(n-1)}{m-n}$.

7. A courier starts from a place and travels a miles a day; n days after he is followed by another, who travels b miles a day; in what time will the second overtake the first?
Ans. $\frac{na}{b-a}$ days.

8. I bought two kinds of sugar—one at a cents a pound, and the other at b cents a pound; how much of each kind must I take to make a mixture of m pounds worth c cents a pound?
Ans. $\frac{m(c-b)}{a-b}$; $\frac{m(a-c)}{a-b}$.

NEGATIVE SOLUTIONS.

197. In the solution of a problem the value of the unknown quantity is sometimes a *negative quantity.*

198. A solution of a problem which gives a negative quantity is called a *Negative Solution.*

1. What number must be added to the number 12 that the result shall be 8?

SOLUTION. Let x equal the number; then, $12+x=8$ or $x=-4$. Now, the result -4 is said to satisfy the question in an *algebraic sense,* since -4 added to 12 equals 8; but the problem is evidently impossible in an *arithmetical sense.* Since, however, adding -4 gives the same result as subtracting $+4$, the negative result indicates that the problem should be, What number must be *subtracted* from 12 that the result shall be 8?

OPERATION.

Let $x=$ the number.

Then, $12+x=8$

$x=-4$

2. A is 45 years old, and B is 15 years old; how many years hence will A be 4 times as old as B?

SOLUTION. Let $x=$ the number of years; then, solving the problem, we have $x=-5$. This result, -5, indicates a reckoning of time *backward* instead of *forward;* hence, it was 5 years *ago* instead of 5 years *hence.* The problem should. therefore, be modified to read, "How many years *since,*" instead of "How many years *hence.*"

OPERATION.

Let $x=$ the number of years.

Then, $(45+x)=4(15+x)$;

whence, $x=-5$

3. A is 45 years old, and B is 15 years old; at what time from the present is A 4 times as old as B?

SOLUTION. Solving this problem by supposing the date to be in the future, we find $x=-5$. Had the result been $+5$, it would have indicated 5 years *hence.* The problem is stated in general terms, so as to admit either result, and need not be modified in its statement. This result, -5, therefore indicates 5 years *since,* or 5 years *ago,* which, by examining the problem, we see is the correct time.

4. There is a fraction such that if 1 be added to its numerator the result is $\frac{1}{3}$, and if 1 be added to the denominator the result is $\frac{1}{2}$; what is the fraction?

SOLUTION. Solving this problem, we find the fraction to be $\frac{-4}{-9}$. This can be verified algebraically, but is absurd arithmetically considered. The problem can be made consistent, however, by changing the word "added" to "subtracted."

199. From these examples and illustrations we derive the following conclusions:

1. *The negative solution indicates some inconsistency or absurdity in the statement of the problem, or a wrong supposition respecting some condition of it.*

2. *A problem which gives a negative solution can usually be so modified that it will become consistent, and the result will be positive.*

3. *The negative solution sometimes merely indicates the direction in which the result is to be reckoned.*

200. The pupils will solve, interpret, and modify the enunciation of the following problems:

5. What number must be *added* to 18 that the result may be 15? *Ans.* -3.

6. What number must be *subtracted* from 12 that the result may be 15? *Ans.* -3.

7. Required a number such that $\frac{2}{3}$ of it shall exceed $\frac{3}{4}$ of it by 2. *Ans.* -24.

8. A man was born in 1825, and his son in 1855; find at what time the father's age is 4 times the son's age, dating from 1870. *Ans.* -5.

9. A man labored 10 days, his little son being with him 8 days, and received $18; at another time he labored 14 days, his son being with him 12 days, and received $25; required the wages of each. *Ans.* Father, $2; son, -25 cts.

DISCUSSION OF PROBLEMS.

201. The **Discussion** of a problem is the process of assigning different conditions to a problem, and interpreting the results which thus arise.

1. If we subtract b from a, by what number must the result be multiplied to give a product of c?

OPERATION.

Let x = the number.
$(a-b)x=c$
$x=\frac{c}{a-b}$

DISCUSSION. Let x represent the number; then we shall have $(a-b)x=c$, from which we find the value of x equal to c divided by $a-b$.

202. This result may have five different forms depending on the values of a, b and c.

I. When a is greater than b.

In this case, $a-b$ is *positive;* and c being *positive*, the quotient is *positive;* hence, the required number is *positive.* This is as it should be, for the problem then is, *By what shall we multiply a positive quantity to produce a positive quantity?* Evidently the multiplier should be *positive.*

II. When a is less than b.

In this case $a-b$ is *negative*, and consequently c divided by $a-b$ is *negative;* hence the required number is *negative.* This is as it should be, for the problem then is, *By what shall we multiply a negative quantity to produce a positive quantity?* Evidently the multiplier should be *negative.*

III. When a is equal to b.

In this case $a-b=0$ and $x=\frac{c}{0}=\infty$ (Prin. 4, Art. 192); that is, *no finite quantity* will answer the conditions. This is correct, since the problem then becomes, *By what must we multiply 0, nothing, to produce c, something?* Evidently by *no finite quantity.*

IV. When c is 0, and a is either greater or less than b.

In this case we have $x=\frac{0}{a-b}$, which equals 0 (Prin. 2, Art. 192); that is, the multiplier is *zero.* This is correct, for the problem then is, *By what must we multiply $a-b$, something, to produce nothing?* Evidently the multiplier should be *zero.*

V. When $c=0$ and $a=b$.

In this case we have $x=\frac{0}{0}$, or *any quantity* whatever (Prin. 3, Art. 192). This is correct, for the problem then is, *By what shall we multiply 0 to produce 0?* Evidently the multiplier may be *any quantity.*

NOTE.—Let the pupils illustrate each form by using particular values for a, b and c.

PROBLEM OF THE COURIERS.

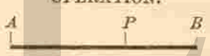
203. The **Problem of the Couriers** is a fine illustration of the subject under consideration. It was originally proposed by *Clairaut*, an eminent French mathematician.

1. Two couriers start at the same time from two places, A and B, a miles apart, the former traveling m miles an hour, and the latter n miles an hour; where will they meet?

There are evidently two cases of the problem.

CASE I. When the couriers travel toward each other.

OPERATION.



$a=AB$
Let $x=AP$;
then $a-x=PB$.
$\frac{x}{m}=\frac{a-x}{n}$
$x=\frac{am}{m+n}$
$a-x=\frac{an}{m+n}$

DISCUSSION. Let A and B represent the two places, and P the point at which they meet. $AB=a$. Let $x=AP$, the distance which the first travels; then $a-x=PB$, the distance which the second travels. Then $\frac{x}{m}$ = the number of hours the first travels, and $\frac{a-x}{n}$ = the number of hours the second travels. They both travel the same time; hence, we have $\frac{x}{m}=\frac{a-x}{n}$, from which we find the values of x and $a-x$.

1st. Suppose $m=n$; then, by substituting, we find $x=\frac{a}{2}$, and $a-x=\frac{a}{2}$; that is, if they travel at the *same rate*, each will travel *half the distance.* This is evident from the nature of the problem.

2d. Suppose $n=0$; then $x=a$ and $a-x=0$; that is, if the second does not travel, the first travels the *whole distance*, and the second *no distance.* This is evident from the conditions of the problem.

3d. Suppose $m=0$; then $x=0$ and $a-x=a$; that is, the first travels *no distance*, and the second travels the *whole distance*. This is evident from the conditions of the problem.

NOTE.—Let the pupil make the suppositions $m=2n$, $m=\frac{1}{2}n$, etc., and interpret the results.

CASE II. When the couriers travel in the same direction.

DISCUSSION. Let A and B represent the two places, and P the point where they meet, each traveling in the direction of P. $AB=a$. Let $x=AP$, the distance the first travels; then $x-a=BP$, the distance the second travels. The times they travel are equal; hence we have $\frac{x}{m}=\frac{x-a}{n}$, from which we find the values of x and $x-a$.

OPERATION.

$a=AB$.

Let $x=AP$;

then $x-a=BP$.

$$\frac{x}{m}=\frac{x-a}{n}$$

$$x=\frac{am}{m-n}$$

$$x-a=\frac{an}{m-n}$$

1st. Suppose $m>n$; then x and $x-a$ will be *positive*.

That is, they will meet at the *right* of B. This is as it should be by the conditions of the problem.

2d. Suppose $n>m$; then x and $x-a$ are both *negative*.

That is, the point of meeting must be at the *left* of A, instead of at the right. Hence, if the second courier travel faster than the first, that they may meet, the *direction* in which they travel *must be changed*. This is evident from the conditions of the problem.

3d. Suppose $m=n$; then $x=\frac{am}{0}$, or ∞, and $x-a=\frac{an}{0}$, or ∞.

That is, if the couriers travel at the same rate, they can meet at no *finite* distance from A; in other words, the one can *never* overtake the other. This is evident from the circumstances of the problem.

4th. Suppose $a=0$; then $x=\frac{0}{m-n}$, or 0, and $x-a=\frac{0}{m-n}$, or 0.

That is, if the couriers are *no distance* apart, they will have to travel *no distance* to be together. This is evident from the circumstances of the problem.

5th. Suppose $m=n$ and $a=0$; then $x=\frac{0}{0}$ and $x-a=\frac{0}{0}$, or *anything*.

That is, if the couriers are *no* distance apart, and travel at the *same rate*, they will *always* be together. This is evident from the nature of the problem.

6th. Suppose $n=0$; then $x=\frac{am}{m}=a$ and $x-a=0$.

That is, if the rate at which the second travels is zero, the first courier

travels the whole distance, and the second *no* distance. This is evident from the circumstances of the problem.

7th. Suppose $m=2n$; then $x=\frac{2am}{m}=2a$ and $x-a=a$.

That is, if the rate at which the first travels is twice the rate at which the second travels, the first courier travels *twice* the distance from A to B, overtaking the second a miles from B. This is evident from the circumstances of the problem.

8th. Suppose m is plus and n is minus; then

$$x=\frac{am}{m+n} \text{ and } x-a=\frac{-an}{m+n}.$$

Here the value of x is *positive*, and the value of $x-a$ is *negative;* hence, they meet at the *right* of A and at the *left* of B, or between A and B. This is evident, since *minus n* indicates that the second courier travels toward A; hence they must meet between A and B.

Changing the signs of the last expression, we have $a-x=\frac{an}{m+n}$, which is the same as the expression for the distance the second courier travels, obtained in the first case of this problem. Case I. is therefore but a special case of the general problem just discussed.

PROBLEMS FOR DISCUSSION.

1. Required a number that, being successively multiplied by m and n, the difference of the products shall equal a.

Ans. $\frac{a}{m-n}$.

When will the result be negative? When indeterminate? When infinite? Illustrate with numbers.

2. B is a years old, and A is m times as old; at what time will A be n times as old as B? *Ans.* $x=\frac{(m-n)a}{n-1}$.

Interpret the result when $m>n$; $m<n$; when $n<1$.

3. The hour- and minute-hands of a watch are a minute-spaces apart between m and $m+1$ o'clock; what is the time?

Ans. $12\frac{(5m-a)}{11}$.

INDETERMINATE AND IMPOSSIBLE PROBLEMS.

204. An **Indeterminate Problem** is one whose conditions can be satisfied by different values of the unknown quantity.

I. *A problem giving only one equation containing two unknown quantities is indeterminate.*

Thus, the equation $x+y=12$ is indeterminate; for, by transposing, we have $x=12-y$; and this equation can be verified by any number of values of x and y.

II. *A problem which gives only two equations containing three unknown quantities is indeterminate.*

Problem.—Given $x+3y-z=8$ and $x+2y-2z=2$, to find x and y.

Solution. By elimination, we find $y+z=6$, which may be verified by any number of values of y and z, and is therefore indeterminate.

III. *A problem is sometimes indeterminate when it contains only one unknown quantity.*

Problem.—What number is that whose $\frac{5}{6}$, diminished by its $\frac{3}{4}$, will equal its $\frac{1}{20}$ increased by its $\frac{1}{30}$?

Solution.

Let $x=$ the number;

then $\frac{5x}{6}-\frac{3x}{4}=\frac{x}{20}+\frac{x}{30}$.

Clearing of fractions, we have $50x-45x=3x+2x$;

transposing, $5x-5x=0$;

factoring, $(5-5)x=0$;

whence, $0\times x=0$,

and $x=\frac{0}{0}$.

The value of x thus found is *indeterminate* (Prin. 3, Art. 192); the problem is therefore indeterminate.

205. An **Impossible Problem** is one whose conditions are impossible or contradictory.

I. *A problem is impossible when its conditions are contradictory*

Problem.—Given the equations $x+y=7$, $x-y=1$ and $xy=16$, to find the values of x and y.

Solution. Uniting equations first and second, we find $x=4$ and $y=3$. But the third equation requires their product to be 16, which is impossible for those values of x and y. Hence the problem is impossible.

II. *A problem that contains only one unknown quantity is sometimes impossible.*

Problem.—Required a number whose $\frac{1}{3}$, plus its $\frac{1}{4}$, diminished by 3, equals its $\frac{7}{12}$, increased by 5.

Solution.

Let $x=$ the number.

Then $\frac{x}{3}+\frac{x}{4}-3=\frac{7x}{12}+5$;

whence, $7x-36=7x+60$,

and $0\times x=96$,

or $x=\frac{96}{0}=\infty$.

This value of x is *infinite*, which shows that no *finite number* will answer the conditions of the problem; it is therefore *impossible*.

Note.—When a problem contains more conditions than unknown quantities, the conditions which are unnecessary are said to be *redundant*.

EXAMPLES.

1. Find the value of x in the equation $\frac{3x+5}{x+2}=\frac{3x-6}{x-2}$.

Solution. Reducing the second member, we have equation (2); clearing of fractions, we have (3); transposing terms and factoring, we have (4); dividing by the coefficient of x, we have $x=\frac{1}{0}$, or ∞, which indicates that no finite value will answer the conditions.

Operation 1st.

$$\frac{3x+5}{x+2}=\frac{3x-6}{x-2} \quad (1)$$

$$\frac{3x+5}{x+2}=3 \quad (2)$$

$$3x+5=3x+6 \quad (3)$$

$$(3-3)x=6-5 \quad (4)$$

$$0\times x=1$$

$$x=\frac{1}{0}=\infty \quad (5)$$

SOLUTION 2D. Clearing the equation of fractions and reducing partly, we have (2); transposing and uniting, we have (3); whence we have (4). This apparent value of x cannot be verified, as the pupil may see by substitution.

OPERATION 2D.

$$\frac{3x+5}{x+2}=\frac{3x-6}{x-2} \quad (1)$$

$$3x^2-x-10=3x^2-12 \quad (2)$$

$$-x=-2 \quad (3)$$

$$x=2 \quad (4)$$

2. Required a number such that its $\frac{1}{4}$, increased by its $\frac{1}{6}$, is equal to its $\frac{11}{20}$, diminished by its $\frac{2}{15}$.

Ans. $\left(x=\frac{0}{0}\right)$. Indeterminate.

3 Required a number whose $\frac{5}{6}$, diminished by 4, is equal to the sum of its $\frac{1}{2}$ and $\frac{1}{3}$, diminished by 3.

Ans. ($x=\infty$). Impossible.

4. A and B dug a ditch for \$20, A receiving \$2, and B \$3, a day; how many days did each labor, if they did not labor the same number of days? *Ans.* Indeterminate.

5. Twenty years ago, A was 40 years old, and his son was only $\frac{1}{4}$ as old; now the son is $\frac{1}{2}$ as old as the father; gaining thus, when will the son be as old as the father? *Ans.* ($x=\infty$).

6. Find a fraction such that if 2 be subtracted from the numerator, or if 3 be added to the denominator, the resulting fractions will equal $\frac{2}{3}$. *Ans.* Indeterminate.

7. Required a number such that 4 times the number, diminished by 12, divided by the number minus 3, may equal 4 times the number, plus 9, divided by the number plus 3.

Ans. Impossible.

NOTE.—The 6th reduces to the indeterminate form, although $\frac{4}{3}$, $\frac{6}{6}$, $\frac{12}{15}$, etc., will answer the conditions of the problem.

REVIEW QUESTIONS.

State the principles of Zero and Infinity. Define Generalization. A General Problem. A Formula. A Negative Solution. State the principles of negative solutions. Define the Discussion of a Problem. An Indeterminate Problem. An Impossible Problem. When is a problem indeterminate? When in possible?

SECTION VI.

INVOLUTION, EVOLUTION AND RADICALS

INVOLUTION.

206. Involution is the process of raising a quantity to any given power.

207. A Power of a quantity is the product obtained by using the quantity as a factor any number of times.

208. An Exponent of a quantity is a number which indicates the power to which the quantity is to be raised.

Thus, let a represent any quantity;

then, $a=a^1$, is the *first* power of a.

$aa=a^2$, is the *second* power of a.

$aaa=a^3$, is the *third* power of a.

$aaaa=a^4$, is the *fourth* power of a.

When the exponent is n, as a^n, it indicates the nth power of a.

209. The **Exponent** (called also the *Index*) indicates how many times the quantity is used as a factor.

The FIRST POWER of a quantity is the quantity itself.

The SECOND POWER of a quantity is called its *square.*

The THIRD POWER of a quantity is called its *cube.*

PRINCIPLES.

1. *The square of a quantity is the product obtained by using the quantity as a factor twice.*
2. *The cube of a quantity is the product obtained by using the quantity as a factor three times.*
3. *All the powers of a positive quantity are positive.*

For, the square of a positive quantity is positive, since it is the product of two positive quantities; and its cube is positive, since it is also the product of two positive quantities, etc.

SOLUTION 2D. Clearing the equation of fractions and reducing partly, we have (2); transposing and uniting, we have (3); whence we have (4). This apparent value of x cannot be verified, as the pupil may see by substitution.

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$$\frac{3x+5}{x+2}=\frac{3x-6}{x-2} \quad (1)$$
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Ans. $\left(x=\frac{0}{0}\right)$. Indeterminate.

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3. *All the powers of a positive quantity are positive.*

For, the square of a positive quantity is positive, since it is the product of two positive quantities; and its cube is positive, since it is also the product of two positive quantities, etc.

4. *The* EVEN *powers of a* NEGATIVE *quantity are* POSITIVE, *and the* ODD *powers are* NEGATIVE.

The square is positive, since it is the product of two negative quantities; the *cube* is *negative*, since it is the square, which is *positive*, multiplied by the quantity, which is *negative*; the *fourth* power is *positive*, since it is the cube, a *negative*, multiplied by the quantity, a *negative*; etc.

CASE I.

210. To raise a monomial to a given power.

1. Raise $4a^2b$ to the third power.

SOLUTION. Multiplying $4a^2b$ by itself, we have $16a^4b^2$, and multiplying the square by $4a^2b$, we have $64a^6b^3$. Examining the result, we see we have the *cube* of the coefficient, and the letters of the given quantity with *three* times the exponents which they have in the root.

OPERATION

$$\begin{array}{r} 4a^2b \\ 4a^2b \\ \hline 16a^4b^2 \\ 4a^2b \\ \hline 64a^6b^3 \end{array}$$

Rule.—I. *Raise the coefficient to the required power, and multiply the exponent of each letter by the index of the power.*

II. *When the quantity is positive, the power will be positive; when the quantity is negative, the even powers will be positive and the odd powers negative.*

EXAMPLES.

2. Square of $3ab^2$. *Ans.* $9a^2b^4$.
3. Square of $5a^3c^2$. *Ans.* $25a^6c^4$.
4. Square of $-6a^2x^3$. *Ans.* $36a^4x^6$.
5. Cube of $3a^3x$. *Ans.* $27a^9x^3$.
6. Cube of $-4a^2c^3$. *Ans.* $-64a^6c^9$.
7. Fourth power of $3a^nb^3$. *Ans.* $81a^{4n}b^{12}$.
8. Fourth power of $-2a^mc^n$. *Ans.* $16a^{4m}c^{4n}$.
9. Fifth power of $-a^2b^3c^4$. *Ans.* $-a^{10}b^{15}c^{20}$.
10. Fifth power of $2a^{-3}b^6c^n$. *Ans.* $32a^{-15}b^{30}c^{5n}$.
11. Sixth power of $2a^3c^2$. *Ans.* $64a^{18}c^{12}$.
12. Seventh power of $-2a^3c^2$. *Ans.* $-128a^{21}c^{14}$
13. Eighth power of $-a^{2n}b^3c^{-4}d^{-n}$. *Ans.* $a^{16n}b^{24}c^{-32}d^{-8n}$

14. nth power of $a^2b^3c^4$. *Ans.* $a^{2n}b^{3n}c^{4n}$.
15. nth power of $-2x^2y^{-3}z^4$. *Ans.* $\pm 2^nx^{2n}y^{-3n}z^{4n}$.
16. Value of $(-2a^2b^3)^5$. *Ans.* $-32a^{10}b^{15}$.
17. Value of $(-a^2b^nc)^4$. *Ans.* $a^8b^{4n}c^4$.
18. Value of $(-2b^{-3}m^{-n})^7$. *Ans.* $-128b^{-21}m^{-7n}$.
19. Value of $(-a^nb^{2n})^n$. *Ans.* $\pm a^{n^2}b^{2n^2}$.
20. Value of $(-x^{2n}b^{-2n}c^m)^n$. *Ans.* $\pm x^{2n^2}b^{-2n^2}c^{mn}$.

CASE II.

211. To raise a fraction to a given power.

1. Find the third power of $\frac{a^2}{c^3}$.

SOLUTION. Using the quantity three times as a factor, we have $\frac{a^2}{c^3}\times\frac{a^2}{c^3}\times\frac{a^2}{c^3}=\frac{a^6}{c^9}$.

OPERATION.

$$\frac{a^2}{c^3}\times\frac{a^2}{c^3}\times\frac{a^2}{c^3}=\frac{a^6}{c^9}$$

Rule.—*Raise both numerator and denominator to the required power.*

EXAMPLES.

2. Square of $\frac{2a}{3c}$. *Ans.* $\frac{4a^2}{9c^2}$.
3 Square of $\frac{3a^2}{4c^2}$. *Ans.* $\frac{9a^4}{16c^4}$.
4. Cube of $\frac{a^2b}{cd^3}$. *Ans.* $\frac{a^6b^3}{c^3d^9}$.
5. Cube of $-\frac{2ac^2}{3x^2}$. *Ans.* $-\frac{8a^3c^6}{27x^6}$
6. Square of $-\frac{3a^2c^n}{4a^nc^2}$. *Ans.* $\frac{9c^{2n-4}}{16a^{2n-4}}$
7. Fourth power of $\frac{am^2}{2c^3}$. *Ans.* $\frac{a^4m^8}{16c^{12}}$
8. Fifth power of $-\frac{2xy}{ac^2}$. *Ans.* $-\frac{32x^5y^5}{a^5c^{10}}$.
9. Sixth power of $-\frac{a^2c^n}{a^nbc}$ *Ans.* $\frac{c^{6n-6}}{a^{6n-12}b^6}$.

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10. Second power of $\frac{2a^{-2}x}{3bz^{-3}}$. *Ans.* $\frac{4a^{-4}x^2}{9b^2z^{-6}}$.

11. Third power of $-\frac{a^3b^2c^{-n}}{a^nb^mc^{-2}}$. *Ans.* $-\frac{c^{6-3n}}{a^{3n-9}b^{3m-6}}$.

12. Find value of $\left(\frac{-ax^n}{2a^nx^2}\right)^4$. *Ans.* $\frac{x^{4n-8}}{16a^{4n-4}}$.

13. Find value of $\left(-\frac{mx^n}{a^nb^{-n}}\right)^5$. *Ans.* $-\frac{m^5x^{5n}}{a^{5n}b^{-5n}}$.

14. Find value of $\left(\frac{a^2b^{-3}}{2cd}\right)^n$. *Ans.* $\frac{a^{2n}b^{-3n}}{2^nc^nd^n}$.

15. Find value of $\left(-\frac{a^nc^2}{a^2c^m}\right)^m$, when m is even. *Ans.* $\frac{a^{m(n-2)}}{c^{m(m-2)}}$.

16. Find value of $\left(-\frac{a^nb^3c^2}{a^2b^nx^2}\right)^m$, when m is odd.

Ans. $-\frac{a^{m(n-2)}c^{2m}}{b^{m(n-3)}x^{2m}}$.

17. Find the square of the fraction $\frac{a^{2n}(x-2)}{a^n(x-3)}$.

Ans. $\frac{a^{2n}(x-2)^2}{(x-3)^2}$.

18. Find the cube of the fraction $\frac{a^3(a^2-4)}{a^3-5a^2+6a}$.

Ans. $\frac{a^6(a+2)^3}{(a-3)^3}$.

CASE III.

212. To raise a polynomial to any given power.

1. To find the square of $a+b$.

SOLUTION. $(a+b)^2=(a+b)(a+b)$, which by multiplying we find to be equal to $a^2+2ab+b^2$.

Rule.—*Find the product of the quantity taken as a factor as many times as there are units in the exponent of the power.*

EXAMPLES.

2. Square of $x-1$. *Ans.* x^2-2x+1.
3. Cube of $a-c$. *Ans.* $a^3-3a^2c+3ac^2-c^3$.
4. Square of $2a^2-3c$. *Ans.* $4a^4-12a^2c+9c^2$.
5. Cube of $1-x$. *Ans.* $1-3x+3x^2-x^3$.
6. Cube of $2a-b^2$. *Ans.* $8a^3-12a^2b^2+$, etc.
7. Fourth power of $a-b$. *Ans.* $a^4-4a^3b+6a^2b^2-$, etc.
8. Fourth power of $2a-c^2$. *Ans.* $16a^4-32a^3c^2+$, etc.
9. Square of $a-b+c$. *Ans.* $a^2-2ab+b^2+2ac-2bc+c^2$.
10. Square of a^2-2b+c^2. *Ans.* $a^4-4a^2b+4b^2+2a^2c^2-4bc^2+c^4$.
11. Cube of $a+b+c$.
    Ans. $a^3+3a^2b+3ab^2+b^3+3a^2c+6abc+3b^2c+3ac^2+3bc^2+c^3$.
12. Cube of $2a-3b^2+c^3$.
    Ans. $8a^3-36a^2b^2+54ab^4-27b^6+12a^2c^3-36ab^2c^3+27b^4c^3+6ac^6-9b^2c^6+c^9$.

CASE IV.

213. Special methods of squaring a polynomial.

1. Find the square of $a+b+c+d$.

SOLUTION. Squaring the polynomial by actual multiplication, and arranging the terms, we shall have the square as written in the margin. Examining this, we see a certain law which may be stated as follows:

OPERATION.
$(a+b+c+d)^2=$
$a^2+b^2+c^2+d^2+2ab+2ac+2ad+2bc+2bd+2cd.$

Rule.—*The square of a polynomial equals the square of each term and twice the product of the terms taken two and two.*

NOTES.—1. The rule may be briefly stated thus: *The square of every one, twice the product of every two.*

2. The pupils will readily solve the given problems mentally by means of this rule.

EXAMPLES.

2. Square of $a+b+c$. *Ans.* $a^2+b^2+c^2+2ab+2ac+2bc$.
3. Square of $a+b-c+d$.
Ans. $a^2+b^2+c^2+d^2+2ab-2ac+2ad$, etc.
4. Square of $u-x+y-z$.
Ans. $u^2+x^2+y^2+z^2-2ux+2uy-2uz$, etc.
5. Square of $a+b+c+d+e$.
Ans. $a^2+b^2+c^2+d^2+e^2+2ab+2ac$, etc.
6. Square of $a+b+c+d+e+f+g$.
Ans. $a^2+b^2+c+d^2+e^2+f^2+g^2$, etc.
7. Square of $2a+3b+4c+d$.
Ans. $4a^2+9b^2+16c^2+d^2+12ab+16ac$, etc.
8. Square of $l-m+n-o+p-q+r$.
Ans. $l^2+m^2+n^2+o^2+p^2+q^2+r^2-2lm$, etc.
9. Square of $a+b-c-d+e+f-g-h+i+j$.
Ans. $a^2+b^2+c^2+d^2+e^2+f^2+g^2$, etc.
10. Square a polynomial consisting of the letters of the alphabet to m. *Ans.* $a^2+b^2+c^2+d^2+e^2+f^2+g^2$, etc.

214. SECOND METHOD.—Arranging the terms in another order and factoring, we have the following:

$$(a+b+c+d)^2=a^2+2ab+b^2+2(a+b)c+c^2+2(a+b+c)d+d^2.$$

Stating this in ordinary language, we have the following rule:

Rule.—*The square of a polynomial equals the square of the first term, plus twice the product of the first term into the second, plus the square of the second, plus twice the sum of the first two terms into the third, plus the square of the third,* etc.

215. THIRD METHOD.—Still another form is the following, which pupils may translate into common language:

$$(a+b+c+d)^2=a^2+(2a+b)b+[2(a+b)+c]c+[2(a+b+c)+d]d$$

CASE V.

216. Special method of cubing a polynomial.

1. Find the cube of $a+b+c$.

SOLUTION. Cubing the polynomial by actual multiplication, we have the expression (1). Factoring a part of this expression, we have the cube in the form of expression (2). Examining the last expression, we perceive a law in its formation which may be expressed as follows:

OPERATION.

$(a+b+c)^3=$

(1) $a^3+3a^2b+3ab^2+b^3+3a^2c+6abc+3b^2c+3ac^2+3bc^2+c^3$;

(2) $a^3+3a^2b+3ab^2+b^3+3(a+b)^2c+3(a+b)c^2+c^3$.

Rule.—*The cube of a polynomial equals the* CUBE *of the* FIRST *term, plus* THREE *times the* SQUARE *of the* FIRST *into the* SECOND, *plus* THREE *times the* FIRST *into the* SQUARE *of the* SECOND, *plus the* CUBE *of the* SECOND; *plus* THREE *times the* SQUARE *of the* SUM *of the* FIRST *and* SECOND *into the* THIRD, *plus* THREE *times the* SUM *of the* FIRST *and* SECOND *into the* SQUARE *of the* THIRD, *plus the* CUBE *of the* THIRD, etc.

EXAMPLES.

2. Cube $b+c+d$. *Ans.* $b^3+3b^2c+3bc^2+c^3+3(b+c)^2d+$, etc.
3. Cube $x+y+z$. *Ans.* x^3+, etc., $+3(x+y)^2z+3(x+y)z^2+z^3$.
4. Cube $a+b+c+d$.
Ans. a^3+, etc., $+3(a+b+c)^2d+3(a+b+c)d^2+d^3$
5. Cube $a+b+c+d+e$.
Ans. a^3+, etc., $+3(a+b+c+d)^2e+3(a+b+c+d)e^2+e^3$.
6. Cube $a+b-c+d-e$. *Ans.* a^3+, etc., $+3(a+b-c+d)e^2-e^3$.
7. Cube x^2-2y+z^3.
Ans. $x^6-6x^4y+12x^2y^2-8y^3+3(x^2-2y)^2z^3+$, etc.

NOTE.—In solving the 7th, expand $a+b+c$, and then substitute x^2 for a, $-2y$ for b, and z^3 for c, or involve it directly.

217. ANOTHER FORM.—The formula for cubing a polynomial may be put in another form, which is sometimes more convenient, as follows:

$$(a+b+c)^3=a^3+(3a^2+3ab+b^2)b+[3(a+b)^2+3(a+b)c+c^2]c$$

THE BINOMIAL THEOREM.

218. The **Binomial Theorem** expresses a general method of raising a binomial to any power.

219. This theorem affords a much shorter method of raising binomials to required powers than the tedious process of multiplication.

NOTE.—The Theorem was discovered by *Sir Isaac Newton*. It was considered of so much importance that the formula expressing it was engraved upon his monument in Westminster Abbey.

220. To derive the binomial theorem, we will raise two binomials to different powers by actual multiplication, and then examine these powers to discover the law of their formation.

Let us raise $a+b$ to the 2d, 3d, 4th and 5th powers.

$$a+b$$
$$a+b$$
$$a^2+ab$$
$$ab+b^2$$
2d power, $a^2+2ab+b^2$
$$a+b$$
$$a^3+2a^2b+ab^2$$
$$a^2b+2ab^2+b^3$$
3d power, $a^3+3a^2b+3ab^2+b^3$
$$a+b$$
$$a^4+3a^3b+3a^2b^2+ab^3$$
$$a^3b+3a^2b^2+3ab^3+b^4$$
4th power, $a^4+4a^3b+6a^2b^2+4ab^3+b^4$
$$a+b$$
$$a^5+4a^4b+6a^3b^2+4a^2b^3+ab^4$$
$$a^4b+4a^3b^2+6a^2b^3+4ab^4+b^5$$
5th power, $a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5$

Raising $a-b$ to the 2d, 3d, 4th and 5th powers, we have—

$$(a-b)^2=a^2-2ab+b^2;$$
$$(a-b)^3=a^3-3a^2b+3ab^2-b^3;$$
$$(a-b)^4=a^4-4a^3b+6a^2b^2-4ab^3+b^4;$$
$$(a-b)^5=a^5-5a^4b+10a^3b^2-10a^2b^3+5ab^4-b^5.$$

In deriving a law from these examples there are five things to be considered:

1st. The *number* of terms; 2d. The *signs* of the terms; 3d. The *letters* in the terms; 4th. The *exponents* of the letters; 5th. The *coefficients* of the terms.

I. NUMBER OF TERMS.—Examining the results in the given examples, we see that the *second* power has *three* terms, the *third* power *four* terms, the *fourth* power *five* terms, the *fifth* power *six* terms; hence we infer that

The number of terms in any power of a binomial is one greater than the exponent of the power.

II. SIGNS OF THE TERMS.—By examining the signs of the terms in the different powers, we infer the following principles:

1. *When both terms of the binomial are positive, all the terms will be positive.*

2. *When the first term is positive and the second negative, all the* ODD *terms, counting from the left, will be* POSITIVE, *and all the* EVEN *terms will be* NEGATIVE.

III. THE LETTERS.—By examining the letters in the different powers we infer the following principle:

The first letter of the binomial appears in all the terms except the last; the second letter appears in all except the first; and their product appears in all except the first and the last.

IV. THE EXPONENTS.—By examining the exponents of the terms in the different powers, we infer the following principles:

1. *The exponent of the leading letter or quantity in the first term is the same as the exponent of the power, and decreases by unity in each successive term toward the right.*

2. *The exponent of the second letter or quantity in the second term is one, and increases by unity in each successive term toward the right, until in the last term it is the same as the exponent of the power.*

3. *The sum of the exponents in any term is equal to the exponent of the power.*

V. THE COEFFICIENTS.—By examining the coefficients of the terms in the different powers, we infer the following principles:

1. *The coefficient of the first and the last term is* 1.

2. *The coefficient of the second term is the exponent of the power.*

Thus, in the *second* power it is 2; in the *third* power, 3; in the *fourth* power, 4; and in the *fifth* power, 5.

3. *The coefficient of any term, multiplied by the exponent of the leading letter in that term, and divided by the number of the term, equals the coefficient of the next term.*

Thus, in examining the powers of $a+b$ or $a-b$, we see that in the 4th power the coefficient of the second term, 4, multiplied by 3, the exponent of a in that term, and divided by 2, the number of the term, equals 6, the coefficient of the following term; in the 5th power we have 5, the coefficient of the 2d term, multiplied by 4, the exponent of a in that term, and divided by 2, the number of the term, equals 10, the coefficient of the 3d term; also 10, multiplied by 3 and divided by 3, equals 10, the coefficient of the 4th term, etc.

NOTES.—1. We see that the coefficients of the last half of the terms when *even*, or the terms after the middle when *odd*, are the same as the coefficients of the preceding terms, inversely; hence we may write the coefficients of the terms after the middle term without actual calculation.

2. The above method of deriving the Binomial Theorem is by observation and induction from particular cases. For a more rigid demonstration see Supplement, page 330.

EXAMPLES.

1. Raise $x-y$ to the fourth power.

SOLUTION.

Letters and exponents,	x^4	x^3y	x^2y^2	xy^3	y^4
Coefficients and signs,	1	-4	$+6$	-4	$+1$
Combining,	$x^4-4x^3y+6x^2y^2-4xy^3+y^4$				

NOTE.—In practice, we first write the literal part of the development, and then, commencing at the first term, insert the coefficients with their signs.

2. Develop $(a+x)^3$. *Ans.* $a^3+3a^2x+3ax^2+x^3$.

3. Develop $(a+c)^4$. *Ans.* $a^4+4a^3c+6a^2c^2+4ac^3+c^4$.

4. Develop $(a-b)^5$.
Ans. $a^5-5a^4b+10a^3b^2-10a^2b^3+5ab^4-b^5$.

5. Develop $(x-y)^6$.
Ans. $x^6-6x^5y+15x^4y^2-20x^3y^3+15x^2y^4-6xy^5+y^6$.

6. Develop $(a+b)^7$.
Ans. $a^7+7a^6b+21a^5b^2+35a^4b^3+35a^3b^4+21a^2b^5+7ab^6+b^7$.

7. Develop $(a-x)^8$.
Ans. $a^8-8a^7x+28a^6x^2-56a^5b^3+70a^4x^4-56a^3x^5+28a^2x^6-8ax^7+x^8$.

8. Develop $(1-x)^6$.
Ans. $1-6x+15x^2-20x^3+15x^4-6x^5+x^6$.

9. Develop $(a-c)^9$.
Ans. $a^9-9a^8c+36a^7c^2-84a^6c^3+126a^5c^4-126a^4c^5+84a^3c^6-36a^2c^7+9ac^8-c^9$.

10. Develop $(a+c)^{10}$.
Ans. $a^{10}+10a^9c+45a^8c^2+120a^7c^3+210a^6c^4+252a^5c^5+210a^4c^6+120a^3c^7+45a^2c^8+10ac^9+c^{10}$.

11. Develop $(a+x)^n$.
Ans. $a^n+na^{n-1}x+\frac{n(n-1)}{2}a^{n-2}x^2+\frac{n(n-1)(n-2)}{2\ 3}a^{n-3}x^3+\dots nax^{n-1}+x^n$.

BINOMIALS, WITH COEFFICIENTS AND EXPONENTS.

221. The **Binomial Theorem** can also be applied to binomials when one or both terms have coefficients and exponents.

1. Raise $2a^2+3b$ to the fourth power.

SOLUTION.

Let $2a^2=m$ and $3b=n$; then $m+n$ will equal $2a^2+3b$.
We have $(m+n)^4=m^4+4m^3n+6m^2n^2+4mn^3+n^4$.
Substituting, $(2a^2)^4+4(2a^2)^3 3b+6(2a^2)^2(3b)^2+4(2a^2)(3b)^3+(3b)^4$.
Reducing, $16a^8+96a^6b+216a^4b^2+216a^2b^3+81b^4$.

NOTE.—It can also be solved by writing the second expression directly and then reducing, without making the substitution.

2. Develop $(2a-3b)^3$. *Ans.* $8a^3-36a^2b+54ab^2-27b^3$.

3. Develop $(a^2+3x)^4$. *Ans.* $a^8+12a^6x+54a^4x^2+108a^2x^3+81x^4$.

4. Develop $(4a-3a^2x^3)^3$.
Ans. $64a^3-144a^4x^3+108a^5x^6-27a^6x^9$.

5. Develop $\left(x-\frac{a}{c}\right)^4$. *Ans.* $x^4-\frac{4ax^3}{c}+\frac{6a^2x^2}{c^2}-\frac{4a^3x}{c^3}+\frac{a^4}{c^4}$.

6. Develop $\left(a^2-\frac{x}{a}\right)^5$.
Ans. $a^{10}-5a^7x+10a^4x^2-10ax^3+\frac{5x^4}{a^2}-\frac{x^5}{a^5}$.

7. Develop $\left(3z-\frac{1}{z}\right)^4$. *Ans.* $81z^4-108z^2+54-\frac{12}{z^2}+\frac{1}{z^4}$.

8. Develop $(\frac{1}{2}a-\frac{2}{3}c)^5$.
Ans. $\frac{a^5}{32}-\frac{5a^4c}{24}+\frac{5a^3c^2}{9}-\frac{20a^2c^3}{27}+\frac{40ac^4}{81}-\frac{32c^5}{243}$.

9. Develop $(n^2-n^{-2})^6$.
Ans. $n^{12}-6n^8+15n^4-20+15n^{-4}-6n^{-8}+n^{-12}$.

10. Develop $(1+\frac{3}{2}x)^5$.
Ans. $1+\frac{15x}{2}+\frac{45x^2}{2}+\frac{135x^3}{4}+\frac{405x^4}{16}+\frac{243x^5}{32}$.

11. Develop $(2a^2x-4az^3)^6$.
Ans. $64a^{12}x^6-768a^{11}x^5z^3+3840a^{10}x^4z^6\cdots$, etc.

APPLIED TO POLYNOMIALS.

222. The **Binomial Theorem** may also be applied to polynomials by regarding them as binomials.

1. Find the second power of $a+b+c+d$.

SOLUTION.

Let $x=a+b$ and $y=c+d$; then $(x+y)^2=(a+b+c+d)^2$.
By the Theorem, $(x+y)^2=x^2+2xy+y^2$.
Substituting, $(a+b)^2+2(a+b)(c+d)+(c+d)^2$.
Expanding, $a^2+2ab+b^2+2ac+2bc+c^2+2ad+2bd+2cd+d^2$.

2. Expand $(a-b+c)^2$. *Ans.* $a^2-2ab+b^2+2ac-2bc+c^2$.

3. Expand $(a+b-c)^3$.
Ans. $a^3+3a^2b+3ab^2+b^3-3a^2c-6abc-3b^2c+3ac^2+3bc^2-c^3$.

4. Expand $(a+b+c+d)^3$.
Ans. $a^3+3a^2b+3ab^2+b^3+3a^2c+6abc+3b^2c+3ac^2+3bc^2+c^3+$, etc.

MISCELLANEOUS EXAMPLES.

1. Prove that the square of the sum of two numbers exceeds the square of their difference by 4 times their product.

2. Show how much the square of the sum of two numbers exceeds the product of their sum and difference.

3. What is the difference between one-half the square of a number and the square of one-half a number?

4. Prove that the difference between the squares of two consecutive numbers equals twice the less number, plus 1.

5. If two numbers differ by unity, prove that the difference of their squares equals the sum of the numbers.

6. Of three consecutive numbers, what is the difference between the square of the second and the product of the first and third?

7. Prove that the sum of the cubes of three consecutive numbers is divisible by the sum of the numbers.

EVOLUTION.

223. Evolution is the process of extracting the root of a quantity.

224. A Root of a quantity is *one* of its *several equal* factors; or, it is a quantity of which the given quantity is a power.

225. The Square Root of a quantity is *one* of its *two equal* factors; the *cube root* is *one* of its *three equal* factors, etc.

226. The Symbol $\sqrt{\ }$, called the *radical sign*, indicates the root of a quantity; thus, $\sqrt{4}$ indicates the square root of 4; $\sqrt[3]{8}$ indicates the cube root of 8.

227. The Index of the root is the figure placed above the radical sign to denote what root is required.

228. A Fractional Exponent is also used to indicate a root of a quantity; thus, $a^{\frac{1}{2}}$ indicates the square root of a; $8^{\frac{1}{3}}$ indicates the cube root of 8.

229. In fractional exponents the numerator indicates a power and the denominator a root of the quantity; thus,

$a^{\frac{2}{3}} = \sqrt[3]{a^2}$, or the cube root of a squared;

$a^{\frac{m}{n}} = \sqrt[n]{a^m}$, or the nth root of the mth power of a.

230. A Perfect Power is a quantity whose required root can be exactly obtained. An *Imperfect Power* is a quantity whose required root cannot be exactly obtained.

PRINCIPLES.

1. *The odd roots of a positive quantity are positive.*

For, the odd powers of a positive quantity are positive, while the odd powers of a negative quantity are negative. Thus, $\sqrt[3]{8} = +2$, and $\sqrt[5]{a^5} = +a$.

2. *The even roots of a positive quantity are either positive or negative.*

For, the *even* powers of either a positive or a negative quantity are positive; thus, $(+3)^2 = 9$ and $(-3)^2 = 9$; hence, $\sqrt{9} = +3$ or -3; also, $\sqrt{a^2} = +a$ or $-a$, since $(+a)^2$ and $(-a)^2$ both equal a^2.

NOTE.—The symbol $\pm$ means *plus* or *minus*; thus, $\sqrt{a^2} = \pm a$ is read the square root of a^2 is *plus* or *minus* a.

3. *The odd roots of a negative quantity are negative.*

For, the odd powers of a negative quantity are negative, while all the powers of a *positive* quantity are positive. Thus, $\sqrt[3]{-8} = -2$ and $\sqrt[5]{-a^5} = -a$.

4. *The even roots of a negative quantity are impossible.*

For, the *even* powers of both positive and negative quantities are positive; hence no quantity raised to an even power can produce a negative quantity. Thus, $\sqrt{-4}$, $\sqrt[4]{-16}$ and $\sqrt{-a^2}$, are all impossible.

NOTE.—The expression of the even root of a negative quantity is called an *Imaginary Quantity*.

CASE I.

231. To extract any root of a monomial.

232. The methods of extracting the roots of monomials are derived from those of involution, the one being the reverse of the other.

PRINCIPLES.

1. *To find any root of a monomial, we extract the root of the numeral coefficient and divide the exponents of the letters by the index of the root.*

This is evident, since in raising a monomial to any power we raise the coefficient to the required power and multiply the exponents of the letters by the index of the power. (Art. 210.)

2. *To find any root of a fraction, we extract the root of both terms.*

This is evident since to raise a fraction to any power we raise both terms to the required power. (Art. 211.)

1. Find the square root of $9a^2b^4$.

OPERATION.

$\sqrt{9a^2b^4} = \pm 3ab^2$

SOLUTION. Since to square a monomial we square the coefficient and multiply the exponents of the letters by 2, to find the square root of a monomial we reverse the process, and extract the square root of the coefficient and divide the exponents of the letters by 2. Doing this, we have $3ab^2$, and this is *plus* or *minus* (Prin. 2, Art. 230). Hence, $\sqrt{9a^2b^4}$ equals $\pm 3ab^2$.

2. Find the cube root of $8a^6b^3$.

SOLUTION. Since extracting the cube root of a quantity is the reverse of raising it to the third power, we extract the cube root of the coefficient 8, and divide the exponents of a^6 and b^3 by 3, and we have $2a^2b$; and the sign is plus (Prin. 1, Art. 230).

OPERATION.

$$\sqrt[3]{8a^6b^3}=2a^2b$$

Rule.—I. *Extract the root of the numeral coefficient and divide the exponents by the index of the root.*

II. *Prefix the double sign, $\pm$, to all even roots, and the minus sign to odd roots of negative quantities.*

EXAMPLES.

3. Square root of $16a^4b^6$. *Ans.* $\pm 4a^2b^3$.
4. Square root of $25x^6y^2$. *Ans.* $\pm 5x^3y$.
5. Cube root of $27a^6b^9$. *Ans.* $3a^2b^3$.
6. Cube root of $-64a^3x^6$. *Ans.* $-4ax^2$.
7. Fifth root of $-32a^5c^{10}$. *Ans.* $-2ac^2$.
8. Fifth root of $-243a^{10}x^{15}z^{20}$. *Ans.* $-3a^2x^3z^4$.
9. Cube root of $216a^3x^6y^9$. *Ans.* $6ax^2y^3$.
10. Square root of $144a^{2n}b^{4n}c^6$. *Ans.* $\pm 12a^nb^{2n}c^3$.
11. Square root of $64ab^3c^6$. *Ans.* $\pm 8a^{\frac{1}{2}}b^{\frac{3}{2}}c^3$.
12. Cube root of $125a^2x^4z^6$. *Ans.* $5a^{\frac{2}{3}}x^{\frac{4}{3}}z^2$.
13. Value of $\sqrt[5]{(a^2c^3x^5)}$. *Ans.* $a^{\frac{2}{5}}c^{\frac{3}{5}}x$.
14. Value of $\sqrt[4]{(16a^{4n}b^{8n})}$. *Ans.* $\pm 2a^nb^{2n}$.
15. Value of $(a^{2n}x^{3n})^{\frac{1}{n}}$. *Ans.* a^2x^3.
16. Value of $(\frac{4}{9}a^4x^6)^{\frac{1}{2}}$. *Ans.* $\pm\frac{2}{3}a^2x^3$.
17. Square root of $\frac{16a^3}{25x^5}$. *Ans.* $\pm\frac{4a^{\frac{3}{2}}}{5x^{\frac{5}{2}}}$.
18. Cube root of $\frac{8a^3}{27m^6}$. *Ans.* $\frac{2a}{3m^2}$.
19. Cube root of $\frac{64a^{3n}}{125c^{6n}}$. *Ans.* $\frac{4a^n}{5c^{2n}}$.
20. Value of $\sqrt[n]{(a^nc^{2n}x^3z^m)}$. *Ans.* $ac^2x^{\frac{3}{n}}z^{\frac{m}{n}}$.

CASE II.

233. To extract the square root of a polynomial.

234. The method of extracting the square root of a polynomial is derived from the law for the square of a polynomial.

1. Find the square root of $a^2+2ab+b^2$.

SOLUTION. The first term, a^2, is the square of the first term of the root, hence the first term of the root is the square root of a^2, which is a. Squaring a and subtracting it from the polynomial, we have $2ab+b^2$, which equals *twice the first term, plus the second term, multiplied by the second term* (Art. 214); hence, if we divide by twice the first term of the root, we will obtain the second term. Dividing by $2a$, we have b, the second term of the root; adding b to $2a$, the *trial divisor*, we have $2a+b$; multiplying by b and subtracting, there is no remainder. Hence, the square root of $a^2+2ab+b^2$ is $a+b$.

OPERATION.

$$\begin{array}{l|l} & a^2+2ab+b^2 \;\lfloor a+b \\ & a^2 \\ 2a & 2ab+b^2 \\ 2a+b & 2ab+b^2 \end{array}$$

2. Extract the square root of $a^2+2ab+2b^2+2ac+2bc+c^2$.

SOLUTION. Proceeding as in Prob. 1, we obtain the first two terms of the root, $a+b$, with a remainder $2ac+2bc+c^2$. This remainder equals twice the sum of the first and second terms, plus the third term, multiplied by the third term (Art. 215); hence, if we divide by twice the sum of the first and second terms, we will obtain the third term. Twice $a+b$ are $2a+2b$; dividing the remainder by $2a+2b$, we obtain c, the third term. Adding c to the *trial divisor*, we have $2a+2b+c$; multiplying by c and subtracting, nothing remains. Hence, etc.

OPERATION.

$$\begin{array}{l|l} & a^2+2ab+b^2+2ac+2bc+c^2 \;\lfloor a+b+c \\ & a^2 \\ 2a & 2ab+b^2 \\ 2a+b & 2ab+b^2 \\ 2a+2b & 2ac+2bc+c^2 \\ 2a+2b+c & 2ac+2bc+c^2 \end{array}$$

Rule.—I. *Arrange the terms of the polynomial with reference to the powers of some letter.*

II. *Extract the square root of the first term; write the result as the first term of the root; subtract the square from the given polynomial, and bring down the next two terms for a dividend.*

III. *Double the root already found, for a trial divisor; divide the first term of the dividend by the result; annex the quotient to the root and also to the trial divisor for a complete divisor. Multiply the complete divisor by the second figure of the root, and subtract the product from the dividend.*

IV. *If there are other terms of the polynomial remaining, proceed in a similar manner until the work is completed.*

NOTES.—1. If the first term of any remainder, when properly arranged, is not divisible by double the first term of the root, the polynomial is not a perfect square.

2. When the trial divisor consists of two or more terms, it is necessary to divide only the first term of the dividend by the first term of the divisor.

EXAMPLES.

Find the square root—

3. Of a^2+4a+4. *Ans.* $a+2$.

4. Of $a^2-4ac+4c^2$. *Ans.* $a-2c$.

5. Of $4a^2-12ax+9x^2$. *Ans.* $2a-3x$.

6. Of $x^2-xy+\frac{1}{4}y^2$. *Ans.* $x-\frac{1}{2}y$.

7. Of $a^{2n}+2a^nb^n+b^{2n}$. *Ans.* a^n+b^n.

8. Of $a^2+b^2+c^2+2ab+2ac+2bc$. *Ans.* $a+b+c$.

9. Of $a^2+x^2+z^2-2ax-2az+2xz$. *Ans.* $a-x-z$.

10. Of $a^2+4ab+4b^2-2ac-4bc+c^2$. *Ans.* $a+2b-c$.

11. Of $a^2-4ac+4c^2+6a-12c+9$. *Ans.* $a-2c+3$.

12. Of $a-2a^{\frac{3}{2}}b+a^2b^2+2a^{\frac{1}{2}}b^{\frac{1}{3}}-2ab^{\frac{4}{3}}+b^{\frac{2}{3}}$. *Ans.* $a^{\frac{1}{2}}-ab+b^{\frac{1}{3}}$.

13. Of $m^4+4m^3n+10m^2n^2+12mn^3+9n^4$. *Ans.* $m^2+2mn+3n^2$.

14. Of $4x^2-16x+16+12xy-24y+9y^2$. *Ans.* $2x-4+3y$.

15. Of $10n^2+25n^4-20n^3+1-4n+16n^6-24n^5$.
Ans. $1-2n+3n^2-4n^3$.

16. Of $a^6+4a^4x^2+4a^2x^4+x^6-4a^5x+4a^4x^3-2a^3x^3-8a^3x^3+4a^2x^4-4ax^5$. *Ans.* $a^3-2a^2x+2ax^2-x^3$.

SQUARE ROOT OF NUMBERS.

235. The method of extracting the *Square Root of Numbers* is most satisfactorily explained by Algebra.

PRINCIPLES.

1. *The square of a number consists of twice as many figures as the number, or of twice as many less one.*

Any integral number between 1 and 10 consists of one figure, and any number between their squares, 1 and 100, consists of *one* or *two* figures; hence the square of a number of *one* figure is a number of *one* or *two* figures.	$1^2=1$ $10^2=100$ $100^2=10,000$ $1000^2=1,000,000$

Any number between 10 and 100 consists of *two* figures, and any number between their respective squares, 100 and 10,000, consists of *three* or *four* figures; hence, the square of a number of *two* figures is a number of *three* or *four* figures, etc. Therefore, etc.

2. *If a number be pointed off into periods of two figures each, beginning at units' place, the number of full periods, together with the partial period at the left, if there be one, will equal the number of places in the square root.*

This is evident from Prin. 1, since the square of a number contains *twice* as many places as the number, or *twice* as many, *less one.*

3. *If we represent the units by* u, *the tens by* t, *the hundreds by* h, *the thousands by* T, *we will have the following formulas:*

$(t+u)^2=t^2+2tu+u^2;$

$(h+t+u)^2=h^2+2ht+t^2+2(h+t)u+u^2;$

$(T+h+t+u)^2=T^2+2Th+h^2+2(T+h)t+t^2+2(T+h+t)u+u^2.$

1. Extract the square root of 2025.

SOLUTION. Separating the number into periods of two figures each, we find there are two figures in the root (Prin. 2); hence the root consists of *tens* and *units*, and the number equals the *square of the tens, plus twice the tens into the units, plus the square of the units.*

OPERATION.

$t^2+2tu+u^2=$	20·25	(40
$t^2=40^2$ =	1600	5
$2tu+u^2=$	425	45
$2t=40\times2=80$		
$u=$ 5		
$(2t+u)u=85\times5=425$		

The greatest number of *tens* whose square is contained in 2025 is 4 *tens*; squaring the *tens* and subtracting, we have

425, which equals *twice the tens into the units, plus the square of the units* Now, since *twice the tens into the units* is generally greater than the *units into the units*, 425 consists principally of *twice the tens into the units;* hence if we divide by *twice the tens*, we can ascertain the *units*. *Twice the tens* equals 40×2, or 80; dividing 425 by 80, we find the units to be 5; then $(2t+u)u$, which equals $2tu+u^2$, equals $(80+5) \times 5$, or 425; subtracting, nothing remains. Hence the square root of 2025 is 4 tens and 5 units, or 45.

2. Extract the square root of 104976.

Solution. Separating the number into periods of two figures each, beginning at the right, we find there are three figures in the root, and the root consists of *hundreds, tens* and *units*, and the number equals $h^2+2ht+t^2+(2h+t)u+u^2$.

OPERATION.

$h^2+2ht+t^2+2(h+t)u+u^2=$		10·49·76(300
$h^2=$	$300^2=$	9 00 00 20
$2ht+t^2+2(h+t)u+u^2$		14976 4
$2h=$	$300 \times 2=600$	324
$t=$	20	
$(2h+t)t=$	$620 \times 20=$	12400
$2(h+t)u+u^2=$		2576
$2(h+t)=$	$320 \times 2=640$	
$u=$	4	
$[2(h+t)+u]u=$	$(644) \times 4=$	2576

The greatest number of *hundreds* whose square is contained in 104976 is 3 *hundreds;* squaring and subtracting, we have 14976 remaining, which equals $2ht+t^2+$, etc. Now, since *twice the hundreds into the tens* is generally much greater than t^2+, etc., 14976 must consist principally of *twice the hundreds into the tens;* hence, if we divide by *twice the hundreds*, we can ascertain the *tens*. *Twice the hundreds* equals $300 \times 2=600$; dividing by 600, we find the tens to be 2, etc.

Note.—In practice we abbreviate the work by omitting the ciphers and condensing the other parts, preserving merely the *trial* and *true* divisors, as is indicated in the margin. Here 6 is the first *trial divisor* and 62 the first *true divisor;* 64 is the second *trial*, and 644 the second *true divisor.*

OPERATION AS IN PRACTICE

	10·49·76(324
3	9
62	149
	124
644	2576
	2576

From these explanations we derive the following rule:

Rule.—I. *Commence at units' place and separate the number into periods of two figures each.*

II. *Find the greatest number whose square is contained in the left-hand period, place it at the right as a quotient and its square under the left-hand period, subtract and annex the next period to the remainder for a dividend.*

III. *Double the root found, and place it at the left for a trial divisor, divide the dividend exclusive of the right-hand figure by it; the quotient will be the second term of the root.*

IV. *Annex the second term of the root to the trial divisor for the true divisor, multiply the result by the second term of the root, subtract and bring down the next period for the next dividend.*

V. *Double the root now found, find the third term of the root as before, and thus proceed until all the periods have been used.*

Notes.—1. If the product of any true divisor by the corresponding term of the root exceeds the dividend, the term of the root must be diminished by a unit.

2. When a dividend, exclusive of the right-hand figure, will not contain the trial divisor, place a cipher in the root and at the right of the trial divisor, then bring down the next period and proceed as before.

3. To extract the square root of a decimal, *we point off the decimal into periods of two figures each, counting from units' place, and proceed as with whole numbers;* the reason of which may be easily seen. When a number is not a perfect square, annex ciphers and find the root on to decimals.

4. To find the square root of a common fraction, it is evident that we extract the square root of both numerator and denominator. When these terms are not perfect squares, the shortest way is to reduce the fraction to a decimal and extract the root.

EXAMPLES.

Extract the square root—

3. 576.	*Ans.* 24.	11. 5503716.	*Ans.* 2346.
4. 1296.	*Ans.* 36.	12. 4137156.	*Ans.* 2034.
5. 2401.	*Ans.* 49.	13. 11594025	*Ans.* 3405.
6. 56644.	*Ans.* 238.	14. $\frac{196}{625}$.	*Ans.* $\frac{14}{25}$.
7. 119025.	*Ans.* 345.	15. $164\frac{50}{72}$.	*Ans.* $12\frac{5}{6}$.
8. 207936.	*Ans.* 456.	16. 12.96.	*Ans.* 3.6.
9. 321489.	*Ans.* 567.	17. .0064.	*Ans.* .08.
10. 6421156.	*Ans.* 2534.	18. 10.6929.	*Ans.* 3.27.

CASE III.

236. To extract the cube root of polynomials.

237. The method of extracting the cube root of a polyno mial is derived from the law for the cube of a polynomial.

1. Find the cube root of $a^3+3a^2b+3ab^2+b^3$.

OPERATION.

$$\begin{array}{l|l} & a^3+3a^2b+3ab^2+b^3 \;\big(\, a+b \\ & a^3 \\ \hline 3a^2 & 3a^2b+3ab^2+b^3 \\ 3a^2+3ab+b^2 & 3a^2b+3ab^2+b^3 \end{array}$$

Solution. The first term, a^3, is the cube of the first term of the root (Art. 220); hence, the first term of the root is the cube root of a^3, or a. Cubing a and subtracting it from the polynomial, we have $3a^2b+3ab^2+b^3$, which equals *three times the square of the first term of the root into the second, plus, etc.*; hence, if we divide the first term of the remainder by *three* times *the first term* of the root, we can ascertain the *second term.* Three times a squared equals $3a^2$, the *trial divisor*; dividing $3a^2b$ by $3a^2$, we obtain b, the *second term* of the root. Adding to the trial divisor *three times the product of the first term of the root by the last term*, and the *square of the last term*, we have for a complete divisor $3a^2+3ab+b^2$; multiplying this by b, we have $3a^2b+3ab^2+b^3$ subtracting this, nothing remains. Hence the cube root of the given polynomial is $a+b$.

Rule.—I. *Arrange the terms of the polynomial with reference to the powers of some letter.*

II. *Extract the cube root of the first term; write the result as the first term of the root; subtract its cube from the given polynomial by bringing down the next three terms for a dividend.*

III. *Take three times the square of the root found for a trial divisor; divide the first term of the dividend by it, and write the quotient for the next term of the root.*

IV. *Add to the trial divisor three times the product of the first and last terms of the root, and the square of the last term, for a complete divisor. Multiply the complete divisor by the last term of the root, and subtract the product from the dividend.*

V. *If there are other terms of the polynomial remaining, proceed in a similar manner until the work is completed.*

Notes.—1. Arrange the terms of each new dividend, when necessary, with reference to the powers of the leading letter of the root.

2 When there are three terms in the root, to obtain the third, we use the first two terms as we did the first term in obtaining the second.

EXAMPLES.

Find the cube root—

2. Of $a^3-3a^2x+3ax^2-x^3$. *Ans.* $a-x$.
3. Of $a^3+6a^2b+12ab^2+8b^3$. *Ans.* $a+2b$.
4. Of $x^6-6x^4+12x^2-8$. *Ans.* x^2-2.
5. Of $a^3+3a^2b+3ab^2+b^3+3a^2c+6abc+3b^2c+3ac^2+3bc^2+c^3$. *Ans.* $a+b+c$.
6. Of $a^6-3a^5+5a^3-3a-1$. *Ans.* a^2-a-1.
7. Of $a^6-6a^5+15a^4-20a^3+15a^2-6a+1$. *Ans.* a^2-2a+1.
8. Of $m^6+6m^5-40m^3+96m-64$. *Ans.* m^2+2m-4.
9. Of $x^6-3ax^5+5a^3x^3-3a^5x-a^6$. *Ans.* x^2-ax-a^2.

CUBE ROOT OF NUMBERS.

238. The method of extracting the *Cube Root of Numbers* is most satisfactorily explained by means of Algebra.

PRINCIPLES.

1. *The cube of a number consists of three times as many figures as the number, or of three times as many less one or two.*

$1^3=1$

$10^3=1000$

$100^3=1,000,000$

Any integral number between 1 and 10 consists of one figure, and any integral number between their cubes, 1 and 1000, consists of one, two or three figures; hence the *cube* of a number of *one* figure is a number of *one, two* or *three* figures. Any number between 10 and 100 consists of *two* figures, and any number between their cubes, 1000 and 1,000,000, consists of *four, five* or *six* figures; hence the cube of a number of *two* figures consists of *three times two* figures, or *three times two*, less *one* or *two* figures. Therefore, etc.

2. *If a number be pointed off into periods of three figures each, beginning at units' place, the number of full periods, together with the partial period at the left, if there be one, will equal the number of figures in the root.*

This is evident from Prin. 1, since the cube of a number contains *three times* as many places as the number, or *three times* as many, *less one or two.*

3. *If we represent the units by* u, *the tens by* t, *the hundreds by* h, etc., *we will have the following formulas:*

1. $(t+u)^3 = t^3 + 3t^2u + 3tu^2 + u^3$;
2. $(h+t+u)^3 = h^3 + 3h^2t + 3ht^2 + t^3 + 3(h+t)^2u + 3(h+t)u^2 + u^3$.

1. Extract the cube root of 15625.

SOLUTION. Separating the number into periods of three figures each, we find there are two figures in the root (Prin. 2); hence the root consists of *tens* and *units*, and the number equals the $tens^3 + 3 \times tens^2 \times units + 3 \times tens \times units^2 + units^3$.

OPERATION.

$t^3+3t^2u+3tu^2+u^3=$		15·625	(20
$t^3=$	$20^3=$	8000	5
$3t^2u+3tu^2+u^3=$		7625	25
$3t^2=$	$3\times20^2=1200$		
$3tu=$	$3\times20\times5=\ \ 300$		
$u^2=$	$5^2=\ \ 25$		
$(3t^2+3tu+u^2)u$	$1525\times5=$	7625	

The greatest number of *tens* whose cube is contained in 15625 is 2 *tens;* cubing the tens and subtracting, we have 7625, which equals $3\times tens^2\times units + 3\times tens\times units^2 + units^3$. Now, since $3\times tens^2\times units$ is generally greater than $3\times tens\times units^2+$, etc., 7625 consists principally of $3\times tens^2\times units$; hence, if we divide 7625 by $3\times tens^2$, we can ascertain the *units.* $3\times tens^2 = 20^2\times3 = 1200$; dividing 7625 by 1200, we find the *units* to be 5. We then find $3\times tens\times units$ equals $3\times20\times5 = 300$, and $units^2$ equals $5^2 = 25$; taking their sum, we have $3t^2+3tu+u^2 = 1525$; and multiplying by the *units*, we have $(3t^2+3tu+u^2)u = 1525\times5$, or 7625; and subtracting, nothing remains. Hence the cube root of 15625 is 25.

AS IN PRACTICE.

	15·625 (25	
$2^3=$	8	
$3\times20^2=$	1200	7625
$3\times20\times5=$	300	
$5^2=$	25	
	1525	7625

NOTE.—In practice we omit the naughts and abbreviate as is seen in the margin.

2. Extract the cube root of 14706125.

SHOWN BY LETTERS.

		14·706·125 (245
$h^3=200^3=$		8 000 000
$3h^2=3\times200^2=$	120000	6 706 125
$3ht=3\times200\times4=$	24000	
$t^2=\quad 40^2=$	1600	
	145600	5 824 000
$3(h+t)^2u+3(h+t)u^2+u^3=$		882125
$3(h+t)^2=3\times240^2=$	172800	
$3(h+t)u=3\times240\times5=$	3600	
$u^2=\quad 5^2=$	25	
	176425	882125

AS IN PRACTICE.

		14·706·125 (245
$2^3=$		8
$20^2\times3=$	1200	6706
$20\times4\times3=$	240	
$4^2=$	16	
	1456	5824
		882125
$240^2\times3=$	172800	
$240\times5\times3=$	3600	
$5^2=$	25	
	176425	882125

NOTE.—In practice, abbreviate by omitting ciphers and using periods instead of the whole number each time.

Rule.—I. *Separate the number into periods of three figures each, beginning at units' place.*

II. *Find the greatest number whose cube is contained in the left-hand period; place it at the right and subtract its cube from the period, and annex the next period to the remainder for a dividend.*

III. *Take 3 times the square of the first term of the root regarded as tens for a* TRIAL DIVISOR; *divide the dividend by it, and place the quotient as the second term of the root.*

IV. *Take 3 times the last term of the root multiplied by the preceding part regarded as tens; write the result under the trial divisor, and under this write the square of the last term of the root; their sum will be the* COMPLETE DIVISOR.

V. *Multiply the* COMPLETE DIVISOR *by the last term of the root; subtract the product from the dividend, and to the remainder annex the next period for a new dividend. Take 3 times the square of the root now found, regarded as tens, for a trial divisor, and find the third term of the root as before; and thus continue until all the periods have been used.*

NOTES.—1. If the product of the complete divisor by any term of the root exceeds the dividend, that term must be diminished by a unit.

2. When a dividend will not contain a trial divisor, place a cipher in the root and two ciphers at the right of the trial divisor, and proceed as before.

3. The cube root of a fraction equals the cube root of the numerator

and denominator. When these are not perfect cubes, reduce to a decimal and then extract the root.

4. From the work in the margin we see that the cube root of a decimal of *one*, *two* or *three* places is a decimal of *one* place; of *four*, *five* or *six* places, is a decimal of *two* places, etc.; hence, to extract the cube root of a decimal, we point off in periods of three figures each, commencing at units' place and counting to the right.

$\sqrt[3]{1}=1$
$\sqrt[3]{.001}=.1$
$\sqrt[3]{.000001}=.01$
etc., etc.

EXAMPLES.

Find the cube root of—

3. 24389.	*Ans.* 29.	11. 41063.625.	*Ans.* 34.5.
4. 300763.	*Ans.* 67.	12. 130.323843.	*Ans.* 5.07.
5. 405224.	*Ans.* 74.	13. 95256152263.	*Ans.* 4567.
6. 18191447.	*Ans.* 263.	14. $34\frac{21}{64}$.	*Ans.* $3\frac{1}{4}$.
7. 44361864.	*Ans.* 354.	15. 6.	*Ans.* 1.8171+.
8. 82881856.	*Ans.* 436.	16. 7.	*Ans.* 1.9129+.
9. 66923416.	*Ans.* 406.	17. 9.	*Ans.* 2.08008+.
10. 1879080904.	*Ans.* 1234.	18. 10.	*Ans.* 2.15443+.

NEW METHOD OF CUBE ROOT.

239. The method here presented is supposed to be simpler and more convenient in its application than those usually given.

240. In the operation, indicate the *trial divisor* by t. d., and the *complete divisor* by C. D., and use dots, thus, . . , to indicate the local value of the figures.

1. Extract the cube root of 14706125

SOLUTION. We find the number of figures in the root, and the first term of the root, as in the preceding method.

We write 2, the first term of the root, at the left at the head of Col. 1st; three times its square with two dots annexed at the head of Col. 2d; its cube under the first period; then subtract and annex the next period for a dividend;

OPERATION.

1st Col.	2d Col.	14·706·125 (245
2	12 . . t. d.	8
4	256	6706
64	1456 C. D.	
8	16	5824
725	1728 . . t. d.	882125
	3625	
	176425 C. D.	882125

and divide by the number in Col. 2d as a *trial divisor*, for the second term of the root.

We then take 2 times 2, the first term, and write the product, 4, in Col. 1st, under the 2, and add; then annex the second term of the root to the 6 in Col. 1st, making 64, and multiply 64 by 4 for a *correction*, which we write under the trial divisor; and adding the *correction* to the *trial divisor*, we have the *complete divisor*, 1456. We then multiply the complete divisor by 4, subtract the product from the dividend, and annex the next period for a new dividend.

We then square 4, the second figure of the root, write the *square* under the *complete divisor*, and add the *correction*, the *complete divisor* and the *square* for the next *trial divisor*, which we find to be 1728. Dividing by the trial divisor, we find the next term of the root to be 5.

We then take 2 times 4, the second term, write the product 8 under the 64, add it to 64, and annex the third term of the root to the sum, 72, making 725, and then multiply 725 by 5, giving us 3625 for the next *correction*. We then find the *complete divisor* by adding the *correction* to the *trial divisor*; multiply the true divisor by 5, and subtract and have no remainder.

SHOWN BY ALGEBRA.

1st Col.		2d Col.	
2	h	12	$3h^2$
4	$2h$	256	$3ht+t^2$
64	$3h+t$	1456	$3h^2+3ht+t^2$
8	$2t$	16	t^2
72	$3h+3t$	1728	$3h^2+6ht+3t^2$
725	$3h+3t+u$	3625	$3hu+3tu+u^2$
		176425	$3h^2+6ht+3t^2+3hu+3tu+u^2$

Rule.—I. *Separate the number into periods of three figures each; find the greatest number whose cube is contained in the first period, and write it in the root.*

II. *Write the first term of the root at the head of* 1st *Col.,* 3 *times its square, with two dots annexed, for a* TRIAL DIVISOR, *at the head of 2d Col., and its cube under the first period; subtract and annex the next period to the remainder for a dividend; divide the dividend by the trial divisor, and place the quotient as the second term of the root.*

III. *Add twice the first term of the root to the number in the first column; annex the second term of the root, multiply the result by the second term, and write the product under the trial divisor* *for a* CORRECTION; *add the* CORRECTION *to the* TRIAL DIVISOR, *and the result will be the* COMPLETE DIVISOR; *multiply the* COMPLETE DIVISOR *by the last term of the root, subtract the product from the dividend, and annex the next period to the result for a new dividend.*

IV. *Square the last term of the root, and take the sum of this* SQUARE, *the last* COMPLETE DIVISOR *and the last* CORRECTION, *and annex two dots, for a new* TRIAL DIVISOR; *divide the dividend by it and obtain the next term of the root.*

V. *Add twice the second term of the root to the last number in the first column; annex the last term to the sum, multiply the result by the last term, and write the product under the last trial divisor for a* CORRECTION; *add the* CORRECTION *to the* TRIAL DIVISOR, *and the result will be the* COMPLETE DIVISOR; *use this as before, and thus continue until all the periods have been used.*

NOTE.—A part of this method can be easily remembered by means of the following formulas, which show the formation of the trial and *complete divisors:*

1. Trial Divisor + Correction = Complete Divisor.
2. Correction + Complete Divisor + Square = Trial Divisor.

2. Extract the cube root of 41673648563.

1ST COL.	2D COL.	41·673·648·563 (3467
3	27 .. t.d.	27
6	376	14673
94	3076 C.D.	
8	16	12304
1026	3468 .. t.d.	2369648
12	6156	
10387	352956 C.D.	
	36	2117736
	359148 .. t.d.	251912563
	72709	
	35987509 C.D.	
		251912563

NOTE.—Let the pupils apply this to the problems under the **previous method.**

RADICALS.

241. A **Radical** is an indicated root of a quantity; as $\sqrt{a}$, $a^{\frac{1}{3}}$, $\sqrt{24}$, $3\sqrt{(a-x)}$, etc.

242. The **Coefficient** of a radical is the quantity which indicates the number of times it is taken. Thus, in $3\sqrt{(a^3x)}$ and $m(a^2-x^2)^{\frac{1}{2}}$, 3 and m are the coefficients.

243. The **Degree** of a radical is indicated by the index of the radical sign, or by the denominator of the fractional exponent. Thus,

$\sqrt{a}$; $b^{\frac{1}{2}}$; $(2a^2x^3)^{\frac{1}{2}}$ are radicals of the *second* degree.

$\sqrt[3]{a^2}$; $b^{\frac{1}{3}}$; $(a^2x^3)^{\frac{1}{3}}$ are radicals of the *third* degree.

$\sqrt[n]{a^3}$; $(2b)^{\frac{1}{n}}$; $(a-b)^{\frac{1}{n}}$ are radicals of the nth degree.

244. Similar Radicals are those which have the same quantity under the same radical sign; as $\sqrt{(a^2c)}$ and $4\sqrt{(a^2c)}$; also $a\sqrt[n]{c}$ and $bc^{\frac{1}{n}}$.

NOTE.—A quantity whose root cannot be expressed without a radical sign or fractional exponent is called an *irrational quantity* or a *surd*; when the root expressed can be obtained exactly, it is called a *rational quantity.*

REDUCTION OF RADICALS.

245. Reduction of Radicals is the process of changing their forms without altering their values.

246. The reduction of radicals depends upon the following principles:

PRIN. 1. *Any root of the product of two or more quantities equals the product of the same roots of those quantities.*

Thus, $\sqrt{(ab)}=\sqrt{a}\times\sqrt{b}$. For, $\sqrt{(ab)}$ equals $a^{\frac{1}{2}}b^{\frac{1}{2}}$ (Art. 231), and $a^{\frac{1}{2}}b^{\frac{1}{2}}$ equals $\sqrt{a}\times\sqrt{b}$. In the same way we may prove that $\sqrt[n]{(ab)}=\sqrt[n]{a}\times\sqrt[n]{b}$.

PRIN. 2. *Fractional exponents may be added, subtracted, multiplied, and divided the same as integral exponents.*

This may be readily inferred from the relation of integers and fractions; it will be rigidly demonstrated in the article on the *Theory of Exponents.*

CASE I.

247. To reduce a radical to its simplest form.

248. A radical is in its *simplest form* when the radical part is integral, and contains no factor of which the given root can be extracted.

1. Reduce $\sqrt{(48a^3x)}$ to its simplest form.

SOLUTION. Resolving the quantity under the radical sign into two factors, one of which, $16a^2$, is a *perfect square*, and the other, $3ax$, not a perfect square, we find $\sqrt{(48a^3x)}$ equals $\sqrt{(16a^2\times 3ax)}$, which (Prin., Art. 246) equals $\sqrt{(16a^2)}\times\sqrt{(3ax)}$; which, extracting the square root of $16a^2$, equals $4a\sqrt{(3ax)}$.

OPERATION.

$$\sqrt{(48a^3x)}=\sqrt{(16a^2\times 3ax)}$$
$$=\sqrt{(16a^2)}\sqrt{(3ax)}$$
$$=4a\sqrt{(3ax)}$$

Rule.—I. *Separate the quantity under the radical into two factors, one of which shall contain all the perfect powers of the same degree as the radical.*

II. *Extract the root of the rational part, and prefix it as a coefficient to the other part placed under the radical sign.*

EXAMPLES.

Reduce the following radicals to their simplest form:

2. $\sqrt{(4a^2x)}$. *Ans.* $2a\sqrt{x}$.
3. $\sqrt{(9a^3c)}$. *Ans.* $3a\sqrt{(ac)}$.
4. $\sqrt{(8a^2x^3)}$. *Ans.* $2ax\sqrt{(2x)}$.
5. $3\sqrt{(12a^5)}$. *Ans.* $6a^2\sqrt{(3a)}$.
6. $a\sqrt{(48a^3b)}$. *Ans.* $4a^2\sqrt{(3ab)}$.
7. $\sqrt{(32a^6b^4c^5)}$. *Ans.* $4a^3b^2c^2\sqrt{(2c)}$.
8. $\sqrt{\{4a^2(a-b)\}}$. *Ans.* $2a(a-b)^{\frac{1}{2}}$.
9. $2\sqrt{\{9(a^3-a^2c)\}}$. *Ans.* $6a\sqrt{(a-c)}$.
10. $a\sqrt[3]{(x^3-x^3z)}$. *Ans.* $ax\sqrt[3]{(1-z)}$.
11. $2(ax^2-bx^3)^{\frac{1}{2}}$. *Ans.* $2x(a-bx)^{\frac{1}{2}}$.
12. $5a\sqrt[3]{(54a^2b^3c^4)}$. *Ans.* $15abc\sqrt[3]{(2a^2c)}$.

13. $(75a^3x^5y)^{\frac{1}{2}}$. *Ans.* $5ax^2(3axy)^{\frac{1}{2}}$.
14. $(24a^4b^5c^2)^{\frac{1}{3}}$. *Ans.* $2ab(3ab^2c^2)^{\frac{1}{3}}$.
15. $\sqrt[4]{\{162a^4(b^5-b^4c)^{\frac{1}{4}}\}}$. *Ans.* $3a\sqrt[4]{\{2b(b-c)^{\frac{1}{4}}\}}$.
16. $(a+b)(a^3-2a^2b+ab^2)^{\frac{1}{2}}$. *Ans.* $(a^2-b^2)\sqrt{a}$.
17. $(m-n)(2am^2+4amn+2an^2)^{\frac{1}{2}}$. *Ans.* $(m^2-n^2)\sqrt{2a}$.
18 $2a\sqrt{(8x^3y+16x^2y^2+8xy^3)}$. *Ans.* $4a(x+y)\sqrt{(2xy)}$.

WHEN THE RADICAL IS FRACTIONAL.

249. A **Fractional Radical** is reduced to its simplest form by changing it to an entire quantity.

1. Reduce $2\sqrt{\frac{2}{3}}$ to its simplest form.

SOLUTION. Multiplying both terms of the radical by 3 to make the denominator a perfect square, we have $2\sqrt{\frac{6}{9}}$; factoring, we have $2\sqrt{\frac{1}{9}\times 6}$, which equals $2\sqrt{\frac{1}{9}}\times\sqrt{6}$ (Art. 246), which, extracting the square root of $\frac{1}{9}$ and multiplying, equals $\frac{2}{3}\sqrt{6}$.

OPERATION

$$2\sqrt{\tfrac{2}{3}}=2\sqrt{\frac{2\times 3}{3\times 3}}=2\sqrt{\tfrac{6}{9}}$$
$$=2\sqrt{\tfrac{1}{9}\times 6}=2\sqrt{\tfrac{1}{9}}\times\sqrt{6}$$
$$=2\times\tfrac{1}{3}\sqrt{6}=\tfrac{2}{3}\sqrt{6}$$

Rule.—I. *Multiply both terms of the fraction by such a quantity as will render its denominator a perfect power of the degree indicated.*

II. *Resolve the quantity under the radical into two factors, one of which is a fraction and a perfect power of the degree indicated, and proceed as before.*

EXAMPLES.

Reduce the following to their simplest forms:

2. $\sqrt{\frac{4}{5}}$. *Ans.* $\frac{2}{5}\sqrt{5}$.
3. $\sqrt{\frac{7}{8}}$. *Ans.* $\frac{1}{4}\sqrt{14}$.
4. $\sqrt{\dfrac{45}{49}}$. *Ans.* $\frac{3}{7}\sqrt{5}$.
5. $2\sqrt{\dfrac{4a^3}{5}}$. *Ans.* $\dfrac{4a}{5}\sqrt{5a}$.
6. $2\sqrt{\frac{5}{8}}$. *Ans.* $\frac{1}{2}\sqrt{10}$.
7. $2\sqrt[3]{\dfrac{4a^4}{9}}$. *Ans.* $\dfrac{2a}{3}\sqrt[3]{12a}$.
8. $4b\sqrt[3]{\dfrac{3a^4}{4b^2}}$. *Ans.* $2a\sqrt[3]{6ab}$.
9. $6z\left(\dfrac{8x^5y^7}{27z^3}\right)^{\frac{1}{4}}$. *Ans.* $2xz(24xy^2z)^{\frac{1}{4}}$.

CASE II.

250. To reduce a rational quantity to the form of a radical.

1. Reduce $2a^2x$ to the form of the cube root.

SOLUTION. Since any quantity is equal to the cube root of its cube, $2a^2x$ is equal to the cube root of $(2a^2x)^3$, which equals the cube root of $8a^6x^3$.

OPERATION.

$$2a^2x = \sqrt[3]{(2a^2x)^3}.$$
$$= \sqrt[3]{(8a^6x^3)}$$

Rule.—*Raise the quantity to the power indicated by the given root, and place the result under the corresponding radical sign.*

NOTE.—The coefficient of a radical or any factor of a coefficient may be placed under the radical sign by raising it to the power indicated by the radical, and multiplying the quantity under the sign by the result.

EXAMPLES.

2. Reduce $2a^2c$ to the form of the square root. *Ans.* $\sqrt{(4a^4c^2)}$.
3. Reduce $3a^2b^3$ to the form of the cube root. *Ans.* $\sqrt[3]{(27a^6b^9)}$.
4. Reduce $a+2b$ to the form of the square root. *Ans.* $\sqrt{(a^2+4ab+4b^2)}$.
5. Reduce $2a^{\frac{1}{2}}b^{\frac{1}{3}}c^{\frac{1}{4}}$ to the form of the fourth root. *Ans.* $\sqrt[4]{(16a^2b^{\frac{4}{3}}c)}$.
6. Express $2a\sqrt{b}$ without a coefficient. *Ans.* $\sqrt{(4a^2b)}$.
7. Express $3a^2\sqrt[3]{(2ac^2)}$ without a coefficient. *Ans.* $\sqrt[3]{(54a^7c^2)}$.
8. Express $(x+y)\sqrt{z}$ without a coefficient. *Ans.* $\sqrt{z(x+y)^2}$.
9. Express $\frac{2}{3}\sqrt{(3ax)}$ without a coefficient. *Ans.* $\sqrt{\frac{4ax}{3}}$.
10. Express $6a\sqrt{(cx)}$ with a coefficient of 2. *Ans.* $2\sqrt{(9a^2cx)}$.
11. Express $\frac{2a}{b}\sqrt{(2ab^3)}$ without a literal coefficient. *Ans.* $2\sqrt{(2a^3b)}$.
12. Express $\frac{3c}{4}\sqrt[3]{\frac{4ax}{9c^2}}$ without a coefficient. *Ans.* $\sqrt[3]{\frac{3acx}{16}}$.

CASE III.

251. To reduce radicals of different degrees to a common radical index.

1. Reduce $a^{\frac{1}{2}}$ and $b^{\frac{1}{3}}$ to a common index.

SOLUTION. $\frac{1}{2}$ equals $\frac{3}{6}$; hence $a^{\frac{1}{2}} = a^{\frac{3}{6}}$, which equals $(a^3)^{\frac{1}{6}}$, which equals $\sqrt[6]{a^3}$. $\frac{1}{3}$ equals $\frac{2}{6}$; hence $b^{\frac{1}{3}} = b^{\frac{2}{6}}$, which equals $(b^2)^{\frac{1}{6}}$, which equals $\sqrt[6]{b^2}$.

OPERATION.

$$a^{\frac{1}{2}} = a^{\frac{3}{6}} = (a^3)^{\frac{1}{6}}, \text{ or } \sqrt[6]{a^3}$$
$$b^{\frac{1}{3}} = b^{\frac{2}{6}} = (b^2)^{\frac{1}{6}}, \text{ or } \sqrt[6]{b^2}$$

Rule.—I. *Reduce the exponents to a common denominator.*

II. *Raise each quantity to the power indicated by the numerator of its reduced exponent, and indicate the root denoted by the common denominator.*

EXAMPLES.

2. Reduce $a^{\frac{1}{2}}$ and $c^{\frac{2}{3}}$ to a common index. *Ans.* $\sqrt[6]{a^3}$; $\sqrt[6]{c^4}$.
3. Reduce $4^{\frac{1}{2}}$ and $6^{\frac{3}{4}}$ to a common index. *Ans.* $\sqrt[4]{16}$; $\sqrt[4]{216}$.
4. Reduce $a^{\frac{1}{n}}$ and $b^{\frac{1}{m}}$ to a common index. *Ans.* $\sqrt[mn]{a^m}$; $\sqrt[mn]{b^n}$.
5. Reduce $\sqrt{a}$, $\sqrt[3]{b^2}$ and $\sqrt[4]{c^3}$ to a common index. *Ans.* $\sqrt[12]{a^6}$; $\sqrt[12]{b^8}$; $\sqrt[12]{c^9}$.
6. Reduce $\sqrt{3}$, $\sqrt[3]{4}$ and $\sqrt[4]{5}$ to a common index. *Ans.* $\sqrt[12]{729}$; $\sqrt[12]{256}$; $\sqrt[12]{125}$.
7. Reduce $2\sqrt{6}$, $3\sqrt[3]{9}$ and $5\sqrt[6]{8}$ to a common index. *Ans.* $2\sqrt[6]{216}$; $3\sqrt[6]{81}$; $5\sqrt[6]{8}$.
8. Reduce $\sqrt{2a}$, $\sqrt[4]{3a^3}$ and $\sqrt[6]{4a^3}$ to a common index. *Ans.* $\sqrt[12]{(64a^6)}$; $\sqrt[12]{(27a^9)}$; $\sqrt[12]{(16a^6)}$.

ADDITION OF RADICALS.

252. Addition of Radicals is the process of finding the sum of two or more radical quantities.

1. What is the sum of $2\sqrt{8}$ and $3\sqrt{18}$?

SOLUTION. Factoring and reducing, we have $2\sqrt{8}$ equal to $4\sqrt{2}$, and $3\sqrt{18}$ equal to $9\sqrt{2}$; 9 times $\sqrt{2}$ plus 4 times $\sqrt{2}$ equals 13 times $\sqrt{2}$.

OPERATION.

$$2\sqrt{8} = 2\sqrt{(4\times 2)} = 4\sqrt{2}$$
$$3\sqrt{18} = 3\sqrt{(9\times 2)} = \underline{9\sqrt{2}}$$
$$13\sqrt{2}$$

2. What is the sum of $\sqrt{(2a^3x)}$ and $\sqrt{(8a^3x)}$.

SOLUTION. Reducing the radicals to their simplest form, we have $\sqrt{(2a^3x)}=a\sqrt{(2ax)}$; and $\sqrt{(8a^3x)}=2a\sqrt{(2ax)}$; $2a\sqrt{(2ax)}$ plus $a\sqrt{(2ax)}$ equals $3a\sqrt{(2ax)}$.

OPERATION.

$$\sqrt{(2a^3x)}=\sqrt{(a^2\times 2ax)}=a\sqrt{(2ax)}$$
$$\sqrt{(8a^3x)}=\sqrt{(4a^2\times 2ax)}=\underline{2a\sqrt{(2ax)}}$$
$$3a\sqrt{(2ax)}$$

Rule.—I. *Reduce the radicals to their simplest form.*

II. *If the radicals are then similar, add their coefficients and annex the common radical.*

III. *If the radicals are not similar, indicate their sum by the proper signs.*

EXAMPLES.

Find the sum—

3. Of $\sqrt{12}$ and $\sqrt{27}$. *Ans.* $5\sqrt{3}$.
4. Of $\sqrt{20}$ and $\sqrt{45}$. *Ans.* $5\sqrt{5}$.
5. Of $b\sqrt{(a^2b)}$ and $a\sqrt{b^3}$. *Ans.* $2ab\sqrt{b}$.
6. Of $\sqrt{(a^2b)}$ and $a\sqrt{b^3}$. *Ans.* $(a+ab)\sqrt{b}$.
7. Of $3\sqrt{(3a^3x)}$ and $a\sqrt{(27ax)}$. *Ans.* $6a\sqrt{(3ax)}$.
8. Of $a\sqrt{(18ax^2)}$ and $x\sqrt{(32a^3)}$. *Ans.* $7ax\sqrt{(2a)}$.
9. Of $a\sqrt{(ac^3)}$ and $c\sqrt{(a^3c)}$. *Ans.* $2ac\sqrt{(ac)}$.
10. Of $2\sqrt[3]{(16a)}$ and $3\sqrt[3]{(54a)}$. *Ans.* $13\sqrt[3]{(2a)}$.
11. Of $\sqrt{50}$, $\sqrt{72}$ and $\sqrt{128}$. *Ans.* $19\sqrt{2}$.
12. Of $\sqrt{(28a^2c^3)}$ and $c\sqrt{(112a^2c)}$. *Ans.* $6ac\sqrt{(7c)}$.
13. Of $\sqrt{2}$ and $2\sqrt{\frac{1}{2}}$. *Ans.* $2\sqrt{2}$.
14. Of $2\sqrt{3}$ and $3\sqrt{\frac{1}{3}}$. *Ans.* $3\sqrt{3}$.
15. Of $4\sqrt[3]{\frac{1}{4}}$ and $6\sqrt[3]{\frac{1}{32}}$. *Ans.* $\frac{7}{2}\sqrt[3]{2}$.
16. Of $\sqrt{(2a^2x)}$ and $\sqrt{(2b^2x)}$. *Ans.* $(a+b)\sqrt{2x}$.
17. Of $\sqrt{(a^2m)}$ and $\sqrt{(a^2n)}$. *Ans.* $a(\sqrt{m}+\sqrt{n})$.
18. Of $\sqrt{(a^4c)}$, $2\sqrt{(a^2b^2c)}$ and $\sqrt{(b^4c)}$. *Ans.* $(a+b)^2\sqrt{c}$.
19. Find the sum of $2x\sqrt{(50a^3c)}$, $3\sqrt[3]{(24a^4x^3)}$, $\frac{1}{3}a\sqrt{(72acx^2)}$ and $2x\sqrt[3]{(81x^4)}$. *Ans.* $12ax(\sqrt{2ac}+\sqrt[3]{3a})$.

SUBTRACTION OF RADICALS.

253. Subtraction of Radicals is the process of finding the difference between two radicals.

1. Subtract $3\sqrt{8}$ from $2\sqrt{32}$.

SOLUTION. Reducing the radicals to their simplest form, we have $2\sqrt{32}=8\sqrt{2}$, and $3\sqrt{8}=6\sqrt{2}$. $6\sqrt{2}$ subtracted from $8\sqrt{2}$ leaves $2\sqrt{2}$.

OPERATION.

$$2\sqrt{32}=2\sqrt{(16\times 2)}=8\sqrt{2}$$
$$3\sqrt{8}=3\sqrt{(4\times 2)}=\underline{6\sqrt{2}}$$
$$2\sqrt{2}$$

Rule.—I. *Reduce the radicals to their simplest form.*

II. *If the radicals are then similar, subtract the coefficient of the subtrahend from the coefficient of the minuend, and annex the common radical.*

III. *If the radicals are not similar, indicate their difference by the proper sign.*

EXAMPLES.

2. From $\sqrt{(20a)}$ take $\sqrt{(5a)}$. *Ans.* $\sqrt{(5a)}$.
3. From $\sqrt{(49ax^3)}$ take $\sqrt{(25ax^3)}$. *Ans.* $2x\sqrt{(ax)}$.
4. From $3\sqrt{(12a^3)}$ take $a\sqrt{(27a)}$. *Ans.* $3a\sqrt{(3a)}$.
5. From $\sqrt[3]{(125a^2)}$ take $\sqrt[3]{(8a^2)}$. *Ans.* $3\sqrt[3]{a^2}$.
6. From $2\sqrt{(a^2c)}$ take $a\sqrt{c^3}$. *Ans.* $(2a-ac)\sqrt{c}$.
7. From $\sqrt{(12a)}$ take $2\sqrt{\frac{3a}{4}}$. *Ans.* $\sqrt{3a}$.
8. From $\sqrt[3]{(250a^4x)}$ take $\sqrt[3]{(54a^4x)}$. *Ans.* $2a\sqrt[3]{(2ax)}$.
9. From $3\sqrt{\frac{3}{4}}$ take $2\sqrt{\frac{1}{3}}$. *Ans.* $\frac{5}{6}\sqrt{3}$.
10. From $4\sqrt[4]{32}$ take $4\sqrt[4]{\frac{1}{8}}$. *Ans.* $6\sqrt[4]{2}$.
11. From $\sqrt{(a^3+a^2x)}$ take $\sqrt{(9ab^2+9b^2x)}$. *Ans.* $(a-3b)\sqrt{(a+x)}$.
12. From $\sqrt{(2a^3+4a^2b+2ab^2)}$ take $\sqrt{(2a^3-4a^2b+2ab^2)}$. *Ans.* $2b\sqrt{(2a)}$.
13. Find the value of $7b\sqrt{(a^3x)}-a\sqrt{(9ab^2x)}+5\sqrt{(a^3c^2x)}-3c\sqrt{(9a^3x)}$. *Ans.* $4a(b-c)\sqrt{ax}$.

MULTIPLICATION OF RADICALS.

254. Multiplication of Radicals is the process of finding the product of two or more radicals.

255. PRINCIPLE.—*The product of the same root of two quantities is equal to the same root of their product.*

For (Art. 246, Prin.), $\sqrt{ab}=\sqrt{a}\times\sqrt{b}$; hence, transposing, $\sqrt{a}\times\sqrt{b}=\sqrt{(ab)}$. Similarly, we may prove that $\sqrt[n]{a}\times\sqrt[n]{b}=\sqrt[n]{(ab)}$.

CASE I.

256. To multiply radicals of the same degree.

1. Multiply $2a\sqrt{(ab)}$ by $3\sqrt{(ac)}$.

SOLUTION. We multiply the coefficients together, and the radical parts together. $2a$ multiplied by 3 equals $6a$; $\sqrt{(ab)}$ multiplied by $\sqrt{(ac)}$ equals $\sqrt{(a^2bc)}$ (Art. 255); hence, the product is $6a\sqrt{(a^2bc)}$, which, reduced, equals $6a^2\sqrt{(bc)}$.

OPERATION.

$$2a\sqrt{(ab)}\times 3\sqrt{(ac)}=$$
$$6a\sqrt{(a^2bc)}=6a^2\sqrt{(bc)}$$

Rule.—*Multiply the coefficients together for the coefficient of the product, and the quantities under the radical sign for the radical part of the product.*

NOTE.—We may reduce to the form of fractional exponents and add the exponents of the similar letters as in simple multiplication.

EXAMPLES.

2. Multiply $3\sqrt{(ac)}$ by $4\sqrt{(acx)}$. *Ans.* $12ac\sqrt{x}$.
3. Multiply $2\sqrt{6}$ by $3\sqrt{8}$. *Ans.* $24\sqrt{3}$.
4. Multiply $2\sqrt{x}$ by $3\sqrt{(ax)}$. *Ans.* $6x\sqrt{a}$.
5. Multiply $3\sqrt{(2a)}$ by $2\sqrt{(3c)}$. *Ans.* $6\sqrt{(6ac)}$.
6. Multiply $2\sqrt{(3a)}$ by $\sqrt{(6ax)}$. *Ans.* $6a\sqrt{(2x)}$.
7. Multiply $2\sqrt{27}$ by $3\sqrt{3}$. *Ans.* 54.
8. Multiply $5\sqrt[3]{(4a)}$ by $3\sqrt[3]{(2a)}$. *Ans.* $30\sqrt[3]{a^2}$.
9. Multiply $2\sqrt{\frac{1}{2}}$ by $2\sqrt{\frac{5}{8}}$. *Ans.* $\sqrt{5}$.

10. Multiply $3\sqrt{\frac{a}{3}}$ by $2\sqrt{\frac{a}{6}}$. *Ans.* $a\sqrt{2}$.
11. Multiply $3\sqrt[n]{(a^2x)}$ by $\sqrt[n]{(ax^3)}$. *Ans.* $3\sqrt[n]{(a^3x^4)}$.
12. Multiply $\sqrt[n]{(a^{n-1}c^{n+1})}$ by $\sqrt[n]{(ac^n)}$. *Ans.* $ac^2\sqrt[n]{c}$.
13. Multiply $a\sqrt[n]{(b^{n+1}c^{n-1})}$ by $\sqrt[n]{(a^{n+1}c^3)}$. *Ans.* $a^2bc\sqrt[n]{(abc^2)}$.
14. Multiply $\sqrt{a}+\sqrt{b}$ by $\sqrt{a}-\sqrt{b}$. *Ans.* $a-b$.
15. Multiply $\sqrt{(a+b)}$ by $\sqrt{(a-b)}$. *Ans.* $\sqrt{(a^2-b^2)}$.
16. Multiply $(m+n)^{\frac{2}{3}}$ by $(m-n)^{\frac{2}{3}}$. *Ans.* $(m^2-n^2)^{\frac{2}{3}}$.
17. Multiply $(a+c)^{\frac{1}{n}}$ by $(a-c)^{\frac{1}{n}}$. *Ans.* $(a^2-c^2)^{\frac{1}{n}}$.
18. Multiply $(a+b)^{\frac{n}{2}}$ by $(a+b)^{\frac{n}{2}}$. *Ans.* $(a+b)^n$.

CASE II.

257. To multiply radicals of different degrees.

1. Multiply $\sqrt{a}$ by $\sqrt[3]{b}$.

SOLUTION. We reduce the radicals to a common index, and then multiply. $\sqrt{a}$ equals $\sqrt[6]{a^3}$; $\sqrt[3]{b}$ equals $\sqrt[6]{b^2}$; $\sqrt[6]{a^3}$ multiplied by $\sqrt[6]{b^2}$ equals $\sqrt[6]{(a^3b^2)}$. (Case I.)

OPERATION.

$$\sqrt{a}=\sqrt[6]{a^3}$$
$$\sqrt[3]{b}=\sqrt[6]{b^2}$$
$$\sqrt[6]{(a^3b^2)}$$

Rule.—I. *Reduce the radicals to a common index, and multiply as in Case I.*; or,

II. *Reduce the radicals to the form of fractional exponents, and multiply as in simple multiplication.*

EXAMPLES.

2. Multiply $a^{\frac{1}{3}}$ by $c^{\frac{1}{2}}$. *Ans.* $a^{\frac{1}{3}}c^{\frac{1}{2}}$, or $\sqrt[6]{(a^2c^3)}$.
3. Multiply $\sqrt{a}$ by $\sqrt[3]{a}$. *Ans.* $a^{\frac{5}{6}}$, or $\sqrt[6]{a^5}$.
4. Multiply $3a^{\frac{1}{3}}$ by $4(ab)^{\frac{2}{3}}$. *Ans.* $12ab^{\frac{2}{3}}$.
5. Multiply $a\sqrt[n]{b}$ by $b\sqrt[m]{c}$. *Ans.* $ab\sqrt[mn]{(b^mc^n)}$.
6. Multiply $3\sqrt[m]{a}$ by $4\sqrt[n]{a}$. *Ans.* $12a^{\frac{m+n}{mn}}$.
7. Multiply $a\sqrt[n]{c^{2n}}$ by $a\sqrt[m]{(c^{3m}x)}$. *Ans.* $a^2c^5\sqrt[m]{x}$.
8. Multiply $\sqrt{(a+c)}$ by $\sqrt[3]{(a+c)}$. *Ans.* $\sqrt[6]{(a+c)^5}$.
9. Multiply $2\sqrt[n]{(a-c)}$ by $3\sqrt{a}$. *Ans.* $6\sqrt[2n]{\{a^n(a-c)^2\}}$.
10. Multiply $\sqrt{(m+n)}$ by $\sqrt[3]{(m-n)}$.
Ans. $\sqrt[6]{(m^2-n^2)^2(m+n)}$.

DIVISION OF RADICALS.

258. Division of Radicals is the process of dividing when one or both terms are radicals.

259. Principle.—*The quotient of the same roots of two quantities is equal to the same root of their quotient.*

For, by Art. 231, $\sqrt[n]{\frac{a}{b}}=\frac{\sqrt[n]{a}}{\sqrt[n]{b}}$;

hence, transposing, $\frac{\sqrt[n]{a}}{\sqrt[n]{b}}=\sqrt[n]{\frac{a}{b}}$.

CASE I.

260. To divide radicals of the same degree.

1. Divide $6\sqrt{(24ax)}$ by $2\sqrt{3a}$.

Solution. We first divide the coefficients, and then the radical parts. 6 divided by 2 equals 3; $\sqrt{(24ax)}$ divided by $\sqrt{3a}$ equals $\sqrt{8x}$, according to the principle above; hence the entire quotient is $3\sqrt{8x}$, which, reduced, equals $6\sqrt{2x}$.

OPERATION.

$$\frac{6\sqrt{(24ax)}}{2\sqrt{3a}}=3\sqrt{8x}=6\sqrt{2x}$$

Rule.—I. *Divide the coefficient of the dividend by the coefficient of the divisor, and the radical part of the dividend by the radical part of the divisor.*

II. *Annex the latter quotient to the former, and reduce the result to its simplest form.*

EXAMPLES.

2. Divide $6\sqrt{(ax)}$ by $2\sqrt{a}$. *Ans.* $3\sqrt{x}$.
3. Divide $4\sqrt{27}$ by $2\sqrt{3}$. *Ans.* 6.
4. Divide $5\sqrt{(27ac)}$ by $3\sqrt{(3a)}$. *Ans.* $5\sqrt{c}$.
5. Divide $6\sqrt{54a}$ by $3\sqrt{27}$. *Ans.* $2\sqrt{(2a)}$.
6. Divide $3\sqrt{(72ab)}$ by $2\sqrt{(6b)}$. *Ans.* $3\sqrt{(3a)}$.
7. Divide $6\sqrt{(20a)}$ by $2\sqrt{(30a)}$. *Ans.* $\sqrt{6}$.

8. Divide $2\sqrt{a^{\frac{7}{3}}}$ by $\sqrt{2a^{\frac{1}{3}}}$. *Ans.* $a\sqrt{2}$.
9. Divide $15(a^3b^5)^{\frac{1}{2}}$ by $3(ab^2)^{\frac{1}{2}}$. *Ans.* $5ab\sqrt{b}$.
10. Divide $5\sqrt[3]{(16a^2x^4)}$ by $2\sqrt[3]{(2ax)}$. *Ans.* $5x\sqrt[3]{a}$.
11. Divide $\frac{3}{4}\sqrt{\frac{1}{3a}}$ by $\frac{1}{2}\sqrt{\frac{3a}{5}}$. *Ans.* $\frac{1}{2a}\sqrt{5}$.
12. Divide $(1-a^2)^{\frac{1}{2}}$ by $(1-a)^{\frac{1}{2}}$. *Ans.* $(1+a)^{\frac{1}{2}}$.

CASE II.

261. To divide radicals of different degrees.

1. Divide $4\sqrt{(ax)}$ by $2\sqrt[3]{a}$.

Solution. We reduce the radicals to a common index, and then divide $4\sqrt{(ax)}$ equals $4\sqrt[6]{(a^3x^3)}$; $2\sqrt[3]{a}$ equals $2\sqrt[6]{a^2}$; $4\sqrt[6]{(a^3x^3)}$ divided by $2\sqrt[6]{a^2}$ equals $2\sqrt[6]{(ax^3)}$, according to Case I.

OPERATION.

$$\frac{4\sqrt{(ax)}}{2\sqrt[3]{a}}=\frac{4\sqrt[6]{(a^3x^3)}}{2\sqrt[6]{a^2}}=2\sqrt[6]{(ax^3)}$$

Rule.—I. *Reduce the radical parts of the dividend and divisor to a common index, and divide as in Case I.*; or,

II. *Reduce the radicals to the form of fractional exponents, and divide as in simple division.*

EXAMPLES.

2. Divide $3\sqrt[3]{(a^2c)}$ by $\sqrt{a}$. *Ans.* $3\sqrt[6]{(ac^2)}$.
3. Divide $6\sqrt[3]{(a^2x^4)}$ by $2\sqrt{(ax)}$. *Ans.* $3\sqrt[6]{(ax^5)}$.
4. Divide $2\sqrt[2]{(ab)}$ by $2\sqrt[4]{(ab)}$. *Ans.* $\sqrt[4]{(ab)}$.
5. Divide $8\sqrt{(ax)}$ by $4\sqrt{(ax^2)}$. *Ans.* $2\sqrt{x^{-1}}$.
6. Divide $6\sqrt{3}$ by $3\sqrt[3]{3}$. *Ans.* $2\sqrt[6]{3}$.
7. Divide 12 by $\sqrt{3}$. *Ans.* $4\sqrt{3}$.
8. Divide $4\sqrt[3]{(cz)}$ by $6\sqrt{(ac)}$. *Ans.* $\frac{2}{3}\sqrt[6]{\frac{z^2}{a^3c}}$.
9. Divide $a\sqrt[n]{x}$ by $c\sqrt[m]{x}$. *Ans.* $\frac{a}{c}\sqrt[mn]{x^{m-n}}$.
10. Divide $\frac{4}{5}\sqrt{\frac{a}{c}}$ by $\frac{2}{3}\sqrt[3]{\frac{a}{c}}$. *Ans.* $\frac{6}{5}\sqrt[6]{\frac{a}{c}}$.

INVOLUTION OF RADICALS.

262. Involution of Radicals is the process of raising radical quantities to any required power.

1. Find the cube of $2\sqrt{a}$.

SOLUTION. Expressing the radical with a fractional exponent, we have $(2a^{\frac{1}{2}})^3$; raising it to the third power, we have $2^3\times a^{\frac{3}{2}}$; which reduced gives $8a\sqrt{a}$.

OPERATION.

$$(2\sqrt{a})^3=(2\times a^{\frac{1}{2}})^3$$
$$=2^3\times a^{\frac{3}{2}}=8\sqrt{a^3}$$
$$=8a\sqrt{a}$$

Rule.—*Raise the rational part to the required power, and multiply the fractional exponent by the index of the power;* or,

Raise the rational and radical parts to the required power, and reduce the result to its simplest form.

NOTE.—*Dividing the index of the radical by any number raises the radical to a power denoted by the number.* Thus, the square of $\sqrt[6]{a}$ is $\sqrt[3]{a}$.

EXAMPLES.

Find the—

2. Square of $3\sqrt[3]{(ac^2)}$. *Ans.* $9c\sqrt[3]{(a^2c)}$.
3. Cube of $2\sqrt{(ax)}$. *Ans.* $8ax\sqrt{(ax)}$.
4. Cube of $3\sqrt{(2x)}$. *Ans.* $54x\sqrt{(2x)}$.
5. Square of $\frac{a}{2}\sqrt{(2a)}$. *Ans.* $\frac{a^3}{2}$.
6. Cube of $4\sqrt[4]{\frac{ax^3}{4}}$. *Ans.* $16x^2\sqrt[4]{(4a^3x)}$.
7. Fourth power of $3\sqrt{\frac{a}{3}}$. *Ans.* $9a^2$.
8. Fourth power of $2a\sqrt[3]{\frac{x}{a}}$. *Ans.* $16a^2x\sqrt[3]{(a^2x)}$.
9. nth power of $a\sqrt[3]{x}$. *Ans.* $a^n\sqrt[3]{x^n}$.
10. Third power of $\sqrt[3]{2a^2}\times\sqrt[6]{(ax^3)}$. *Ans.* $2a^2x\sqrt{(ax)}$.
11. Square of $\sqrt{a}-\sqrt{x}$. *Ans.* $a-2\sqrt{(ax)}+x$.
12. Square of $\sqrt{2}+a\sqrt{2}$. *Ans.* $2+4a+2a^2$.

EVOLUTION OF RADICALS.

263. Evolution of Radicals is the process of extracting any required root of radical quantities.

1. Find the cube root of $a^3\sqrt[4]{c^3}$.

SOLUTION. Reducing the radical to the form of a fractional exponent, we have $a^3c^{\frac{3}{4}}$; extracting the cube root by dividing the exponents by 3, we have $ac^{\frac{1}{4}}$, which equals $a\sqrt[4]{c}$.

OPERATION.

$$\sqrt[3]{a^3\sqrt[4]{c^3}}=\sqrt[3]{(a^3c^{\frac{3}{4}})}=$$
$$ac^{\frac{1}{4}}=a\sqrt[4]{c}$$

Rule.—*Extract the required root of the rational part, and divide the fractional exponent by the index of the root;* or,

Extract the required root of the rational and radical parts, and reduce the result to its simplest form.

NOTE.—*Multiplying the index of the radical by a number extracts a root of the radical denoted by the number.* Thus, the square root of $\sqrt[3]{a}$ is $\sqrt[6]{a}$.

EXAMPLES.

Find the—

2. Square root of $4\sqrt[3]{a^2}$. *Ans.* $\pm 2\sqrt[3]{a}$.
3. Square root of $9\sqrt[3]{(4x^2)}$. *Ans.* $\pm 3\sqrt[3]{(2x)}$.
4. Square root of $16\sqrt[3]{(3x)}$. *Ans.* $\pm 4\sqrt[6]{(3x)}$.
5. Cube root of $2\sqrt{(ax)}$. *Ans.* $\sqrt[6]{(4ax)}$.
6. Cube root of $2a\sqrt{(2a)}$. *Ans.* $\sqrt{(2a)}$.
7. Fourth root of $3a\sqrt[3]{(3a)}$. *Ans.* $\sqrt[3]{(3a)}$.
8. Fourth root of $\frac{1}{2}\sqrt{(2a)}$. *Ans.* $\sqrt[8]{(\frac{1}{2}a)}$.
9. Fifth root of $4c^2\sqrt{(2c)}$. *Ans.* $\sqrt{(2c)}$.
10. Fifth root of $4x\sqrt[4]{(4x)}$. *Ans.* $\sqrt[4]{(4x)}$.
11. Cube root of $\frac{a}{2}\sqrt{\frac{a}{2}}$. *Ans.* $\frac{1}{2}\sqrt{(2a)}$.
12. Square root of $\frac{a}{3}\sqrt[3]{\frac{a}{3}}$. *Ans.* $\frac{1}{3}\sqrt[3]{(3a^2)}$.
13. Fourth root of $\frac{a}{4}\sqrt[3]{\frac{a}{4}}$. *Ans.* $\frac{1}{2}\sqrt[3]{(2a)}$.

RATIONALIZATION.

264. Rationalization is the process of removing the radical sign from a quantity.

CASE I.

265. To rationalize any monomial surd.

1. Rationalize $\sqrt{a}$, also $a^{\frac{1}{3}}$.

SOLUTION. Multiplying $\sqrt{a}$ by $\sqrt{a}$, we have a, a rational quantity.

OPERATIONS.

$\sqrt{a}\times\sqrt{a}=a$

SOLUTION. Multiplying $a^{\frac{1}{3}}$ by $a^{\frac{2}{3}}$, we have a, a rational quantity.

$a^{\frac{1}{3}}\times a^{\frac{2}{3}}=a$

Rule.—*Multiply the surd by the same quantity with a fractional exponent which added to the given exponent shall equal unity.*

EXAMPLES.

2. What factor will rationalize $a^{\frac{3}{5}}$? *Ans.* $a^{\frac{2}{5}}$.
3. What factor will rationalize $\sqrt[3]{(a^2c)}$? *Ans.* $\sqrt[3]{ac^2}$.
4. What factor will rationalize $2\sqrt[4]{(ab)}$? *Ans.* $\sqrt[4]{(a^3b^3)}$.
5. What factor will rationalize $3\sqrt[3]{(a^2b)}$? *Ans.* $\sqrt[3]{(ab^2)}$.

CASE II.

266. To rationalize a binomial surd of the second degree.

1. Rationalize $\sqrt{a}-\sqrt{b}$.

SOLUTION. Since the product of the sum and difference of two quantities equals the difference of their squares, if we multiply $\sqrt{a}-\sqrt{b}$ by $\sqrt{a}+\sqrt{b}$, we obtain $a-b$, a rational quantity.

OPERATION.

$$\begin{array}{c}\sqrt{a}-\sqrt{b}\\ \underline{\sqrt{a}+\sqrt{b}}\\ a-b\end{array}$$

Rule.—*Multiply the given binomial by the same binomial, with the signs of one of the terms changed.*

EXAMPLES.

What factor will—

2. Rationalize $\sqrt{a}+\sqrt{x}$? *Ans.* $\sqrt{a}-\sqrt{x}$.
3. Rationalize $\sqrt{2}-\sqrt{3}$? *Ans.* $\sqrt{2}+\sqrt{3}$.
4. Rationalize $2\sqrt{a}+\sqrt{5}$? *Ans.* $2\sqrt{a}-\sqrt{5}$.
5. Rationalize $a+\sqrt{b}$? *Ans.* $a-\sqrt{b}$.

CASE III.

267. To rationalize either of the terms of a fractional surd.

1. Rationalize the denominator of $\frac{a}{\sqrt{x}}$.

SOLUTION. Multiplying both terms of the fraction by $\sqrt{x}$, we have $\frac{a\sqrt{x}}{x}$, in which the denominator is rational, and the value of the fraction is not changed.

OPERATION.

$$\frac{a}{\sqrt{x}}\times\frac{\sqrt{x}}{\sqrt{x}}=\frac{a\sqrt{x}}{x}$$

Rule.—*Multiply both terms of the fraction by a factor that will render either term rational which may be required.*

EXAMPLES.

2. Rationalize the denominator of $\frac{1}{\sqrt{2}}$. *Ans.* $\frac{1}{2}\sqrt{2}$.
3. Rationalize the denominator of $\frac{a}{2\sqrt{3}}$. *Ans.* $\frac{a\sqrt{3}}{6}$.
4. Rationalize the denominator of $\frac{1}{1+\sqrt{3}}$. *Ans.* $\frac{\sqrt{3}-1}{2}$.
5. Rationalize the denominator of $\frac{\sqrt{3}}{3-\sqrt{3}}$. *Ans.* $\frac{\sqrt{3}+1}{2}$.
6. Rationalize the denominator of $\frac{a}{\sqrt{x}-\sqrt{y}}$. *Ans.* $\frac{a(\sqrt{x}+\sqrt{y})}{x-y}$.
7. Rationalize the denominator of $\frac{\sqrt{a}+\sqrt{c}}{\sqrt{a}-\sqrt{c}}$. *Ans.* $\frac{(\sqrt{a}+\sqrt{c})^2}{a-c}$.

IMAGINARY QUANTITIES.

268. An **Imaginary Quantity** is an indicated even root of a negative quantity.

269. Imaginary quantities, though they represent impossible operations, are of use in some departments of mathematics.

PRINCIPLE.—*Every imaginary quantity may be reduced to the form of* $a\sqrt{-1}$; or $a\sqrt[2n]{-1}$.

Thus, $\sqrt{-a^2}=\sqrt{a^2\times -1}=\sqrt{a^2}\times\sqrt{-1}=\pm a\sqrt{-1}$;

Also, $\sqrt{-n}=\sqrt{n\times -1}=\pm\sqrt{n}\times\sqrt{-1}$; or, putting $a=\sqrt{n}$, we have $\pm a\sqrt{-1}$.

In all higher powers the form will be $\pm a\sqrt[2n]{-1}$.

REDUCTION OF IMAGINARY QUANTITIES.

Reduce to simplest form—

1. $\sqrt{-n^2}$. *Ans.* $\pm n\sqrt{-1}$.
2. $\sqrt{-4}$. *Ans.* $\pm 2\sqrt{-1}$.
3. $\sqrt{-9a^4}$. *Ans.* $\pm 3a^2\sqrt{-1}$.
4. $\sqrt{-16a^6}$. *Ans.* $\pm 4a^3\sqrt{-1}$.
5. $\sqrt{-n^3}$. *Ans.* $\pm n^{\frac{3}{2}}\sqrt{-1}$.
6. $\sqrt{-4a^2c^4}$. *Ans.* $\pm 2ac^2\sqrt{-1}$.
7. $\sqrt[4]{-16a^2b^4}$. *Ans.* $\pm 2a^{\frac{1}{2}}b\sqrt{-1}$.
8. $\sqrt{-8a^2c^3}$. *Ans.* $\pm 2ac^{\frac{3}{2}}\sqrt{-2}$.

ADDITION AND SUBTRACTION OF IMAGINARY QUANTITIES.

1. Add $\sqrt{-a^2}$ and $\sqrt{-b^2}$.

SOLUTION. $\sqrt{-a^2}=a\sqrt{-1}$; $\sqrt{-b^2}=b\sqrt{-1}$; adding these two quantities, we have $(a+b)\sqrt{-1}$.

OPERATION.

$$\sqrt{-a^2}=a\sqrt{-1}$$
$$\sqrt{-b^2}=b\sqrt{-1}$$
$$\overline{(a+b)\sqrt{-1}}$$

EXAMPLES.

2. Add $\sqrt{-a^2}$ and $\sqrt{-c^2}$. *Ans.* $(a+c)\sqrt{-1}$.
3. Add $\sqrt{-4}$ and $\sqrt{-9}$. *Ans.* $5\sqrt{-1}$.

4. Add $\sqrt{-8}$ and $\sqrt{-18}$. *Ans.* $5\sqrt{-2}$.
5. Subtract $\sqrt{-4}$ from $\sqrt{-16}$. *Ans.* $2\sqrt{-1}$.
6. Subtract $\sqrt{-2m^2}$ from $\sqrt{-8m^2}$. *Ans.* $m\sqrt{-2}$.

MULTIPLICATION OF IMAGINARY QUANTITIES.

1. Multiply $3\sqrt{-2}$ by $\sqrt{-3}$.

SOLUTION. $3\sqrt{-2}=3\sqrt{2}\times\sqrt{-1}$, and $\sqrt{-3}=\sqrt{3}\times\sqrt{-1}$; multiplying, we have $3\sqrt{6}\times(\sqrt{-1})^2$, which equals $3\sqrt{6}\times -1$, or $-3\sqrt{6}$.

OPERATION.

$$3\sqrt{-2}=3\sqrt{2}\times\sqrt{-1}$$
$$\sqrt{-3}=\sqrt{3}\times\sqrt{-1}$$
$$\overline{3\sqrt{6}\times(\sqrt{-1})^2=-3\sqrt{6}}$$

NOTE.—Had we multiplied the quantities under the radical sign at first, we could not have determined the sign of the product.

EXAMPLES.

2. Multiply $\sqrt{-3}$ by $2\sqrt{-2}$. *Ans.* $-2\sqrt{6}$.
3. Multiply $a\sqrt{-b^2}$ by $2\sqrt{-b^2}$. *Ans.* $-2ab^2$.
4. Multiply $1+\sqrt{-1}$ by $1-\sqrt{-1}$. *Ans.* 2.
5. Multiply $1+\sqrt{-1}$ by $1+\sqrt{-1}$. *Ans.* $2\sqrt{-1}$.

DIVISION OF IMAGINARY QUANTITIES.

1. Divide $4\sqrt{-6}$ by $2\sqrt{-3}$.

SOLUTION. $4\sqrt{-6}=4\sqrt{6}\times\sqrt{-1}$, and $2\sqrt{-3}=2\sqrt{3}\times\sqrt{-1}$; dividing the dividend by the divisor and canceling the common factors, we have $2\sqrt{2}$.

OPERATION.

$$\frac{4\sqrt{-6}=4\sqrt{6}\times\sqrt{-1}}{2\sqrt{-3}=2\sqrt{3}\times\sqrt{-1}}=2\sqrt{2}$$

NOTE.—In dividing, it is not necessary to reduce to the general form, though it is sometimes convenient.

EXAMPLES.

2. Divide $6\sqrt{-3}$ by $2\sqrt{-4}$. *Ans.* $\frac{3}{2}\sqrt{3}$.
3. Divide $4\sqrt{-a^2}$ by $a\sqrt{-2}$. *Ans.* $2\sqrt{2}$.
4. Divide $a\sqrt{-6c}$ by $\sqrt{-2ac}$. *Ans.* $\sqrt{3a}$.
5. Divide $2\sqrt{-1}$ by $1+\sqrt{-1}$. *Ans.* $1+\sqrt{-1}$.

PRINCIPLES.

1. *The* PRODUCT *of two imaginary quantities is real, with the sign before the radical the* REVERSE *of that given by the common rule.*

Thus, $\sqrt{-a^2}\times\sqrt{-b^2}=-ab$; and $-\sqrt{-a^2}\times-\sqrt{-b^2}=-ab$; but $-\sqrt{-a^2}\times\sqrt{-b^2}=+ab$, etc.

2. *The* QUOTIENT *of two imaginary quantities is real, with the sign before the radical the* SAME *as that given by the common rule.*

Thus, $\sqrt{-a^2}\div\sqrt{-c^2}=+\frac{a}{c}$; $-\sqrt{-a^2}\div-\sqrt{-c^2}=+\frac{a}{c}$; also $-\sqrt{-a^2}\div+\sqrt{-c^2}=-\frac{a}{c}$, etc.

SQUARE ROOT OF BINOMIAL SURDS.

270. Some binomials containing a radical quantity are squares of binomials, and will thus admit of the extraction of the square root.

Thus, $(2+\sqrt{3})^2=4+4\sqrt{3}+3$, or $7+4\sqrt{3}$; hence the square root of $7+4\sqrt{3}$ is $2+\sqrt{3}$.

271. Since the second term in the square of a binomial is *twice* the *product* of the other two terms, if we reduce the binomial surd to the form $a+2\sqrt{b}$, a will be the *sum* and b the *product* of two numbers.

1. Find the square root of $11+6\sqrt{2}$.

SOLUTION.—$11+6\sqrt{2}=11+2\sqrt{18}$; now find two numbers whose sum is 11 and product 18. These numbers, we see by inspection, are 9 and 2; then $11+2\sqrt{18}=9+2\sqrt{18}+2$, the square root of which is $3+\sqrt{2}$.

NOTE.—The numbers 9 and 2 can be obtained by letting $x+y=11$ and $xy=18$, and finding x and y.

EXAMPLES.

Find the

2. Square root of $14+6\sqrt{5}$. *Ans.* $3+\sqrt{5}$.
3. Square root of $28+10\sqrt{3}$. *Ans.* $5+\sqrt{3}$.
4. Square root of $11-4\sqrt{7}$. *Ans.* $2-\sqrt{7}$.
5. Square root of $15+6\sqrt{6}$. *Ans.* $3+\sqrt{6}$.
6. Square root of $5+2\sqrt{6}$. *Ans.* $\sqrt{2}+\sqrt{3}$.

7. Square root of $8-2\sqrt{15}$. *Ans.* $\sqrt{3}-\sqrt{5}$.
8. Square root of $9+2\sqrt{14}$. *Ans.* $\sqrt{2}+\sqrt{7}$.
9. Square root of $14-4\sqrt{6}$. *Ans.* $2\sqrt{3}-\sqrt{2}$.
10. Square root of $30+12\sqrt{6}$. *Ans.* $2\sqrt{3}+3\sqrt{2}$.
11. Square root of $3a-2a\sqrt{2}$. *Ans.* $\sqrt{a}-\sqrt{2a}$.
12. Square root of $2a+2\sqrt{a^2-b^2}$. *Ans.* $\sqrt{a+b}+\sqrt{a-b}$.
13. Square root of $m^2-2n\sqrt{m^2-n^2}$. *Ans.* $\sqrt{m^2-n^2}-n$.
14. Square root of $7+30\sqrt{-2}$. *Ans.* $5+3\sqrt{-2}$.
15. Square root of $12\sqrt{-2}-14$. *Ans.* $2+3\sqrt{-2}$.
16. Square root of $24\sqrt{-2}-23$. *Ans.* $3+4\sqrt{-2}$.
17. Square root of $40\sqrt{-3}-23$. *Ans.* $5+4\sqrt{-3}$.
18. Square root of $\sqrt{18}-4$. *Ans.* $\sqrt[4]{2}(\sqrt{2}-1)$.
19. Square root of $4\sqrt{3}-6$. *Ans.* $\sqrt[4]{3}(\sqrt{3}-1)$.
20. Square root of $6\sqrt{5}-10$. *Ans.* $\sqrt[4]{5}(\sqrt{5}-1)$.

NOTE.—In the 18th, let $x+y=\sqrt{18}$ and $xy=4$; in 19th, let $x+y=4\sqrt{3}$, and, since $6=2\sqrt{9}$, let $xy=9$, etc.

RADICAL EQUATIONS.

SOLVED AS SIMPLE EQUATIONS.

272. Radical Equations are those which contain the unknown quantity in the form of a radical.

273. Some radical equations may be solved by the principles of simple equations, examples of which will now be presented.

1. Given $\sqrt{x}-3=2$, to find x.

SOLUTION.—Given, $\sqrt{x}-3=2$,
transposing, $\sqrt{x}=5$,
squaring, $x=25$. *Ans.*

2. Given $\sqrt{(5+2\sqrt[3]{x})}=3$, to find x.

SOLUTION.—Given, $\sqrt{(5+2\sqrt[3]{x})}=3$,
squaring, $5+2\sqrt[3]{x}=9$,
transposing and reducing, $2\sqrt[3]{x}=4$,
dividing and cubing, $x=8$.

3. Given $\sqrt{x+7}+\sqrt{x-5}=6$.

SOLUTION.

Given,	$\sqrt{x+7}+\sqrt{x-5}=6$,
transposing,	$\sqrt{x+7}=6-\sqrt{x-5}$,
squaring,	$x+7=36-12\sqrt{x-5}+x-5$,
transposing and reducing,	$\sqrt{x-5}=2$,
squaring,	$x-5=4$,
transposing,	$x=9$.

EXAMPLES.

4. Given $\sqrt{2x}+5=9$, to find x. *Ans.* $x=8$.

5. Given $\sqrt{(x-3)}+2=5$, to find x. *Ans.* $x=12$.

6. Given $\sqrt{(x+5)}=\sqrt{x}+1$, to find x. *Ans.* $x=4$.

7. Given $3+\sqrt{(2x+4)}=7$, to find x. *Ans.* $x=6$.

8. Given $8-\sqrt{x}=\sqrt{(x-16)}$, to find x. *Ans.* $x=25$.

9. Given $\sqrt{(6+\sqrt[3]{3x})}+5=8$, to find x. *Ans.* $x=9$.

10. Given $\sqrt{x}-2=\sqrt{(x-24)}$, to find x. *Ans.* $x=49$

11. Given $\sqrt{(x+2)}=\dfrac{5}{\sqrt{(x+2)}}$, to find x. *Ans.* $x=3$.

12. Given $\sqrt{(x-9)}+\sqrt{(x+11)}=10$, to find x. *Ans.* $x=25$.

13. Given $\sqrt{(x-a)}=\sqrt{x}-\frac{1}{2}\sqrt{a}$, to find x. *Ans.* $x=\dfrac{25a}{16}$.

14. Given $\dfrac{x-2}{\sqrt{x}}=\dfrac{2\sqrt{x}}{3}$, to find x. *Ans.* $x=6$.

15. Given $\sqrt{(x+4ab)}=2a-\sqrt{x}$, to find x. *Ans.* $x=(a-b)^2$.

16. Given $x+\sqrt{(a-x)}=\dfrac{a}{\sqrt{(a-x)}}$, to find x. *Ans.* $x=a-1$.

17. Given $\sqrt{(x-a)}+\sqrt{(x-b)}=\sqrt{(a-b)}$, to find x. *Ans.* $x=a$.

18. Given $\dfrac{x-ax}{\sqrt{x}}=\dfrac{\sqrt{x}}{x}$, to find x. *Ans.* $x=\dfrac{1}{1-a}$.

SECTION VII

QUADRATIC EQUATIONS.

274. A **Quadratic Equation** is one in which the second power is the highest power of the unknown quantity; as, $x^2=4$, and $2x^2+3x=5$.

275. There are *two classes* of quadratic equations—*Pure* or *Incomplete* quadratics, and *Affected* or *Complete* quadratics.

276. The term which does not contain the unknown quantity is called the *absolute term*, or the term *independent of the unknown quantity.*

NOTES.—1. A quadratic equation is also called an equation of the *second degree.* An equation of the fourth degree is called a *bi-quadratic equation.*

2. When there are two unknown quantities, a quadratic is defined as an equation in which the greatest sum of the exponents in any term is two.

PURE QUADRATICS.

277. A **Pure Quadratic Equation** is one which contains the second power only of the unknown quantity; as $x^2=16$.

278. The **General Form** of a pure quadratic equation, is $ax^2=b$.

1. Given $2x^2+6=24$, to find x.

SOLUTION.

Given,	$2x^2+6=24$,
transposing and reducing,	$x^2=9$,
extracting the square root,	$x=\pm 3$.

3\. Given $\sqrt{x+7}+\sqrt{x-5}=6$.

SOLUTION.

Given,	$\sqrt{x+7}+\sqrt{x-5}=6$,
transposing,	$\sqrt{x+7}=6-\sqrt{x-5}$,
squaring,	$x+7=36-12\sqrt{x-5}+x-5$,
transposing and reducing,	$\sqrt{x-5}=2$,
squaring,	$x-5=4$,
transposing,	$x=9$.

EXAMPLES.

4\. Given $\sqrt{2x}+5=9$, to find x. *Ans.* $x=8$.

5\. Given $\sqrt{(x-3)}+2=5$, to find x. *Ans.* $x=12$.

6\. Given $\sqrt{(x+5)}=\sqrt{x}+1$, to find x. *Ans.* $x=4$.

7\. Given $3+\sqrt{(2x+4)}=7$, to find x. *Ans.* $x=6$.

8\. Given $8-\sqrt{x}=\sqrt{(x-16)}$, to find x. *Ans.* $x=25$.

9\. Given $\sqrt{(6+\sqrt[3]{3x})}+5=8$, to find x. *Ans.* $x=9$.

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13\. Given $\sqrt{(x-a)}=\sqrt{x}-\frac{1}{2}\sqrt{a}$, to find x. *Ans.* $x=\dfrac{25a}{16}$.

14\. Given $\dfrac{x-2}{\sqrt{x}}=\dfrac{2\sqrt{x}}{3}$, to find x. *Ans.* $x=6$.

15\. Given $\sqrt{(x+4ab)}=2a-\sqrt{x}$, to find x. *Ans.* $x=(a-b)^2$.

16\. Given $x+\sqrt{(a-x)}=\dfrac{a}{\sqrt{(a-x)}}$, to find x. *Ans.* $x=a-1$.

17\. Given $\sqrt{(x-a)}+\sqrt{(x-b)}=\sqrt{(a-b)}$, to find x.
Ans. $x=a$.

18\. Given $\dfrac{x-ax}{\sqrt{x}}=\dfrac{\sqrt{x}}{x}$, to find x. *Ans.* $x=\dfrac{1}{1-a}$.

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NOTES.—1. A quadratic equation is also called an equation of the *second degree.* An equation of the fourth degree is called a *bi-quadratic equation.*

2\. When there are two unknown quantities, a quadratic is defined as an equation in which the greatest sum of the exponents in any term is two.

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278. The **General Form** of a pure quadratic equation, is $ax^2=b$.

1\. Given $2x^2+6=24$, to find x.

SOLUTION.

Given,	$2x^2+6=24$,
transposing and reducing,	$x^2=9$,
extracting the square root,	$x=\pm 3$.

2. Given $ax^2+n=cx^2+m$, to find x.

SOLUTION.

Given,	$ax^2+n=cx^2+m,$
transposing,	$ax^2-cx^2=m-n,$
factoring,	$(a-c)x^2=m-n,$
reducing,	$x^2=\frac{m-n}{a-c},$
extracting square root,	$x=\pm\sqrt{\frac{m-n}{a-c}}.$

Rule.—I. *Reduce the equation to the form* $ax^2=b$.

II. *Divide by the coefficient of* x^2, *and extract the square root of both members.*

EXAMPLES.

3. Given $x^2+5=3x^2-13$, to find x. *Ans.* $x=\pm 3$.

4. Given $(x-6)(x+6)=-11$, to find x. *Ans.* $x=\pm 5$.

5. Given $x^2+ab=5x^2$, to find x. *Ans.* $x=\pm\frac{1}{2}\sqrt{(ab)}$.

6. Given $2x^2-3=\frac{x^2}{3}+12$, to find x. *Ans.* $x=\pm 3$.

7. Given $x^2+a^2+b^2=2ab+2x^2$, to find x. *Ans.* $x=\pm(a-b)$.

8. Given $4x+8=(x+2)^2$, to find x. *Ans.* $x=\pm 2$.

9. Given $\frac{1}{1-x}+\frac{1}{1+x}=3$, to find x. *Ans.* $x=\pm\frac{1}{3}\sqrt{3}$.

10. Given $\frac{4}{x-3}-\frac{4}{x+3}=\frac{1}{3}$, to find x. *Ans.* $x=\pm 9$.

11. Given $2x+5x^{-1}=3x-11x^{-1}$, to find x. *Ans.* $x=\pm 4$.

12. Given $\frac{x+3}{x-3}+\frac{x-3}{x+3}=3\frac{1}{3}$, to find x. *Ans.* $x=\pm 6$.

13. Given $\frac{x}{x+1}+\frac{x}{x+4}=1$, to find x. *Ans.* $x=\pm 2$.

14. Given $\frac{x}{a}+\frac{a}{x}=\frac{x}{b}+\frac{b}{x}$, to find x. *Ans.* $x=\pm\sqrt{(ab)}$.

15. Given $(n-x)^3+(n+x)^3=3n^3$, to find x. *Ans.* $x=\pm\frac{n}{6}\sqrt{6}$.

279. Radical Equations sometimes become pure quadratics when cleared of radicals.

1. Given $\sqrt{(x^2+9)}=\sqrt{(2x^2-7)}$, to find x. *Ans.* $x=\pm 4$.

2. Given $\sqrt{(x^2-4)}=2\sqrt{(a-1)}$, to find x. *Ans.* $x=\pm 2\sqrt{a}$.

3. Given $\sqrt{\left(\frac{10x^2-2}{2x}\right)}=\sqrt{x}$, to find x. *Ans.* $x=\pm\frac{1}{2}$.

4. Given $(x+a)^{\frac{1}{2}}=\frac{a}{(x-a)^{\frac{1}{2}}}$, to find x. *Ans.* $x=\pm a\sqrt{2}$.

5. Given $\sqrt{(x+m)}=\sqrt{\{x+\sqrt{(n^2+x^2)}\}}$, to find x.
Ans. $x=\sqrt{(m^2-n^2)}$.

6. Given $\sqrt{(x+a)}=\frac{b}{\sqrt{(x-a)}}$, to find x.
Ans. $x=\pm\sqrt{(a^2+b^2)}$.

7. Given $\sqrt{\{x^2+2ax+\sqrt{(x^2-4)}\}}=a+x$, to find x.
Ans. $x=\pm\sqrt{(a^4+4)}$.

8. Given $\sqrt{\{x^2+\sqrt{(x^4-n^4)}\}}=n$, to find x. *Ans.* $x=\pm n$.

9. Given $\sqrt{(x+m)}=\sqrt[4]{(x^2+n^2)}$, to find x. *Ans.* $x=\frac{n^2-m^2}{2m}$.

10. Given $x+\sqrt{(a^2+x^2)}=\frac{2a^2}{\sqrt{(a^2+x^2)}}$, to find x.
Ans. $x=\pm\frac{1}{3}a\sqrt{3}$.

PRINCIPLES OF PURE QUADRATICS.

280. The **Principles** of Pure Quadratics relate to the form of the equation and the relative value of its roots.

PRINCIPLES.

1. *Every pure quadratic equation may be reduced to the form* $ax^2=b$.

For, it is evident we can reduce all the terms containing x^2 to one term, as ax^2, and all the known terms to one term, as b; hence the form will become $ax^2=b$.

2. *Every pure quadratic equation has two roots, equal in numerical value, but of opposite signs.*

DEM. 1ST. The general form of a pure quadratic is $ax^2=b$; dividing by a, we have $x^2=b$ divided by a; representing the quotient by m^2, we have $x^2=m^2$; extracting the square root, we have $x=\pm m$. Therefore, etc.

OPERATION.

$$ax^2=b$$
$$x^2=\frac{b}{a}=m^2$$
$$x=\pm m$$

DEM. 2D. From the equation $x^2=m^2$, by transposition, we have $x^2-m^2=0$; factoring, we have $(x+m)(x-m)=0$. This equation can be satisfied by making $x-m$ equal 0, or by making $x+m$ equal 0, and in no other way (Art. 192, Prin. 1). Making $x-m=0$, we have $x=+m$; making $x+m=0$, we have $x=-m$. Therefore, etc.

$$x^2=m^2$$
$$x^2-m^2=0$$
$$(x+m)(x-m)=0$$
$$x-m=0$$
$$x=+m$$
$$x+m=0$$
$$x=-m$$

PROBLEMS

PRODUCING PURE QUADRATICS.

1. The product of two numbers is 48, and the greater is 3 times the smaller; required the numbers.

SOLUTION.

Let $x=$ the smaller number,
and $3x=$ the larger number;
then $3x\times x=3x^2=48$;
whence, $x^2=16$;
then $x=\pm 4$,
and $3x=\pm 12$.

2. The sum of two numbers is 10, and their product is 24; required the numbers.

SOLUTION. Let $5+x$ represent the greater number, and $5-x$ the smaller number; their sum will be 10, which satisfies the first condition of the problem. By the second condition, $(5+x)(5-x)=24$, or $25-x^2=24$. Reducing, we have $x=\pm 1$. Using the positive value, we have 6 or 4; using the negative value, we have 4 or 6.

OPERATION.

$5+x=$ the greater number;
$5-x=$ the less number.

$$(5+x)(5-x)=24$$
$$25-x^2=24$$
$$x^2=1$$
$$x=\pm 1$$
$$5+x=6 \text{ or } 4$$
$$5-x=4 \text{ or } 6$$

3. The sum of two numbers is 15, and their product is 54; required the numbers. *Ans.* 9; 6.

4. The sum of two numbers is 12, and the sum of their squares is 74; required the numbers. *Ans.* 7; 5.

5. The difference of two numbers is 5, and their product is 84; required the numbers. *Ans.* 12; 7.

6. Divide the number 24 into two such parts that their product shall be 140. *Ans.* 14; 10.

7. The difference of two numbers is 4 and the sum of their squares is 208; what are the numbers? *Ans.* 12; 8.

8. Required a number whose square is 432 more than the square of $\frac{1}{2}$ the number. *Ans.* 24.

9. What number is that which, if multiplied by $\frac{1}{2}$ of itself, is 72 greater than the square of its $\frac{2}{3}$? *Ans.* 36.

10. What two numbers are to each other as 4 to 5, and the difference of whose squares is 81? *Ans.* 12; 15.

11. Required two numbers whose product is 48, and the quotient of the greater divided by the less, 3. *Ans* 12; 4.

12. There is a rectangular field containing 4 acres whose length is to its breadth as 5 to 2; required its dimensions.
Ans. Length, 40 rods; breadth, 16 rods.

13. There is a number whose third part squared and subtracted from 170 leaves a remainder of 26; what is the number? *Ans.* 36.

14. $\frac{3}{4}$ of the square of twice a number is equal to $\frac{4}{3}$ of the square of $\frac{4}{3}$ of the number, increased by 28; what is the number? *Ans.* 6.

15. A man has two cubical bundles of hay, one of which contains 999 cubic feet more than the other; what are the dimensions of each, if the smaller is $\frac{3}{4}$ as long as the larger? *Ans.* 12 ft.; 9 ft.

16. A merchant bought a piece of cloth for \$216, and the number of dollars he paid for a yard was to the number of yards as 2 to 3; required the price per yard and the number of yards. *Ans.* 18 yards at \$12 a yard.

17. A man bought a field whose length was to its breadth as 5 to 4; the price per acre was equal to the number of rods in the length of the field, and 5 times the distance around the

field equaled the number of dollars that it cost; required the length and breadth of the field.

Ans. Length, 60 rods; breadth, 48 rods.

18. From a cask containing 81 gallons of wine a vintner draws off a certain quantity, and then, filling the cask with water, draws off the same quantity again, and then there remains only 36 gallons of pure wine; how much wine did he draw off each time? *Ans.* First, 27 gals.; second, 18 gals.

AFFECTED QUADRATICS.

281. An **Affected Quadratic Equation** is one that contains both the second and first powers of the unknown quantity; as, $x^2+4x=12$.

282. The **General Forms** of an affected quadratic are $ax^2+bx=c$, and $x^2+2px=q$.

1. Given $x^2+6x=16$, to find the value of x.

OPERATION.

$$x^2+6x=16$$
$$x^2+6x+9=25$$
$$x+3=\pm 5$$
$$x=-3\pm 5$$
$$x=+2$$
$$x=-8$$

SOLUTION. If the first member of this equation were a perfect square, we could extract the square root of both members, and find the value of x from the resulting equation. Let us, therefore, see if we can make the first member a perfect square.

The square of a binomial is equal to the square of the first term, plus twice the product of the first term into the second, plus the square of the second. If now we consider x^2+6x as the first two terms of the *square of a binomial*, the first term of this binomial will be the square root of x^2 or x; and $6x$ will be twice the first term of the binomial into the second; hence, if we divide $6x$ by twice the first term, $2x$, the quotient, 3, must be the second term of the binomial, and its square, 9, added to the first member of the equation, will render it a perfect square.

Adding 9 to the first member to complete the square, and to the second member to preserve the equality, we have $x^2+6x+9=25$. Extracting the square root, we have $x+3=\pm 5$; from which, using the positive value of 5, we have $x=2$; and using the negative value, we have $x=-8$.

Rule.—I. *Reduce the given equation to the general form* $x^2+2px=q$.

II. *Add to both members the square of one-half the coefficient of* x.

III. *Extract the square root of both members, and solve the resulting simple equation.*

2. Given $x^2-8x=-7$, to find x.

OPERATION.

$$x^2-8x=-7 \quad (1)$$
$$x^2-8x+16=9 \quad (2)$$
$$x-4=\pm 3 \quad (3)$$
$$x=7 \quad (4)$$
$$x=1 \quad (5)$$

SOLUTION. Completing the square by adding the square of half the coefficient of x to both members, we have $x^2-8x+16=9$. Extracting the square root of both members, we have $x-4=\pm 3$. Using the positive value of 3, we have $x=7$; using the negative value, we have $x=1$.

VERIFICATION.

$7^2-8\times 7=-7$; or $49-56=-7$;

also, $1^2-8\times 1=-7$; or $1-8=-7$.

3. Given $x^2+4x=32$, to find x.

SOLUTION.

Given the equation,	$x^2+4x=32$,
completing the square,	$x^2+4x+4=36$,
extracting the root,	$x+2=\pm 6$
whence,	$x=-2+6=+4$,
and	$x=-2-6=-8$.

Both of which will verify the equation.

4. Given $x^2-12x=-20$, to find x.

SOLUTION.

Given the equation,	$x^2-12x=-20$,
completing the square,	$x^2-12x+36=16$,
extracting the root,	$x-6=\pm 4$;
whence,	$x=6+4=10$
and	$x=6-4=2$.

5. Given $x^2-5x=6$, to find x.

SOLUTION.

Given the equation,	$x^2-5x=6$,
completing the square,	$x^2-5x+(\frac{5}{2})^2=6+(\frac{5}{2})^2$,
which reduced, equals	$x^2-5x+(\frac{5}{2})^2=6+\frac{25}{4}=\frac{49}{4}$,
extracting the root,	$x-\frac{5}{2}=\pm\frac{7}{2}$;
whence,	$x=\frac{5}{2}+\frac{7}{2}=+6$,
and	$x=\frac{5}{2}-\frac{7}{2}=-1$.

6. Given $3x^2 - 7x = 66$, to find x.

SOLUTION.

Given the equation, $3x^2 - 7x = 66$,

dividing by the coefficient of x^2, $x^2 - \frac{7x}{3} = 22$,

completing the square, $x^2 - \frac{7}{3}x + (\frac{7}{6})^2 = 22 + (\frac{7}{6})^2$,

which reduced, equals $x^2 - \frac{7}{3}x + (\frac{7}{6})^2 = \frac{841}{36}$,

extracting the root, $x - \frac{7}{6} = \pm \frac{29}{6}$;

whence, $x = \frac{7}{6} + \frac{29}{6} = +6$,

and $x = \frac{7}{6} - \frac{29}{6} = -\frac{11}{3}$.

Reduce the following equations:

7. $x^2 + 2x = 24$. *Ans.* $x = 4$, or -6.
8. $x^2 + 2x = 35$. *Ans.* $x = 5$, or -7.
9. $x^2 + 4x = 32$. *Ans.* $x = 4$, or -8.
10. $x^2 - 6x = 16$. *Ans.* $x = 8$, or -2.
11. $x^2 - 3x = 18$. *Ans.* $x = 6$, or -3.
12. $x^2 - 5x = 84$. *Ans.* $x = 12$, or -7.
13. $x^2 - 11x = 60$. *Ans.* $x = 15$, or -4.
14. $x^2 - 13x = 140$. *Ans.* $x = 20$, or -7.
15. $x^2 - 14x = -24$. *Ans.* $x = 12$, or 2.
16. $x^2 - 4x = 1$. *Ans.* $x = 2 \pm \sqrt{5}$
17. $3x^2 - 4x = 39$. *Ans.* $x = 4\frac{1}{3}$, or -3.
18. $(x-1)(x-2) = 20$. *Ans.* $x = 6$, or -3.
19. $x^2 - 6x = -13$. *Ans.* $x = 3 \pm \sqrt{-4}$.
20. $4(x^2 - 1) = 4x - 1$ *Ans.* $x = \frac{3}{2}$, or $-\frac{1}{2}$.
21. $(2x-3)^2 = 8x$. *Ans.* $x = 4\frac{1}{2}$, or $\frac{1}{2}$.
22. $x^2 - 3 = \frac{x-3}{6}$. *Ans.* $x = 1\frac{2}{3}$, or $-1\frac{1}{2}$.
23. $x^2 + 2px = q$. *Ans.* $x = -p \pm \sqrt{(p^2 + q)}$.
24. $x^2 - ax = b$. *Ans.* $x = \frac{1}{2}(a \pm \sqrt{a^2 + 4b})$.
25. $x^2 - 2nx = m^2 - n^2$. *Ans.* $x = n \pm m$.
26. $x^2 - ax - bx = -ab$. *Ans.* $x = a$, or b.

27. $x + \frac{1}{x-3} = 5$. *Ans.* $x = 4$.
28. $x = 2 + \frac{5}{4x}$ *Ans.* $x = 2\frac{1}{2}$, or $-\frac{1}{2}$.
29. $\frac{x-1}{x-3} + 2x = 12$. *Ans.* $x = 5$, or $3\frac{1}{2}$.
30. $\frac{2}{x+3} + \frac{x+3}{2} = \frac{10}{3}$. *Ans.* 3, or $-2\frac{1}{3}$.
31. $\frac{2x}{x+2} + \frac{x+2}{2x} = 2$. *Ans.* $x = 2$.
32. $\frac{x}{x+1} + \frac{x+1}{x} = \frac{13}{6}$. *Ans.* $x = 2$, or -3.
33. $\frac{x+2}{x+1} + \frac{x+1}{x+2} = \frac{13}{6}$. *Ans.* $x = 1$, or -4.
34. $\frac{x+1}{x-1} - \frac{x-2}{x+2} = \frac{9}{5}$ *Ans.* $x = 3$, or $-\frac{2}{3}$.
35. $x^2 + 2ax = a^2$. *Ans.* $x = a(-1 \pm \sqrt{2})$.
36. $3a^2x^{-1} - x = -2a$. *Ans.* $x = 3a$, or $-a$.
37. $x^2 - 2ax = b^2 - a^2$. *Ans.* $x = a + b$, or $a - b$.
38. $x^2 - (a - b + c)x = (b - a)c$. *Ans.* $x = a - b$, or $+c$.

283. The roots of a complete quadratic may be directly written by the following rule:

Rule.—I. *Reduce the quadratic to the form* $x^2 + 2px = q$.

II. *Take, with a contrary sign, one-half of the coefficient of* x, *plus or minus the square root of the sum obtained by adding the square of half the coefficient of* x *to the second member.*

The reason for this rule may be shown as follows: Solving the general quadratic, $x^2 + 2px = q$, we find that $x = -p \pm \sqrt{(q + p^2)}$; comparing the root with the general equation, we see that $-p$ is $\frac{1}{2}$ of $2p$ with the sign changed, and $\sqrt{(q + p^2)}$ is the square root of the second member plus the square of half the coefficient of x.

NOTE.—Pupils should be required to solve some of the previous examples by this rule

SECOND METHOD OF COMPLETING THE SQUARE.

284. A SECOND METHOD of completing the square enables us to avoid fractions in the solutions of all forms of quadratics.

285. The method already given involves fractions when the coefficient of x is an odd number or a fraction.

NOTE.—This method is sometimes called the *Hindoo Method* of solving quadratics.

1. Given $ax^2+bx=c$, to find the value of x.

SOLUTION. To complete the square without using fractions, it is evident that the first term must be a perfect square and the second term must be divisible by 2.

OPERATION.

$$ax^2+bx=c \qquad (1)$$
$$4a^2x^2+4abx=4ac \qquad (2)$$
$$4a^2x^2+4abx+b^2=4ac+b^2 \qquad (3)$$
$$2ax+b=\pm\sqrt{(4ac+b^2)} \qquad (4)$$
$$2ax=-b\pm\sqrt{(4ac+b^2)} \qquad (5)$$
$$x=\frac{-b\pm\sqrt{(4ac+b^2)}}{2a} \qquad (6)$$

To make the first term a perfect square and the second term divisible by 2, we multiply both members by $4a$, which gives eq. (2). If now we consider $4a^2x^2+4abx$ as the first two terms of the square of a binomial, the first term of this binomial will be the square root of $4a^2x^2$, or $2ax$; $4abx$ will be twice the product of the first term by the second; hence, if we divide $4abx$ by two times $2ax$, or $4ax$, the quotient b will be the second term of the binomial, and its square, b^2, added to both members of the equation, will complete the square.

Extracting the square root and reducing, we have the values of x expressed in equation (6).

Rule.—I. *Reduce the equation to the form $ax^2+bx=c$, in which the three terms are integral and prime to each other.*

II. *Multiply the equation by 4 times the coefficient of x^2; add the square of the coefficient of x to both members.*

III. *Extract the square root, and find the value of x in the resulting simple equation.*

NOTES.—1. A quadratic may also be solved by multiplying by the coefficient of x^2, and adding the square of one-half the coefficient of x.

2. Let pupils make a rule from the formula which expresses the root of the general quadratic, $ax^2+bx=c$.

2. Reduce $3x^2+5x=22$.

SOLUTION.

Given the equation,	$3x^2+5x=22$,
multiplying by 4×3 or 12,	$36x^2+60x=264$,
completing the square,	$36x^2+60x+25=264+25$,
extracting the root,	$6x+5=\pm17$,
transposing and reducing,	$6x=-5\pm17$;
whence,	$x=+2$,
and	$x=-3\frac{2}{3}$.

EXAMPLES.

3. Reduce $2x^2-3x=9$. *Ans.* $x=3$, or $-1\frac{1}{2}$.
4. Reduce $3x^2+8x=28$. *Ans.* $x=2$, or $-4\frac{2}{3}$.
5. Reduce $4x^2+7x=11$. *Ans.* $x=1$, or $-2\frac{3}{4}$.
6. Reduce $5x^2+2x=88$. *Ans.* $x=4$, or $-4\frac{2}{5}$.
7. Reduce $5x^2-4x=156$. *Ans.* $x=6$, or $-5\frac{1}{5}$.
8. Reduce $4x^2-45x=36$. *Ans.* $x=12$, or $-\frac{3}{4}$.
9. Reduce $8x^2-7x=165$. *Ans.* $x=5$, or $-4\frac{1}{8}$.
10. Reduce $9x^2-7x=116$. *Ans.* $x=4$, or $-3\frac{2}{9}$.
11. Reduce $2x^2+ax=b$. *Ans.* $x=\frac{1}{4}\{-a\pm\sqrt{(a^2+8b)}\}$.
12. Reduce $\frac{x-3a}{b}=\frac{9(b-a)}{x}$. *Ans.* $x=3b$, or $3(a-b)$.
13. Reduce $x^2+3x=5$. *Ans.* 1.1925+, or −4.1925+.
14. Reduce $x^2+2x=5$. *Ans.* 1.449+, or −3.449+.
15. Reduce $x^2-8x=-8$. *Ans.* 6.828+, or 1.172+.
16. Reduce $5x^2-4x=2$. *Ans.* 1.148, or −0.348.

THIRD METHOD OF COMPLETING THE SQUARE.

286. A THIRD METHOD of completing the square is stated in the following rule:

Rule.—I. *Make the coefficient of the first term of the quadratic a positive square.*

II. *Divide the second term by twice the square root of the first, and add the square of the quotient to both members.*

EQUATIONS IN THE QUADRATIC FORM.

287. Any equation which is in, or may be put in, the quadratic form, may be solved by the following methods.

288. An equation is in the quadratic form when it contains but two powers of an unknown quantity, and the index of one power is twice that of the other as, $x^{2n}+ax^n=b$, or $ax^{2n}+bx^n=c$.

CASE I.

289. When the unknown term of the quadratic is a monomial.

1. Given $x^4-6x^2=-8$, to find x.

SOLUTION.

Given the equation,	$x^4-6x^2=-8,$
completing the square,	$x^4-6x^2+9=1,$
extracting the square root,	$x^2-3=\pm 1,$
transposing and reducing,	$x^2=4$, or 2,
extracting the square root,	$x=\pm 2$, or $\pm\sqrt{2}$.

2. Given $x^6-4x^3=32$, to find x.

SOLUTION.

Given the equation,	$x^6-4x^3=32,$
completing the square,	$x^6-4x^3+4=36,$
extracting the square root,	$x^3-2=\pm 6,$
transposing and reducing,	$x^3=8$, or -4,
extracting the cube root,	$x=2,$
and	$x=\sqrt[3]{(-4)}$.

NOTE.—Since the *odd roots* of a negative quantity are real, by extracting the cube root of 4 and prefixing the minus sign we find the approximate value of the second root of the above equation.

EXAMPLES.

3. Given $x^4+4x^2=32$, to find x.
   Ans. $x=\pm 2$, or $\pm 2\sqrt{(-2)}$.
4. Given $x^4-5x^2=36$, to find x.
   Ans. $x=\pm 3$, or $\pm 2\sqrt{(-1)}$.

5. Given $x^6-7x^3=8$, to find x. *Ans.* $x=2$, or -1.
6. Given $x^2+4x^{-2}=5$, to find x. *Ans.* $x=\pm 2$, or ± 1.
7. Given $x+3\sqrt{x}=18$, to find x. *Ans.* $x=9$, or 36.
8. Given $x^{\frac{2}{3}}+4x^{\frac{1}{3}}=5$, to find x. *Ans.* $x=1$, or -125.
9. Given $x^{2n}-ax^n=b$, to find x. *Ans.* $x=(\frac{1}{2}a\pm\sqrt{b+\frac{1}{4}a^2})^{\frac{1}{n}}$.
10. Given $x^{2n}+4x^n=12$, to find x. *Ans.* $x=\sqrt[n]{2}$, or $\sqrt[n]{-6}$.
11. Given $x^3+7x^{\frac{3}{2}}=3\frac{3}{4}$, to find x. *Ans.* $x=\frac{1}{2}\sqrt[3]{2}$, or $\frac{1}{2}\sqrt[3]{450}$.
12. Given $\sqrt[3]{x}+2\sqrt[3]{x^2}=\frac{3}{8}$, to find x. *Ans.* $x=\frac{1}{64}$, or $-\frac{27}{64}$.
13. Given $x^n-ax^{\frac{n}{2}}=b$, to find x. *Ans.* $x=(\frac{1}{2}a\pm\sqrt{b+\frac{1}{4}a^2})^{\frac{2}{n}}$.
14. Given $\frac{\sqrt{4x}+2}{4+\sqrt{x}}=\frac{4+\sqrt{x}}{\sqrt{x}}$, to find x. *Ans.* $x=64$, or 4.

CASE II.

290. When the unknown term of the quadratic is a polynomial.

291. When a polynomial becomes the basis of the quadratic form, we may consider it as a single quantity, and proceed as in the previous case.

1. Given $(x^2+2x)^2+4(x^2+2x)=96$, to find x.

SOLUTION.

Given the equation,	$(x^2+2x)^2+4(x^2+2x)=96;$
completing the square,	$(x^2+2x)^2+4(x^2+2x)+4=100;$
extracting the sq. root,	$x^2+2x+2=\pm 10;$
transposing,	$x^2+2x=8$, or -12;
completing the square,	$x^2+2x+1=9$, or -11;
extracting the sq. root,	$x+1=\pm 3$, or $\pm\sqrt{(-11)}$;
whence,	$x=2$, or -4,
and	$x=-1\pm\sqrt{(-11)}$.

NOTE.—This problem may also be solved by placing $x^2+2x=y$, giving the equation, $y^2+4y=96$, which may be solved and the value of y thus found placed equal to x^2+2x, from which the value of x can readily be found.

2. Given $(x^2-2)^2+2x^2=67$, to find x.

SOLUTION.

Given the equation,	$(x^2-2)^2+2x^2=67$;
subtracting 4,	$(x^2-2)^2+2x^2-4=63$;
factoring,	$(x^2-2)^2+2(x^2-2)=63$;
completing the square,	$(x^2-2)^2+2(x^2-2)+1=64$;
extracting the sq. root,	$(x^2-2)+1=\pm 8$;
whence,	$x^2=9$, or -7,
and	$x=\pm 3$, or $\pm\sqrt{-7}$.

NOTE.—The 4 was subtracted to put the equation in the quadratic form. Sometimes this can be done by transposing a term or by adding or subtracting some quantity.

EXAMPLES.

3. Given $(x^2-3)^2+4(x^2-3)=5$, to find x. *Ans.* $x=\pm 2$.

4. Given $(x^2+3x)^2-6(x^2+3x)=216$, to find x.
Ans. $x=3$, or -6.

5. Given $(x^2-4x)^{\frac{1}{2}}+(x^2-4x)=3\frac{3}{4}$, to find x.
Ans. $x=4\frac{1}{2}$, or $-\frac{1}{2}$.

6. Given $\sqrt{(5+x)}+\sqrt[4]{(5+x)}=6$, to find x.
Ans. $x=11$, or 76.

7. Given $\left(\frac{4}{x}+x\right)^2+6\left(\frac{4}{x}+x\right)=40$, to find x.
Ans. $x=2$, or $-5\pm\sqrt{(21)}$.

8. Given $x-\sqrt{(x+5)}=1$, to find x. *Ans.* $x=4$, or -1.

9. Given $(x-4)^2-6\sqrt{(x-4)}=\frac{16}{x-4}$, to find x.
Ans. $x=8$, or $4+\sqrt[3]{4}$.

10. Given $(x^2+2x-3)^2+7(x^2+2x-3)=60$, to find x.
Ans. $x=2$, -4, or $-1\pm 2\sqrt{(-2)}$.

11. Given $(x^2-9)^2-11x^2+40=21$, to find x.
Ans. $x=\pm 5$, or ± 2.

12. Given $(x^2-4x+5)^2+4x^2-16x=-8$, to find x.
Ans. $x=3$, or 1, or $2\pm\sqrt{(-7)}$.

NOTE.—Add 59 to both members of Ex. 11; add 20 to both members of Ex. 12. Several of the problems in Art. 290 have other results than those given.

PROBLEMS

PRODUCING AFFECTED QUADRATICS.

1. Find two numbers such that their sum is 16 and their product is 60.

SOLUTION.

Let	$x=$ one number;
then	$16-x=$ the other number.
by the conditions,	$(16-x)x=60$;
which gives	$x^2-16x=-60$;
whence,	$x=10$, or 6,
and	$16-x=6$, or 10.

Hence the two numbers are 10 and 6.

2. A man sold a watch for \$24, and lost as much per cent. as the watch cost him; what did the watch cost him?

SOLUTION.

Let	$x=$ the cost of the watch;
then	$x=$ the loss per cent.,
and	$\frac{x}{100}\times x=\frac{x^2}{100}=$ the loss;
therefore,	$x-\frac{x^2}{100}=24$;
whence,	$x=60$, or 24.

Both of these values will satisfy the conditions of the problem. The pupil will show this by verification.

3. A lady divided \$144 equally among some poor persons: if there had been two more, each would have received \$1 less; required the number of persons.

SOLUTION.

Let $x=$ the number of persons; then $\frac{144}{x}=$ what each received; and $\frac{144}{x+2}=$ what each would have received if there had been two more; then $\frac{144}{x+2}=\frac{144}{x}-1$; whence $x=16$, or -18. The number of persons was therefore 16; the negative result will not satisfy the problem in an arithmetical sense.

NOTE.—If, in the problem, 2 *more* be changed to 2 *less*, and \$1 *less* to \$1 *more*, the correct result will be 18.

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4. A gentleman divided \$50 between his two sons in such a manner that the product of their shares was 600; what was the share of each? *Ans.* \$30; \$20.

5. The wall which encloses a rectangular garden is 128 yards long, and the area of the garden is 1008 square yards; what is its length and breadth? *Ans.* 36 yds.; 28 yds.

6. An officer wishes to arrange 1600 men in a solid body, so that each rank may exceed each file by 60 men; how many must be placed in rank and file? *Ans.* 20; 80.

7. A merchant bought a number of Bibles for \$50, which he sold for \$5.50 a piece, and thus gained as much as one Bible cost; how many Bibles did he buy? *Ans.* 10.

8. The perimeter of a room is 48 feet, and the area of the floor equals 35 times the difference of its length and breadth; what are the dimensions of the room? *Ans.* 14 ft.; 10 ft.

9. Two boys, A and B, bought 10 oranges for 24 cents, each paying 12 cents; if A paid 1 cent more apiece than B, how many oranges did each buy? *Ans.* A, 4; B, 6.

10. A lot of sheep cost \$180, but on 2 of them being stolen, the rest averaged \$1 more a head than at first; find the number of sheep.

11. A man walked 48 miles in a certain time: if he had gone 4 miles more per hour, he would have gone the distance in 6 hours' less time; how many miles did he travel per hour? *Ans.* 4 miles.

12. In an orchard containing 180 trees there are 3 more trees in a row than there are rows; required the number of rows and the number of trees in a row.

Ans. 12 rows, and 15 trees in a row.

13. In a purse containing 52 coins of silver and copper, each silver coin is worth as many cents as there are copper coins, and each copper coin is worth as many cents as there are silver coins, and the whole is worth \$2; how many are there of each? *Ans.* 2 silver; 50 copper.

14. A person distributed \$6 equally among a number of

paupers; and as there were 5 less than he supposed, they each received 10 cents apiece more than they otherwise would; how many paupers were there? *Ans.* 15.

15. The expenses of a party amount to \$10; and if each pays 30 cents more than there are persons, the bill will be settled; how many persons are there? *Ans.* 20.

16. There is a number consisting of two digits whose sum is 10 and the sum of whose squares is 52; it is required to find the number. *Ans.* 46, or 64.

17. Mr. Leslie sold his horse for \$171, and gained as much per cent. as the horse cost him; what was the first cost of the horse? *Ans.* \$90.

18. A person laid out a certain sum of money for goods, which he sold again for \$24, and lost as much per cent. as the goods cost him; what was the first cost? *Ans.* \$40, or \$60.

19. A yacht sails 90 miles down a river whose current moves 3 miles an hour, and is gone 16 hours; required the rate of sailing. *Ans.* 12 miles an hour.

20. A farmer bought a number of sheep for \$80; if he had bought 4 more for the same money, he would have paid \$1 less for each; how many did he buy? *Ans.* 16 sheep.

21. A man bought a quantity of meat for \$2.16. If meat were to rise in price 1 cent per pound, he would get 3 pounds less for the same sum. How much meat did he buy? *Ans.* 27 lbs.

22. The plate of a mirror, 18 inches by 12, is to be set in a frame of uniform width, whose surface is to be equal to the surface of the glass; required the width of the frame. *Ans.* 3 inches.

23. Todhunter gives the following beautiful little problem: Find the price of eggs per dozen when two less for 12 cents raises the price 1 cent per dozen. *Ans.* 8 cts.

24. A and B start at the same time to travel 90 miles; A travels 1 mile an hour faster than B, and arrives 1 hour earlier; at what rate per hour did each travel?

Ans. A, 10 mi.; B, 9 mi.

25. A person bought cloth for \$72, which he sold again at \$$6\frac{1}{2}$ a yard, and gained by the bargain as much as one yard cost him; required the number of yards. *Ans.* 12.

QUADRATIC EQUATIONS

CONTAINING TWO UNKNOWN QUANTITIES.

292. The **Degree** of an equation containing two or more unknown quantities is determined by the greatest sum of the exponents of the unknown quantities contained in any term. Thus,

$2ax+3xy=4a$ is an equation of the 2d degree;

$3xy+4x^2y=12$ is an equation of the 3d degree.

293. A **Homogeneous Equation** is one in which the sum of the exponents of the unknown quantities in each term which contains them is the same. Thus,

$$3x^2+2xy+y^2=31$$

and $$x^3+3x^2y+3xy^2+y^3=27$$

are each homogeneous equations.

294. A **Symmetrical Equation** is one in which the unknown quantities are similarly involved, or one in which they can change places without destroying the equation. Thus,

$$x^2+y^2=13, \text{ and } \frac{x}{y}+\frac{y}{x}=2\tfrac{1}{2}$$

and $$x^2+y^2-xy+3x+3y=22$$

are each symmetrical.

295. Quadratic Equations containing two unknown quantities can generally be solved by the rules for quadratics if they come under one of the following cases:

I. When one of the equations is simple and the other quadratic.

II. When each equation is homogeneous and quadratic.

III. When each equation is symmetrical.

NOTE.—Two quadratics containing two unknown quantities usually produce a biquadratic in elimination; hence all quadratics containing two unknown quantities cannot be solved by the rules for quadratics.

CASE I.

296. When one of the equations is simple and the other quadratic.

297. Equations of this case can generally be solved by substituting in the quadratic equation an expression for the value of one unknown quantity found from the simple equation.

1. Given $\left\{\begin{matrix} 3x+y=9 \\ x^2+y^2=13 \end{matrix}\right\}$, to find x and y.

SOLUTION.

	$3x+y=9,$	(1)
	$x^2+y^2=13.$	(2)
From (1),	$y=9-3x;$	(3)
squaring (3),	$y^2=81-54x+9x^2;$	(4)
substituting in (2),	$x^2+81-54x+9x^2=13;$	(5)
reducing (5),	$x^2-\frac{27}{5}x=-\frac{34}{5};$	
completing the square,	$x^2-\frac{27}{5}x+(\frac{27}{10})^2=\frac{49}{100};$	
extracting sq. root,	$x-\frac{27}{10}=\pm\frac{7}{10};$	
whence,	$x=2, \text{ or } 3\frac{2}{5},$	
and	$y=3, \text{ or } -1\frac{1}{5}.$	

EXAMPLES.

Find the values of x and y in the following equations:

2. Given $\left\{\begin{matrix} xy=15 \\ x+y=8 \end{matrix}\right\}$. *Ans.* $\left\{\begin{matrix} x=5, \text{ or } 3, \\ y=3, \text{ or } 5. \end{matrix}\right.$

3. Given $\left\{\begin{matrix} x-y=3 \\ x^2-y^2=21 \end{matrix}\right\}$. *Ans.* $\left\{\begin{matrix} x=5, \\ y=2. \end{matrix}\right.$

4. Given $\left\{\begin{matrix} x+y=6 \\ x^2+y^2=20 \end{matrix}\right\}$. *Ans.* $\left\{\begin{matrix} x=4, \text{ or } 2, \\ y=2, \text{ or } 4. \end{matrix}\right.$

5. Given $\left\{\begin{matrix} xy=8 \\ 4x-3y=10 \end{matrix}\right\}$. *Ans.* $\left\{\begin{matrix} x=4, \text{ or } -1\frac{1}{2}, \\ y=2, \text{ or } -5\frac{1}{3}. \end{matrix}\right.$

6. Given $\left\{\begin{matrix} 2x+y=11 \\ 3x^2-y^2=2 \end{matrix}\right\}$. *Ans.* $\left\{\begin{matrix} x=3, \text{ or } +41. \\ y=5, \text{ or } -71. \end{matrix}\right.$

7. Given $\left\{\begin{matrix} xy=18 \\ 3y-2x=12 \end{matrix}\right\}$. *Ans.* $\left\{\begin{matrix} x=3, \text{ or } -9, \\ y=6, \text{ or } -2. \end{matrix}\right.$

8. Given $\left\{\begin{array}{l}\frac{1}{x}+\frac{1}{y}=5 \\ \frac{1}{x^2}+\frac{1}{y^2}=13\end{array}\right\}$. *Ans.* $\left\{\begin{array}{l}x=\frac{1}{2}, \text{ or } \frac{1}{3} \\ y=\frac{1}{3}, \text{ or } \frac{1}{2}\end{array}\right.$

9 Given $\left\{\begin{array}{l}xy=35 \\ x^2-y^2=24\end{array}\right\}$. *Ans.* $\left\{\begin{array}{l}x=\pm 7 \text{ or } \pm 5\sqrt{(-1)}, \\ y=\pm 5, \text{ or } \pm 7\sqrt{(-1)}.\end{array}\right.$

10. Given $\left\{\begin{array}{l}x^3-y^3=28(x-y) \\ x+y=6\end{array}\right\}$. *Ans.* $\left\{\begin{array}{l}x=4, \text{ or } 2, \\ y=2, \text{ or } 4\end{array}\right.$

CASE II.

298. When each equation is homogeneous and quadratic.

299. Equations in this case are usually most conveniently solved by substituting for one unknown quantity the product of the other by a third unknown.

1. Given $\left\{\begin{array}{l}x^2+xy=10 \\ xy+2y^2=24\end{array}\right\}$, to find x and y.

SOLUTION.

$x^2+xy=10,$ (1)

$xy+2y^2=24.$ (2)

Let $y=vx,$ (3)

substituting in (1), $x^2+vx^2=10,$ (4)

substituting in (2), $vx^2+2v^2x^2=24,$ (5)

from (4), $x^2=\frac{10}{1+v},$ (6)

from (5), $x^2=\frac{24}{v+2v^2},$ (7)

equating (6) and (7), $\frac{10}{1+v}=\frac{24}{v+2v^2},$ (8)

clearing of fractions, etc., $10v^2-7v=12,$ (9)

solving (9), $v=\frac{3}{2}, \text{ or } -\frac{4}{5}.$ (10)

substituting in (6), $x^2=\frac{10}{1+\frac{3}{2}}=4,$

and $x^2=\frac{10}{1-\frac{4}{5}}=50;$

whence, $x=\pm 2, \text{ or } \pm 5\sqrt{2},$

and $y=\pm 3, \text{ or } \pm 4\sqrt{2}.$

2. Given $\left\{\begin{array}{l}x^2-2xy=5 \\ x^2-y^2=21\end{array}\right\}$. *Ans.* $\left\{\begin{array}{l}x=\pm 5, \\ y=\pm 2.\end{array}\right.$

3. Given $\left\{\begin{array}{l}x^2-y^2=12 \\ x^2-xy+y^2=12\end{array}\right\}$. *Ans.* $\left\{\begin{array}{l}x=\pm 4, \\ y=\pm 2.\end{array}\right.$

4. Given $\left\{\begin{array}{l}x^2y(x+y)=20 \\ x^2y(2x-3y)=20\end{array}\right\}$. *Ans.* $\left\{\begin{array}{l}x=\pm 2\sqrt{2}, \text{ or } \pm 2\sqrt{-2}, \\ y=\pm \frac{1}{2}\sqrt{2}, \text{ or } \pm \frac{1}{2}\sqrt{-2}.\end{array}\right.$

5. Given $\left\{\begin{array}{l}x^2-xy=8 \\ x^2-y^2=12\end{array}\right\}$. *Ans.* $\left\{\begin{array}{l}x=\pm 4, \pm\sqrt{\infty}, \\ y=\pm 2, \pm\sqrt{\infty}.\end{array}\right.$

6. Given $\left\{\begin{array}{l}x^2+xy=10 \\ x^2+y^2=13\end{array}\right\}$. *Ans.* $\left\{\begin{array}{l}x=\pm 2, \text{ or } \pm \frac{5}{2}\sqrt{2}, \\ y=\pm 3, \text{ or } \mp \frac{1}{2}\sqrt{2}.\end{array}\right.$

7. Given $\left\{\begin{array}{l}x^2-y^2=3 \\ x^2-2xy+2y^2=2\end{array}\right\}$. *Ans.* $\left\{\begin{array}{l}x=\pm 2, \text{ or } \pm \frac{4}{5}\sqrt{5}, \\ y=\pm 1, \text{ or } \pm \frac{1}{5}\sqrt{5}.\end{array}\right.$

CASE III.

300. When each equation is symmetrical.

301. There is no general method for the solution of equations in this case. The various expedients employed depend upon the powers of binomials and principles of factoring.

1. Given $\left\{\begin{array}{l}xy=24 \\ x+y=10\end{array}\right\}$, to find x and y.

SOLUTION.

$xy=24,$ (1)

$x+y=10.$ (2)

Squaring (2), $x^2+2xy+y^2=100,$ (3)

multiplying (1) by (4), $4xy=96,$ (4)

subtracting (4) from (3), $x^2-2xy+y^2=4,$ (5)

extracting square root, $x-y=\pm 2,$ (6)

uniting (6) and (2), $x=6, \text{ or } 4,$

and $y=4, \text{ or } 6.$

2. Given $\left\{\begin{array}{l} x+y=5 \\ x^2y^2-4xy=12 \end{array}\right\}$, to find x and y.

SOLUTION.

	$x+y=5,$	(1)
	$x^2y^2-4xy=12.$	(2)
Completing the square,	$x^2y^2-4xy+4=16,$	(3)
extracting square root,	$xy-2=\pm 4;$	(4)
whence,	$xy=6, \text{ or } -2,$	(5)
squaring (1),	$x^2+2xy+y^2=25,$	
subtracting 4 times (5),	$x^2-2xy+y^2=1, \text{ or } 33,$	
extracting square root,	$x-y=\pm 1, \text{ or } \pm\sqrt{33};$	
whence,	$x=3, \text{ or } 2, \text{ or } \frac{1}{2}(5\pm\sqrt{33}),$	
and	$y=2, \text{ or } 3, \text{ or } \frac{1}{2}(5\pm\sqrt{33}).$	

3. Given $\left\{\begin{array}{l} x+y=7 \\ x^3+y^3=91 \end{array}\right\}$, to find x and y.

SOLUTION.

	$x+y=7,$	(1)
	$x^3+y^3=91.$	(2)
Dividing (2) by (1),	$x^2-xy+y^2=13,$	(3)
squaring (1),	$x^2+2xy+y^2=49,$	(4)
subtracting (3) from (4),	$3xy=36,$	(5)
dividing by 3,	$xy=12,$	(6)
subtracting (6) from (3),	$x^2-2xy+y^2=1,$	
extracting square root,	$x-y=\pm 1;$	
whence,	$x=4, \text{ or } 3,$	
and	$y=3, \text{ or } 4.$	

NOTES.—1. Let the pupils see that the values of x and y in these equations are not equal to each other; for when $x=4$, $y=3$; and when $x=3$, $y=4$. Their values are interchangeable.

2. The signs $\pm$ and $\mp$ are equivalent when used independently; but when taken in connection they are the reverse of each other. Thus, if $x=\pm a$ and $y=\mp b$, then when $x=+a$, $y=-b$; and when $x=-a$, $y=+b$.

EXAMPLES.

4. Given $\left\{\begin{array}{l} xy=20 \\ x-y=1 \end{array}\right\}$. *Ans.* $\left\{\begin{array}{l} x=5, \text{ or } -4, \\ y=4, \text{ or } -5. \end{array}\right.$

5. Given $\left\{\begin{array}{l} \sqrt{(xy)}=2 \\ \sqrt{x}+\sqrt{y}=3 \end{array}\right\}$. *Ans.* $\left\{\begin{array}{l} x=4, \text{ or } 1, \\ y=1, \text{ or } 4. \end{array}\right.$

6. Given $\left\{\begin{array}{l} x-y=3 \\ \frac{x^2}{y^2}+\frac{4x}{y}=32 \end{array}\right\}$. *Ans.* $\left\{\begin{array}{l} x=4, \text{ or } 2\frac{2}{3}, \\ y=1, \text{ or } -\frac{1}{3}. \end{array}\right.$

7. Given $\left\{\begin{array}{l} xy=15 \\ (x+y)^2-6(x+y)=16 \end{array}\right\}$.

Ans. $\left\{\begin{array}{l} x=5, 3, \text{ or } -1\pm\sqrt{(-14)}, \\ y=3, 5, \text{ or } -1\mp\sqrt{(-14)}. \end{array}\right.$

8. Given $\left\{\begin{array}{l} x+\sqrt{(xy)}+y=9 \\ x^2+xy+y^2=27 \end{array}\right\}$. *Ans.* $\left\{\begin{array}{l} x=3, \\ y=3. \end{array}\right.$

9. Given $\left\{\begin{array}{l} x-y=2 \\ x^3-y^3=152 \end{array}\right\}$. *Ans.* $\left\{\begin{array}{l} x=6, \text{ or } -4, \\ y=4, \text{ or } -6. \end{array}\right.$

10. Given $\left\{\begin{array}{l} x^3-y^3=19 \\ x^2y-xy^2=6 \end{array}\right\}$. *Ans.* $\left\{\begin{array}{l} x=3, \text{ or } -2, \\ y=2, \text{ or } -3. \end{array}\right.$

11. Given $\left\{\begin{array}{l} \frac{x^2}{y}+\frac{y^2}{x}=9 \\ x+y=6 \end{array}\right\}$. *Ans.* $\left\{\begin{array}{l} x=4, \text{ or } 2, \\ y=2, \text{ or } 4. \end{array}\right.$

302. Equations which are not symmetrical may sometimes be so combined as to produce a symmetrical equation.

303. Equations that are not symmetrical with respect to the unknown quantities themselves may be symmetrical with respect to some multiple or power of these quantities.

1. Given $\left\{\begin{array}{l} x^2+xy=45 \\ y^2+xy=36 \end{array}\right\}$, to find x and y.

SOLUTION.

	$x^2+xy=45;$	(1)
	$y^2+xy=36.$	(2)
Adding,	$x^2+2xy+y^2=81;$	(3)
evolving,	$x+y=\pm 9;$	(4)
subtracting (2) from (1),	$x^2-y^2=9;$	(5)
dividing (5) by (4),	$x-y=\pm 1;$	(6)
adding (4) and (6),	$2x=\pm 10$	(7)
whence	$x=\pm 5;$	(8)
subtracting (6) from (4),	$2y=\pm 8;$	(9)
whence	$y=\pm 4.$	(10)

2. Given $\left\{\begin{array}{l} x+2y=13 \\ x^2+4y^2=109 \end{array}\right\}$, to find x and y.

SOLUTION.

	$x+2y=13$	(1)
	$x^2+4y^2=109$	(2)
Squaring (1),	$x^2+4xy+4y^2=169$;	(3)
subtracting (2) from (3),	$4xy=60$;	(4)
subtracting (4) from (2),	$x^2-4xy+4y^2=49$;	(5)
evolving,	$x-2y=\pm 7$;	(6)
adding (1) and (6),	$2x=20$, or 6;	(7)
whence	$x=10$, or 3;	(8)
subtracting (6) from (1),	$4y=6$, or 20;	(9)
whence	$y=\frac{3}{2}$, or 5.	(10)

NOTE.—In Example 1, the sum of the two equations gives a symmetrical equation. Example 2 is symmetrical with respect to x and $2y$.

EXAMPLES.

3. Given $\left\{\begin{array}{l} x^2+y^2+2x=19 \\ xy+y=8 \end{array}\right\}$. *Ans.* $\left\{\begin{array}{l} x=1, \text{ or } 3, \\ y=4, \text{ or } 2. \end{array}\right.$

4. Given $\left\{\begin{array}{l} x-3y=3 \\ x^2+9y^2=45 \end{array}\right\}$. *Ans.* $\left\{\begin{array}{l} x=6, \text{ or } -3, \\ y=1, \text{ or } -2. \end{array}\right.$

5. Given $\left\{\begin{array}{l} xy=ab \\ \frac{x}{a}+\frac{y}{b}=2 \end{array}\right\}$. *Ans.* $\left\{\begin{array}{l} x=a, \\ y=b. \end{array}\right.$

6. Given $\left\{\begin{array}{l} x^2+y^2=20 \\ xy-x-y=2 \end{array}\right\}$. *Ans.* $\left\{\begin{array}{l} x=4, \text{ or } 2, \\ y=2, \text{ or } 4. \end{array}\right.$

7. Given $\left\{\begin{array}{l} \frac{x}{a}+\frac{y}{b}=1 \\ \frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \end{array}\right\}$. *Ans.* $\left\{\begin{array}{l} x=a, \text{ or } 0, \\ y=0, \text{ or } b. \end{array}\right.$

8. Given $\left\{\begin{array}{l} x^2+y^2=106 \\ x-y+\sqrt{(x-y)}=6 \end{array}\right\}$. *Ans.* $\left\{\begin{array}{l} x=9, \text{ or } -5, \\ y=5, \text{ or } -9. \end{array}\right.$

9. Given $\left\{\begin{array}{l} x+y=a^3-b^3 \\ x^{\frac{1}{3}}+y^{\frac{1}{3}}=a-b \end{array}\right\}$. *Ans.* $\left\{\begin{array}{l} x=a^3, \text{ or } -b^3, \\ y=-b^3, \text{ or } a^3. \end{array}\right.$

10. Given $\left\{\begin{array}{l} xy^2-x^2y=-6 \\ x^3-y^3=19 \end{array}\right\}$. *Ans.* $\left\{\begin{array}{l} x=3, \text{ or } -2, \\ y=2, \text{ or } -3. \end{array}\right.$

11. Given $\left\{\begin{array}{l} x^2y+xy^2=20 \\ x^3+y^3=65 \end{array}\right\}$. *Ans.* $\left\{\begin{array}{l} x=4, \text{ or } 1, \\ y=1, \text{ or } 4. \end{array}\right.$

12. Given $\left\{\begin{array}{l} 2xy=2a^{\frac{1}{2}}b^{\frac{1}{2}} \\ x^4+y^4=a^2+b^2 \end{array}\right\}$. *Ans.* $\left\{\begin{array}{l} x=\sqrt{a}, \text{ or } \sqrt{b}, \\ y=\sqrt{b}, \text{ or } \sqrt{a}. \end{array}\right.$

13. Given $\left\{\begin{array}{l} \sqrt[3]{x}-\sqrt[3]{y}=1 \\ x-y=7 \end{array}\right\}$. *Ans.* $\left\{\begin{array}{l} x=8, \text{ or } -1, \\ y=1, \text{ or } -8. \end{array}\right.$

14. Given $\left\{\begin{array}{l} xy=36 \\ x-y=\sqrt{x}+\sqrt{y} \end{array}\right\}$. *Ans.* $\left\{\begin{array}{l} x=9, \text{ or } 4, \\ y=4, \text{ or } 9. \end{array}\right.$

15. Given $\left\{\begin{array}{l} xy=6 \\ x^2-3x+3y=10-y^2 \end{array}\right\}$. *Ans.* $\left\{\begin{array}{l} x=3, \text{ or } -2, \\ y=2, \text{ or } -3. \end{array}\right.$

16. Given $\left\{\begin{array}{l} x^3+y^3=\frac{7}{16}(x+y)^3 \\ xy=3 \end{array}\right\}$. *Ans.* $\left\{\begin{array}{l} x=3, \text{ or } 1, \\ y=1, \text{ or } 3. \end{array}\right.$

17. Given $\left\{\begin{array}{l} x^{-2}+y^{-2}=\frac{13}{36} \\ x^{-1}+y^{-1}=\frac{5}{6} \end{array}\right\}$. *Ans.* $\left\{\begin{array}{l} x=2, \text{ or } 3, \\ y=3, \text{ or } 2. \end{array}\right.$

18. Given $\left\{\begin{array}{l} x^2+y^2=25 \\ x^4+y^4=337 \end{array}\right\}$. *Ans.* $\left\{\begin{array}{l} x=\pm 3, \text{ or } \pm 4, \\ y=\pm 4, \text{ or } \pm 3. \end{array}\right.$

19. Given $\left\{\begin{array}{l} xy=6 \\ x^4+y^4=97 \end{array}\right\}$. *Ans.* $\left\{\begin{array}{l} x=\pm 3, \text{ or } \pm 2, \\ y=\pm 2, \text{ or } \pm 3. \end{array}\right.$

20. Given $\left\{\begin{array}{l} x+y=5 \\ x^4+y^4=257 \end{array}\right\}$. *Ans.* $\left\{\begin{array}{l} x=4, \text{ or } 1, \\ y=1, \text{ or } 4. \end{array}\right.$

PROBLEMS

PRODUCING QUADRATICS WITH TWO UNKNOWN QUANTITIES.

1. The sum of two numbers is 7, and the sum of their squares is 25; required the numbers. *Ans.* 4 and 3.

2. The difference of two numbers is 2, and the difference of their squares is 20; required the numbers. *Ans.* 6 and 4.

3. Divide 97 into two such parts that the sum of the square roots of those parts may equal 13. *Ans.* 81 and 16.

4. The difference of two numbers is a, and the difference of their square roots is $\frac{1}{2}\sqrt{2a}$; required the numbers. *Ans.* $\frac{9}{8}a$ and $\frac{a}{8}$.

5. Find two numbers whose product is 3 times their sum, and the sum of their squares is 160. *Ans.* 4 and 12.

6. Divide the number 10 into two such parts that the sum of the cubes of the parts may be 280. *Ans.* 6 and 4.

7. The difference of two numbers is 3, and the difference of their cubes is 117; required the numbers. *Ans.* 5 and 2.

8. Find two numbers whose product is 6 times their difference, and the sum of their squares is 13. *Ans.* 3 and 2.

9. The sum of two numbers is a, and the sum of their cubes is $4a^3$; required the numbers. *Ans.* $\frac{a}{2}(1 \pm \sqrt{5})$; $\frac{a}{2}(1 \mp \sqrt{5})$.

10. Two men, A and B, can together do a piece of work in 12 days; in how many days can each do it if it takes B 10 days longer than A? *Ans.* A, 20 days; B, 30 days.

11. A colonel forms his regiment of 1025 men into two squares, one of which has 5 men more in a side than the other; required the number of men in a side of each. *Ans.* 20; 25.

12. A farmer sold 7 calves and 12 sheep for $50; and the price received for each was such that 3 more calves were sold for $10 than sheep for $6; what was the price of each?
Ans. Calves, $2; sheep, $3.

13. Find two numbers such that their difference added to the difference of their squares shall equal 6, and their sum added to the sum of their squares shall equal 18. *Ans.* 3 and 2.

14. The expense of a sociable was $70, but before the bill was paid, 4 of the young men sneaked off, in consequence of which each of the others had to pay $2 more than his proper share; how many young men were there? *Ans.* 14.

15. A merchant sold some cloth for $24, and some silk at $1 less a yard for the same sum; required the number of yards of each, provided there were 2 yards of silk more than of cloth.
Ans. Cloth, 6; silk, 8.

16. A and B run a race; B, who runs slower than A by a mile in 2 hours, starts first by 2 minutes, and they get to the 4-mile stone together · required their rates of running.
Ans. A, 8 mi.; B, $7\frac{1}{2}$ mi. an hour.

17. A certain rectangle contains 300 square feet; a second rectangle is 8 feet shorter and 10 feet broader, and also contains 300 square feet; find the length and breadth of the first rectangle. *Ans.* Length, 20 ft.; breadth, 15 ft.

18. A bought two pieces of cloth of different sorts; the finer cost 1 dollar more a yard than the coarser, and there were 10 yards more of the coarser than the finer; find how many yards there were in each piece, provided the coarser cost $80 and the finer $90. *Ans.* 30 yds.; 40 yds.

19. The area of a rectangular field is 2275 square rods; and if the length of each side is diminished by 5 rods, the area will be 1800 rods; required the dimensions of the field.
Ans. 65 rods; 35 rods.

20. There is a certain number, of two digits; the sum of the squares of the digits is equal to the number increased by the product of the digits, and if 36 be added to the number, the digits will be reversed; what is the number?
Ans. 48.

21. A person bought two cubical stacks of hay for £41, each of which cost as many shillings per cubic yard as there were yards in the side of the other; and the greater stood on more ground than the less by 9 square yards; what was the price of each? *Ans.* £25 and £16.

22. A laborer dug two trenches for £17 16s., one of which was 6 yards longer than the other, and the digging of each trench cost as many shillings a yard as it was yards in length; what was the length of each? *Ans.* 10 yds.; 16 yds.

23. Required two numbers such that their sum, their product and the difference of their squares shall be equal to one another.
Ans. $\frac{3}{2} \pm \frac{1}{2}\sqrt{5}$; $\frac{1}{2} \pm \frac{1}{2}\sqrt{5}$.

24. Two partners, A and B, gained $18 by trade: A's money was in trade 12 months, and he received for his principal and gain $26; B's money, which was $30, was in trade 16 months. How much did A put in trade? *Ans.* $20.

25. The fore wheels of a carriage make 5 revolutions more than the hind wheels in going 60 yards; but if the circumference of each should be increased one yard, the fore wheels will make only 3 more revolutions than the hind wheels in the same distance; required the circumference of each.
Ans. 3 and 4 yards.

26. An English landholder received £7 4s. for a certain quantity of wheat, and an equal sum, at a price less by 1s. 6d.

per bushel, for a quantity of barley which exceeded the quantity of wheat by 16 bushels; how many bushels were there of each? *Ans.* 32 bu. wheat; 48 bu. barley.

27. A and B run a race around a two-mile course. In the first heat B reaches the winning-post 2 minutes before A; in the second heat A increases his speed 2 miles per hour, and B diminishes his as much, and A then arrives at the winning-post 2 minutes before B. Find at what rate each man ran in the first heat. *Ans.* 10 mi. per hour; 12 mi. per hour.

PRINCIPLES OF QUADRATIC EQUATIONS.

304. The PRINCIPLES of Quadratics are the relations which exist between a quadratic and its roots.

NOTE.—This subject may be omitted by young pupils, and even by older pupils until review, if the teacher prefers.

PRINCIPLE I.

Every quadratic equation has two roots, and only two.

FIRST. The general form of the complete quadratic is $x^2+2px=q$. Completing the square of the general quadratic, and finding the value of x, we have two values, $-p+\sqrt{q+p^2}$ and $-p-\sqrt{q+p^2}$, which proves the principle.

OPERATION.

$$x^2+2px=q \quad (1)$$
$$x^2+2px+p^2=q+p^2 \quad (2)$$
$$x=-p+\sqrt{q+p^2} \quad (3)$$
$$x=-p-\sqrt{q+p^2} \quad (4)$$

SECOND. This proposition can also be demonstrated in another way, as follows:

Assume that $m^2=q+p^2$, or $m=\sqrt{q+p^2}$,
then we have $x^2+2px+p^2=m^2$;
or, in another form, $(x+p)^2=m^2$;
transposing, $(x+p)^2-m^2=0$;
factoring, $(x+p+m)(x+p-m)=0$;
making the second factor equal to zero, $x+p-m=0$;
from which we have $x=-p+m=-p+\sqrt{q+p^2}$;
making the first factor equal to zero, $x+p+m=0$;
from which we have $x=-p-m=-p-\sqrt{q+p^2}$;

and since equation (3) can be satisfied in these two ways, and in these two ways only, therefore x can have but two values, and the principle is true.

PRINCIPLE II.

The sum of the two roots of a quadratic of the form $x^2+2px=q$ is equal to the coefficient of the first power of x, with its sign changed.

For, solving the quadratic $x^2+2px=q$, we find the two roots are $-p+\sqrt{q+p^2}$ and $-p-\sqrt{q+p^2}$; taking the sum of the two roots, we have $-2p$, which is the coefficient of the first power of x, with its sign changed. Therefore, etc.

OPERATION.

$$x^2+2px=q$$
$$x=-p+\sqrt{q+p^2}$$
$$x=-p-\sqrt{q+p^2}$$
$$\overline{\qquad -2p \qquad}$$

PRINCIPLE III.

The product of the two roots of a quadratic of the form $x^2+2px=q$ is equal to the known term with its sign changed.

For, multiplying the two roots together, we have the result $p^2-(q+p^2)$, which, reduced, equals $-q$, which is the known term with its sign changed. Therefore, etc.

OPERATION.

$$x=-p+\sqrt{q+p^2}$$
$$x=-p-\sqrt{q+p^2}$$
$$p^2-p\sqrt{q+p^2}$$
$$+p\sqrt{q+p^2}-(q+p^2)$$
$$p^2-(q+p^2)=-q$$

305. Principles II. and III. enable us to construct quadratic equations from their roots.

1. Find the quadratics whose roots are 4 and 3.

SOLUTION. Since the sum of the two roots with its sign changed equals the *coefficient* of x, and their product, with its sign changed, equals the known term, the equation whose roots are 4 and 3 must be $x^2-(4+3)x=-(4\times3)$, which, reduced, becomes $x^2-7x=-12$.

OPERATION.

$$x^2-(4+3)x=-4\times3$$
$$x^2-7x=-12$$

EXAMPLES.

Find the quadratic whose—

2. Roots are 5 and 4. *Ans.* $x^2-9x=-20$
3. Roots are 7 and -3. *Ans.* $x^2-4x=21$.
4. Roots are 4 and -9. *Ans.* $x^2+5x=36$.

5. Roots are -2 and -8. *Ans.* $x^2+10x=-16$.

6. Roots are $2+\sqrt{3}$ and $2-\sqrt{3}$. *Ans.* $x^2-4x=-1$.

7. Roots are a and b. *Ans.* $x^2-(a+b)x=-ab$.

8. Roots are $a+b$ and $a-b$. *Ans.* $x^2-2ax=b^2-a^2$.

9. Roots are $a+b\sqrt{c}$ and $a-b\sqrt{c}$. *Ans.* $x^2-2ax=b^2c-a^2$.

PRINCIPLE IV.

A quadratic equation of the form $x^2+2px=q$ may be resolved into two binomial factors, of which the first term in each is x, *and the second term the roots with their signs changed.*

For, suppose the two roots are r and r'; then (Prin. II. and III.) we have $x^2-(r+r')x=-rr'$; transposing rr' to the first member, we have $x^2-(r+r')x+rr'=0$; factoring, we have $(x-r)(x-r')=0$; which proves the principle.

OPERATION.

Let $x=r$ and r'

$$x^2-(r+r')x=-rr'$$
$$x^2-(r+r')x+rr'=0$$
$$(x-r)(x-r')=0$$

306. Quadratic equations may also be constructed from their roots by Principle IV.

1. Find an equation whose roots are 3 and 5.

SOLUTION. Since the roots are 3 and 5, we have (Prin. IV.) $(x-3)(x-5)=0$. Expanding, we have $x^2-8x+15=0$; transposing, we have $x^2-8x=-15$.

OPERATION.

$$(x-3)(x-5)=0$$
$$x^2-8x+15=0$$
$$x^2-8x=-15$$

EXAMPLES.

Find the equation whose—

2. Roots are 6 and -4. *Ans.* $x^2-2x=24$.

3. Roots are $+3$ and -8. *Ans.* $x^2+5x=24$.

4. Roots are $2a$ and $3a$. *Ans.* $x^2-5ax=-6a^2$.

5. Roots are $+a$ and $-a$. *Ans.* $x^2=a^2$.

6. Roots are $\frac{m}{2}$ and $-\frac{n}{2}$. *Ans.* $x^2+\frac{1}{2}(n-m)x=\frac{1}{4}mn$.

7. Roots are $a+2\sqrt{n}$ and $a-2\sqrt{n}$. *Ans.* $x^2-2ax=4n-a^2$

FORMS OF QUADRATICS.

307. There are four distinct forms of the complete quadratic, depending upon the sign of $2p$ and q. Thus,

1st form,	$x^2+2px=q$.	3d form,	$x^2+2px=-q$.
2d form,	$x^2-2px=q$.	4th form,	$x^2-2px=-q$.

PRINCIPLE V.

In a quadratic of the first form one root is positive and the other negative, the negative root being the greater.

For, since q is the product of the two roots with its sign changed, and is *positive*, one root must be positive and the other negative; and since $2p$ is the sum of the roots with its sign changed, and is *positive*, the *negative* root must be the greater.

PRINCIPLE VI.

In a quadratic of the second form one root is positive and the other negative, the positive root being the greater.

For, since q is the product of the two roots with its sign changed, and is *positive*, one root must be positive and the other negative; and since $2p$ is their sum with its sign changed, and is *negative*, the *positive* root must be the *greater*.

PRINCIPLE VII.

In a quadratic of the third form both roots are negative.

For, since q is the product of the two roots with its sign changed, and is *negative*, the two roots must be *both negative* or *both positive*; and since $2p$ is the sum of the roots with its sign changed, and is *positive*, *both* roots must be *negative*.

PRINCIPLE VIII.

In a quadratic of the fourth form both roots are positive.

For, since q is the product of the two roots with its sign changed, and is *negative*, the roots must be *both positive* or *both negative*; and since $2p$ is the sum of the roots with its sign changed, and is *negative*, *both* roots must be *positive*.

EXAMPLES.

Required the form where the—

1. Roots are -8 and 5. *Ans.* 1st form.
2. Roots are 9 and -6. *Ans.* 2d form.
3. Roots are 6 and 7. *Ans.* 4th form.
4. Roots are -5 and -4. *Ans.* 3d form.
5. Roots are $-3a$ and $2a$. *Ans.* 1st form.

VALUES OF p AND q.

308. The quantities p and q are general, and may therefore have any values whatever. We will now discuss the equation by assigning different values to each.

FIRST.—Suppose $q=0$.

Solving the equation $x^2+2px=q$, and substituting 0 for q in the root, we have $x=-p\pm p$, whence $x=0$, or $-2p$.

1ST OPERATION.

$$x^2+2px=q$$
$$x=-p\pm\sqrt{q+p^2}$$
Let $q=0$
$$x=-p\pm p$$
$$x=0, \text{ or } -2p$$

2D OPERATION.

$$x^2+2px=q$$
Let $q=0$
$$x^2+2px=0$$
$$x(x+2p)=0$$
$$x=0$$
$$x=-2p$$

Making the same substitution in the equation, we have $x^2+2px=0$; factoring, we have $x(x+2p)=0$; dividing by $x+2p$, we have $x=0$; dividing by x, we have $x+2p=0$, or $x=-2p$.

The third form gives the same result; the second and fourth forms give $x=0$ and $x=+2p$.

SECOND.—Suppose $2p=0$.

If $2p=0$, by substituting 0 for $2p$ in the root of the quadratic we have $x=\pm\sqrt{q}$.

1ST OPERATION.

$$x^2+2px=q$$
$$x=-p\pm\sqrt{q+p^2}$$
Let $2p=0$
$$x=\pm\sqrt{q}$$

2D OPERATION.

$$x^2+2px=q$$
Let $2p=0$
$$x^2=q$$
$$x=\pm\sqrt{q}$$

Making the same substitution in the equation, it reduces to a pure

quadratic, $x^2=q$. Solving this, we have the same result as before, $x=\pm\sqrt{q}$.

In the first and second forms this result will be *real*; in the third and fourth forms, in which q is negative, the result will be imaginary.

THIRD.—Suppose $p^2=q$ when q is negative.

Take the quadratic of the third form, $x^2+2px=-q$. Substituting p^2 for q in the root, we have $x=-p+0$, or $-p$; and $x=-p-0$, or $-p$; hence x has two values, both of which are $-p$.

1ST OPERATION.

$$x^2+2px=-q$$
$$x=-p\pm\sqrt{p^2-q}$$
Let $p^2=q$
$$x=-p+0=-p$$
$$x=-p-0=-p$$

2D OPERATION.

$$x^2+2px=-q$$
Let $p^2=q$
$$x^2+2px+p^2=0$$
$$(x+p)^2=0$$
$$(x+p)(x+p)=0$$
$$x=-p$$
$$x=-p$$

Making the same substitution in the equation and reducing, we have $(x+p)(x+p)=0$; dividing by the first factor, we have $x+p=0$, or $x=-p$; dividing by the second factor, we have $x+p=0$, or $x=-p$.

In the first and second forms the results will be different.

FOURTH.—Suppose q to be greater than p^2 when q is negative.

If in the third or fourth form we assume q numerically greater than p^2, the quantity under the radical becomes a negative quantity, and the value of x is therefore imaginary.

OPERATION.

$$x^2+2px=-q$$
$$x=-p\pm\sqrt{p^2-q}$$

Hence, the root of an equation in the third and fourth forms is imaginary when q is numerically greater than p^2.

EXAMPLES.

1. Find a number such that its square increased by four times the number equals zero. *Ans.* -4.
2. Required the number such that its square, plus 6 times that number, shall equal minus 9. *Ans.* -3.
3. Divide 8 into two such parts that their product shall be equal to 20. *Ans.* $4\pm2\sqrt{-1}$.

Why does the root in the last equation become imaginary? Which supposition does Ex. 2 illustrate? What does Ex. 1 illustrate?

IMAGINARY ROOT.

309. An **Imaginary Root** of a quadratic is a root which contains an imaginary quantity.

310. The **Imaginary Root** occurs in the third and fourth forms of a quadratic upon a certain supposition.

311. We shall now discuss the imaginary root under three distinct heads:

FIRST.—When does a quadratic give an imaginary root?

PRIN. 1. *A quadratic gives an imaginary root when the known term is negative and numerically greater than the square of half the coefficient of the first power of* x.

For, if q is negative and numerically greater than p^2, the quantity under the radical is *negative*, and we shall have the square root of a negative quantity, which is imaginary. Therefore, etc.

OPERATION.

$$x^2+2px=-q$$
$$x=-p\pm\sqrt{(p^2-q)}$$

SECOND.—What is assumed by a quadratic which gives an imaginary root?

PRIN. 2. *A quadratic which gives an imaginary root assumes that the product of two quantities is greater than the square of half their sum.*

For, since $2p$ is the *sum* of the two roots with its sign changed, p^2 is the *square of half the sum* of two quantities, and q is the *product* of the two roots, with its sign changed; hence, when q is negative and greater than p^2, the quadratic assumes that the product of two quantities is greater than the square of half their sum.

OPERATION.

$$x^2+2px=-q$$
$$x=-p\pm\sqrt{(p^2-q)}$$

THIRD.—Prove that this assumption is false.

PRIN. 3.—*The product of two quantities can never be greater than the square of half their sum; hence, the above assumption is false.*

Let $2p$ represent any number, and let it be divided into two parts, $p+z$ and $p-z$; the product of the two parts is p^2-z^2; the sum of the parts is $2p$, and the square of half their sum is p^2. Now, p^2 is greater than p^2-z^2; hence the product of two numbers can never be greater than the square of half their sum

OPERATION.

$$2p=(p+z)+(p-z);$$

Product. $(p+z)(p-z)=p^2-z^2$;

Sum, $(p+z)+(p-z)=2p$;

$\left(\frac{\text{Sum}}{2}\right)^2$, $\left(\frac{2p}{2}\right)^2=p^2$.

Now, $p^2>p^2-z^2$;

hence, $\left(\frac{\text{Sum}}{2}\right)^2>$ Product.

From the above discussion we see that a quadratic of the form $x^2\pm2px=-q$, in which q is greater than p^2, assumes that the product of two quantities is greater than the square of half their sum, which is absurd. When a problem furnishes such an equation, the problem is impossible.

EXAMPLES.

1. Divide the number 12 into two parts such that their product shall be 40. *Ans.* $6\pm2\sqrt{-1}$.

2. A farmer thought to enclose 40 square rods in rectangular form by a fence whose entire length shall be 20 rods; required its length and breadth. *Ans.* $5\pm\sqrt{-15}$; $5\mp\sqrt{-15}$.

Why do these problems give an imaginary result? What is incorrect in the first? What is incorrect in the second?

REVIEW QUESTIONS.

Define a Quadratic Equation. State the two classes of Quadratics. Define a Pure Quadratic. An Affected Quadratic. Give examples of each. How do we solve a pure quadratic? How solve an affected quadratic? State each method of completing the square. Explain each method.

Define a Quadratic of two Unknown Quantities. A Homogeneous Equation. A Symmetrical Equation. What cases of quadratics of two unknown quantities can be solved? Give examples. Show the method of solution.

Define principles of Quadratics. State the principles of Pure Quadratics. State the first four principles of Incomplete Quadratics. State the four forms of a quadratic. State the principles of the forms. Define an Imaginary Root. State the principles of an Imaginary Root.

SECTION VIII.

RATIO AND PROPORTION.

RATIO.

312. Ratio is the measure of the relation of two similar quantities. Thus, the ratio of 8 to 4 is 2.

313. The **Symbol** of ratio is the colon, :, read *to*, or *is to* Thus, $a : c$ indicates the ratio of a to c.

314. The **Terms** of a ratio are the two quantities compared. The first term is the *Antecedent;* the second term is the *Consequent.* The two terms together are called a *Couplet.*

315. A ratio is expressed by writing the two quantities with the symbol between them (Art. 27), or by writing the consequent under the antecedent in the form of a fraction.

Thus, the ratio of a to c is $a : c$, or $\frac{a}{c}$.

316. A **Simple Ratio** is the ratio of two quantities. A *Compound Ratio* is the product of two or more simple ratios. Thus, $(a : b)(c : d)$, or $\frac{a}{b} \times \frac{c}{d}$.

317. A **Compound Ratio** is usually expressed by writing the simple ratios one under another.

Thus, $\left\{\begin{matrix} a : b \\ c : d \end{matrix}\right\}$ expresses the ratio compounded of $a : b$ and $c : d$.

318. A **Duplicate Ratio** of two quantities is the ratio of their squares; as, $a^2 : c^2$. A **Triplicate Ratio** of two quantities is the ratio of their cubes; as, $a^3 : c^3$.

319. A **Ratio of Equality** exists when the two terms are equal. When the antecedent is the greater, it is called a ratio of *greater inequality* when less, a ratio of *less inequality.*

NOTES.—1. The symbol of ratio, :, is supposed to be a modification of the symbol of division.

2. Ratio is usually defined as the relation of two numbers. This is indefinite, however, for the ratio is the *measure* of the relation.

3. A few authors divide the second term by the first, calling it the *French Method.* This is wrong in method and name, as nearly all the French mathematicians, like the German, English, etc., divide the first term by the second.

PRINCIPLES.

1. *The ratio equals the quotient of the antecedent divided by the consequent.*

OPERATION.

Thus, if r represents the ratio of a to c, we have $r = a : c$, or r equals a divided by c (Art. 315). Therefore, etc.

$$r = a : c$$
$$r = \frac{a}{c}$$

2. *The antecedent is equal to the product of the consequent and ratio.*

OPERATION.

For, if $r = a$ divided by c, clearing of fractions, we have $a = r . c$. Therefore, etc.

$$r = \frac{a}{c}$$
$$r . c = a$$

3. *The consequent is equal to the quotient of the antecedent divided by the ratio.*

OPERATION.

For, if $r = a$ divided by c, clearing of fractions, we have $r . c = a$; and dividing by r, we have $c = a$ divided by r. Therefore, etc.

$$r = \frac{a}{c}$$
$$r . c = a$$
$$c = \frac{a}{r}$$

4. *Multiplying the antecedent or dividing the consequent multiplies the ratio.*

For, a ratio is expressed by a fraction whose numerator is the antecedent and denominator the consequent; and multiplying the numerator or dividing the denominator multiplies the fraction (Art. 129, Prin. 1). Therefore, etc.

5. *Dividing the antecedent, or multiplying the consequent, divides the ratio.*

For, a ratio is expressed by a fraction whose numerator is the antecedent and denominator the consequent; and dividing the numerator or multiplying the denominator divides the fraction (Art. 129, Prin. 2). Therefore, etc.

6. *Multiplying or dividing both terms of a ratio by any number does not change the ratio.*

For, a ratio is expressed by a fraction whose numerator is the antecedent and denominator the consequent; and multiplying or dividing both terms of a fraction does not change its value (Art. 129, Prin. 3). Therefore, etc.

NOTE.—These principles are restricted to simple ratio. Similar principles may be proved of compound ratio.

CASE I.

320. Problems which arise in simple ratio.

1. Find the ratio of $4a^2$ to $2a$. *Ans.* $2a$.

2. Find the ratio of 3 bushels to 2 pecks. *Ans.* 6.

3. Find the ratio of a^2-x^2 to $a+x$. *Ans.* $a-x$.

4. The ratio is $2a$ and the consequent $3ab$; required the antecedent. *Ans.* $6a^2b$.

5. The antecedent is $6a^3c^2$ and ratio $2a^2$; required the consequent. *Ans.* $3ac^2$.

6. The ratio is $\frac{a}{c}$ and the antecedent is $\frac{c^2}{a}$; required the consequent. *Ans.* $\frac{c^3}{a^2}$.

7. If the ratio of a to b is $\frac{8}{15}$, what is the ratio of $5a$ to $4b$? *Ans.* $\frac{2}{3}$.

8. If the ratio of $3a$ to $2b$ is $\frac{2}{3}$, what is the ratio of a to b? *Ans.* $\frac{4}{9}$.

9. If the ratio of $2m$ to $5n$ is $\frac{4}{5}$, what is the ratio of $5m$ to $2n$? *Ans.* 5.

10. If the ratio of a to c is $\frac{4}{5}$, what is the ratio of $a+c$ to $a-c$? *Ans.* -9.

11. The ratio of two numbers is $a+b$, and the consequent is $a-b$; required the antecedent. *Ans.* a^2-b^2.

CASE II.

321. Problems which arise in compound ratio.

1. Find the ratio compounded of $8:15$ and $21:24$.

SOLUTION. The ratio of 8 to 15 is $\frac{8}{15}$; the ratio of 21 to 24 is $\frac{21}{24}$; compounding them by taking their product, we have $\frac{8}{15}\times\frac{21}{24}=\frac{7}{15}$.

OPERATION. $r=\frac{8}{15}\times\frac{21}{24}=\frac{7}{15}$

2. Find the ratio compounded of $a:b$ and $b^2:3ax$. *Ans.* $\frac{b}{3x}$.

3. Required the value of— $\left\{\begin{matrix}3:6\\8:15\end{matrix}\right\}$; of $\left\{\begin{matrix}a:b\\c:d\end{matrix}\right\}$; of $\left\{\begin{matrix}a^2:c^3\\c^2:d\end{matrix}\right\}$. *Ans.* $\frac{4}{15}$; $\frac{ac}{bd}$; $\frac{a^2}{cd}$.

4. Given the compound ratio of $x:8$ and $6:9$ equals $\frac{8}{15}$, to find the first antecedent. *Ans.* $x=6\frac{2}{5}$.

5. Given the compound ratio $\left\{\begin{matrix}9:12\\a:18\end{matrix}\right\}=\frac{1}{4}$, to find the second antecedent. *Ans.* 6.

6. Given the compound ratio $\left\{\begin{matrix}16:15\\21:c\end{matrix}\right\}=7$, to find the second consequent. *Ans.* $3\frac{1}{5}$.

7. The ratio of $2:3\frac{1}{3}$ equals the compound ratio $\left\{\begin{matrix}16:15\\18:c\end{matrix}\right\}$; required the second consequent. *Ans.* 32.

8. The duplicate of the ratio $x:1$ equals the ratio $27:x$; required the value of x. *Ans.* 3.

9. The ratio $a-x:b-x$ is the duplicate of the ratio $a:b$; required the value of x. *Ans.* $\frac{ab}{a+b}$.

10. The duplicate ratio of $x:a$ equals the compound ratio $\left\{\begin{matrix}x^2:a^2\\b^2:x\end{matrix}\right\}$; required the value of x. *Ans.* $x=b^2$.

PROPORTION.

322. A **Proportion** is an expression of equality between equal ratios. Thus, a formal comparison of the equal ratios 8 to 4 and 12 to 6, as $8 : 4 = 12 : 6$, is a proportion.

323. The **Symbol** of proportion is the double colon, : :. Thus, $a : b :: c : d$ is read the ratio of a to b equals the ratio of c to d; or, a is to b as c is to d.

324. The **Terms** of a proportion are the four quantities compared. The first and fourth terms are the *extremes*, and the second and third are the *means*.

325. The **Couplets** are the two ratios compared. The *first couplet* consists of the first and second terms; the *second couplet* consists of the third and fourth terms.

326. A **Mean Proportional** of two quantities is a quantity which may be made the means of a proportion in which the two quantities are the extremes; as, $a : b :: b : c$.

327. A **Continued Proportion** is one in which each consequent is the same as the next antecedent; as, $a : b :: b : c :: c : d$.

328. Quantities are in proportion by *Alternation* when antecedent is compared with antecedent and consequent with consequent. Thus, if $a : b :: c : d$, by alternation, $a : c :: b : d$.

329. Quantities are in proportion by *Inversion* when the antecedents are made consequents and the consequents antecedents. Thus, if $a : b :: c : d$, by inversion, $b : a :: d : c$.

330. Quantities are in proportion by *Composition* when the sum of antecedent and consequent is compared with either antecedent or consequent. Thus, if $a : b :: c : d$, by composition, $a : a+b :: c : c+d$.

331. Quantities are in proportion by *Division* when the difference of antecedent and consequent is compared with antecedent or consequent. Thus, if $a : b :: c : d$, by division, $a : a-b :: c : c-d$.

NOTE.—Ratio arises from the comparison of two *quantities*; proportion from the comparison of two *ratios*. A proportion is therefore a comparison of the results of two previous comparisons.

SIMPLE PROPORTION.

332. A **Simple Proportion** is an expression of equality between simple ratios; as, $a : b :: c : d$.

333. A **Proportion** may be written in the form of an *equation*. Thus, $a : b :: c : d$ becomes $\frac{a}{b} = \frac{c}{d}$.

334. This **Equation** is called the *fundamental equation* of the proportion. It lies at the basis of the principles of proportion.

335. The **Principles** of simple proportion are expressed in the following theorems:

THEOREM I.

In every proportion the product of the extremes is equal to the product of the means.

Let	$a : b :: c : d$;
then (Art. 333),	$\frac{a}{b} = \frac{c}{d}$;
clearing of fractions	$ad = bc$.

Therefore, etc.

THEOREM II.

Either extreme is equal to the product of the means divided by the other extreme.

Let	$a : b :: c : d$;
then (Theo. I.),	$ad = bc$;
hence,	$a = \frac{bc}{d}$; and $d = \frac{bc}{a}$.

Therefore, etc.

COR.—*Either mean equals the product of the extremes divided by the other mean.*

THEOREM III.

If the product of two quantities equals the product of two other quantities, two of them may be made the extremes, and the other two the means, of a proportion.

Let $ad = bc$;

dividing by bd, $\frac{a}{b} = \frac{c}{d}$;

or (Art. 333), $a : b :: c : a$.

Therefore, etc.

THEOREM IV.

A mean proportional between two quantities equals the square root of their product.

Let $a : b :: b : c$;

then (Theo. I.), $b^2 = ac$,

and $b = \sqrt{ac}$.

Therefore, etc.

THEOREM V.

If four quantities are in proportion, they will be in proportion by ALTERNATION.

Let $a : b :: c : d$;

then, $ad = bc$;

dividing by dc, $\frac{a}{c} = \frac{b}{d}$;

whence, $a : c :: b : d$.

Therefore, etc.

THEOREM VI.

If four quantities are in proportion, they will be in proportion by INVERSION.

Let $a : b :: c : d$;

then (Theo. I.), $bc = ad$;

dividing by ac, $\frac{b}{a} = \frac{d}{c}$;

whence, $b : a :: d : c$.

Therefore, etc.

THEOREM VII.

If four quantities are in proportion, they will be in proportion by COMPOSITION.

Let $a : b :: c : d$;

then will $a+b : b :: c+d : d$.

For, $\frac{a}{b} = \frac{c}{d}$;

adding 1 to each side, $\frac{a}{b} + 1 = \frac{c}{d} + 1$;

reducing, $\frac{a+b}{b} = \frac{c+d}{d}$;

whence, $a+b : b :: c+d : d$.

Therefore, etc.

THEOREM VIII.

If four quantities are in proportion, they will be in proportion by DIVISION.

Let $a : b :: c : d$;

then will $a-b : b :: c-d : d$.

For, $\frac{a}{b} = \frac{c}{d}$;

subtracting 1, $\frac{a}{b} - 1 = \frac{c}{d} - 1$;

reducing, $\frac{a-b}{b} = \frac{c-d}{d}$;

whence, $a-b : b :: c-d : d$.

Therefore, etc.

THEOREM IX.

If four quantities are in proportion, like powers or roots of those quantities will be proportional.

Let $a : b :: c : d$;

then $\frac{a}{b} = \frac{c}{d}$;

raising to nth power, $\frac{a^n}{b^n} = \frac{c^n}{d^n}$;

hence, $a^n : b^n :: c^n : d^n$;

similarly, $a^{\frac{1}{n}} : b^{\frac{1}{n}} :: c^{\frac{1}{n}} : d^{\frac{1}{n}}$.

Therefore, etc.

THEOREM X.

Equimultiples of two quantities are proportional to the quantities themselves.

Let a and b be any two quantities.

Then $\frac{a}{b}=\frac{a}{b}$;

multiplying by m, $\frac{ma}{mb}=\frac{a}{b}$;

whence, $ma : mb :: a : b$.

Therefore, etc.

THEOREM XI.

If four quantities are in proportion, any equimultiples of the first couplet will be proportional to any equimultiples of the second couplet.

Let $a : b :: c : d$;

then $\frac{a}{b}=\frac{c}{d}$,

and $\frac{ma}{mb}=\frac{nc}{nd}$;

whence, $ma : mb :: nc : nd$.

Therefore, etc.

THEOREM XII.

If two proportions have a couplet in each the same, the other couplets will form a proportion.

Let $a : b :: c : d$,

and $a : b :: e : f$;

then, $\frac{a}{b}=\frac{c}{d}$ and $\frac{a}{b}=\frac{e}{f}$;

hence, $\frac{c}{d}=\frac{e}{f}$;

or, $c : d :: e : f$.

Therefore, etc.

THEOREM XIII.

The products of the corresponding terms of two proportions are proportional.

Let $a : b :: c : d$,

and $m : n :: p : q$;

then, $\frac{a}{b}=\frac{c}{d}$,

and $\frac{m}{n}=\frac{p}{q}$;

multiplying, $\frac{am}{bn}=\frac{cp}{dq}$;

whence, $am : bn :: cp : dq$.

Therefore, etc.

THEOREM XIV.

If any number of quantities are in proportion, any antecedent will be to its consequent as the sum of all the antecedents is to the sum of all the consequents.

Let $a : b :: c : d :: e : f$, etc.;

then will $a : b :: a+c+e : b+d+f$.

For Theo. I., $ad=bc$,

and $af=be$;

also, $ab=ba$.

Adding, $ab+ad+af=ba+bc+be$,

factoring, $a(b+d+f)=b(a+c+e)$;

whence, $a : b :: a+c+e : b+d+f$.

Therefore, etc.

ADDITIONAL THEOREMS.

336. These theorems will afford pupils an opportunity to exercise original thought in applying the principles of proportion. Part of them may be omitted until review.

1. If $a : b :: c : d$, prove that $am : bn :: cm : dn$.
2. If $a : b :: c : d$, prove that $a^2 : b^2 :: ac : bd$.

3. If $a:b::c:d$, prove that $a^2:c^2::ab:cd$.

4. If $a:b::c:d$, prove that $a:a-b::c:c-d$.

5 If $a:b::c:d$, prove that $a+b:c+d::a-b:c-d$.

6 If $a:b::c:d$, prove that $a:na+mb::c:nc+md$.

7. If $a:b::b:c$, prove that $a^2:b^2::a:c$.

8. If $a:b::b:c$, prove that $a:c::b^2:c^2$.

9. If $a:b::b:c$, prove that $a^2-b^2:a::b^2-c^2:c$.

10. If $a:b::b:c$, prove that $a^2+b^2:a^2-b^2::a+c:a-c$.

11. If $a:b::b:c$, prove that $(a^2+b^2)(b^2+c^2)=(ab+bc)^2$.

PROBLEMS IN RATIO AND PROPORTION.

337. These problems in ratio and proportion can be readily solved by an application of the previous principles.

1. The product of two numbers is 15, and the difference of their squares is to the square of their difference as 4 to 1; required the numbers.

SOLUTION.

Let	$x=$ the greater number,	
and	$y=$ the less number;	
then,	$xy=15$,	(1)
and	$x^2-y^2:(x-y)^2::4:1$.	(2)
Dividing 1st couplet by $x-y$,	$x+y:x-y::4:1$,	
by composition and division,	$2x:2y::5:3$,	
dividing 1st couplet by 2,	$x:y::5:3$;	
hence, Theo. I.,	$3x=5y$,	
substituting in eq. (1),	$\frac{5y^2}{3}=15$;	
whence,	$y=3$,	
and	$x=5$.	

2. The product of two numbers is 24, and the sum of their squares is to the square of their sum as 13 to 25; what are the numbers?

SOLUTION.

Let x and y represent the numbers.

Then,	$xy=24$,	(1)
and	$x^2+y^2:x^2+2xy+y^2::13:25$.	
By division, Theo. VIII.,	$2xy:(x+y)^2::12:25$,	
substituting for $2xy$,	$48:(x+y)^2::12:25$,	
dividing antecedents by 12,	$4:(x+y)^2::1:25$;	
extracting the square root,	$2:x+y::1:5$;	
from Theo. I.,	$x+y=10$.	

From which the values of x and y can readily be found.

3. What is the ratio of $6a$ inches to b yards? *Ans.* $\frac{a}{6b}$.

4. Two numbers are in the ratio of 2 to 3, and if 3 be added to each, the ratio is that of 5 to 7; find the numbers. *Ans.* 12 and 18.

5. Two numbers are in the ratio of 4 to 5, and if 6 be taken from each, the ratio is that of 3 to 4; find the numbers. *Ans.* 24 and 30.

6. Two numbers are in the ratio of 3 to 5, and if 2 be taken from the less and 5 be added to the greater, the ratio is that of 2 : 5; find the numbers. *Ans.* 12 and 20.

7. Find the number which added to each term of the ratio 5 : 3 makes it $\frac{3}{4}$ of what it would have been if the same number had been taken from each term. *Ans.* 1.

8. Find two numbers in the ratio of 2 to 3, such that their difference bears the same relation to the difference of their squares as 1 to 25. *Ans.* 10 and 15.

9. Find two numbers in the ratio of 3 to 4, such that their sum has to the sum of their squares the ratio of 7 to 50. *Ans.* 6 and 8.

10. Find two numbers in the ratio of 5 to 6, such that their sum has to the difference of their squares the ratio of 1 to 7. *Ans.* 35 and 42.

11. The sum of two numbers is 10, and the sum of their squares is to the difference of their squares as 13 to 5; required the numbers. *Ans.* 6 and 4.

12. The difference of two numbers is 6, and their product is to the sum of their squares as 2 to 5; what are the numbers? *Ans.* 12 and 6.

13. Two numbers are to each other as 3 to 2, and if 6 be added to the greater and subtracted from the less, the results will be as 3 to 1; what are the numbers? *Ans.* 24 and 16.

14. The product of two numbers is 12, and the difference of their cubes is to the sum of their cubes as 13 to 14; required the numbers. *Ans.* 6 and 2.

15. There are three numbers in continued proportion: the middle number is 60, and the sum of the others is 125; what are the numbers? *Ans.* 45; 60; 80.

16. A quantity of milk is increased by water in the ratio of 7 : 6, and then 8 gallons are sold; the remainder, when mixed with 8 gallons of water, is increased in the ratio of 7 to 5; how much milk was there at first? *Ans.* 24 gallons.

REVIEW QUESTIONS.

1. Define Ratio. The Terms. A Simple Ratio. A Compound Ratio. A Duplicate Ratio. A Triplicate Ratio. A Ratio of Equality. Of Inequality. State the Principles of Ratio.

2. Define a Proportion. The Terms of a Proportion; Extremes, Means; Couplets. A Mean Proportional. A Continued Proportion. Proportion by Alternation. By Inversion. By Composition. By Division. State the Fundamental Equation of Proportion. Enunciate the Theorems.

SECTION IX.

PROGRESSIONS.

338. A **Progression** is a series of quantities in which the terms vary according to some fixed law.

339. The **Terms** of a progression are the quantities of which it is composed.

340. The **Extremes** of a progression are the first and last terms; the *Means* are the terms between the extremes.

NOTE.—The general term for *Progression* is *series.* There are many different kinds of series; the only two appropriate for an elementary algebra are *arithmetical* and *geometrical progression.*

ARITHMETICAL PROGRESSION.

341. An **Arithmetical Progression** is a series of quantities which vary by a common difference.

342. The **Common Difference** is the quantity which, added to any term, will give the following term: thus, in 1, 3, 5, 7, the *common difference* is 2.

343. An **Ascending Progression** is one in which the quantities increase from left to right; as, 1, 3, 5, 7, 9, etc.

344. A **Descending Progression** is one in which the terms decrease from left to right; as, 12, 10, 8, 6, etc.

345. The **Terms** considered in Arithmetical Progression are *five,* any three of which being given the other two may be found.

THE FIVE TERMS.

1. The first term, a;
2. The last term, l;
3. The number of terms, n;
4. The common difference, d;
5. The sum of the terms, S.

11. The sum of two numbers is 10, and the sum of their squares is to the difference of their squares as 13 to 5; required the numbers. *Ans.* 6 and 4.

12. The difference of two numbers is 6, and their product is to the sum of their squares as 2 to 5; what are the numbers? *Ans.* 12 and 6.

13. Two numbers are to each other as 3 to 2, and if 6 be added to the greater and subtracted from the less, the results will be as 3 to 1; what are the numbers? *Ans.* 24 and 16.

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THE FIVE TERMS.

1. The first term, a;
2. The last term, l;
3. The number of terms, n;
4. The common difference, d;
5. The sum of the terms, S.

346. Prin.—*The common difference is positive in an ascending series, and negative in a descending series.*

CASE I.

347. Given the first term, the common difference and the number of terms, to find the last term.

1. Given a the first term, d the common difference, n the number of terms, to find the expression for l, the last term.

Solution. The 1st term is a; the 2d term is $a+d$; the 3d term is $a+d$ plus d, or $a+2d$; the 4th term is $a+3d$, and so on. By examining these terms, we see that *any term equals a, plus the product of d taken as many times as the number of terms less one;* hence the nth term equals $a+(n-1)d$; or, representing the nth term by l, we have $l=a+(n-1)d$.

OPERATION.

1st term $=a$
2d term $=a+d$
3d term $=a+2d$
4th term $=a+3d$
$\therefore$ nth term $=a+(n-1)d$
or, $l=a+(n-1)d$

Rule.—*To the first term add the product of the common difference multiplied by the number of terms less one.*

Note.—An ascending series of n terms may be written as follows:

$$a,\ a+d,\ a+2d,\ a+3d\ \dots\ a+(n-1)d.$$

A descending series of n terms may be written as follows:

$$a,\ a-d,\ a-2d,\ a-3d\ \dots\ a-(n-1)d.$$

EXAMPLES.

2. Find the 12th term of the series 2, 5, 8, 11, etc.

Solution. In this problem $a=2$, $d=3$ and $n=12$. The formula for the last term is $l=a+(n-1)d$; substituting the values of a, d and n, we have $l=2+(12-1)3$; and reducing, we have $l=35$.

OPERATION.

$l=a+(n-1)a$
$l=2+(12-1)3$
$l=2+33=35$

Note.—The problem may also be solved by the rule instead of substituting in the formula.

3. Find the 18th term of the series 1, 4, 7, 10, etc. *Ans.* 52.

4. Find the 17th term of the series 3, 7, 11, 15, etc. *Ans.* 67.

5. Find the 20th term of the series 1, $2\frac{1}{3}$, $3\frac{2}{3}$, 5, etc. *Ans.* $26\frac{1}{3}$.

6. Find the 14th term of the series 29, 27, 25, 23, etc. *Ans.* 3.

7. Find the 40th term of the series 1, $2\frac{2}{3}$, $4\frac{1}{3}$, 6, etc. *Ans.* 66

8. Find the 15th term of the series $\frac{7}{8}$, $\frac{13}{16}$, $\frac{3}{4}$, $\frac{11}{16}$, etc. *Ans.* 0.

9. Find the 30th term of the series a, $3a$, $5a$, $7a$, etc. *Ans.* $59a$.

10. Find the nth term of the series 2, 4, 6, 8, etc. *Ans.* $2n$.

11. Find the nth term of the series $2b$, $4b$, $6b$, etc. *Ans.* $2bn$.

12. Find the nth term of the series 1, 3, 5, 7, etc. *Ans.* $2n-1$.

13. Find the nth term of the series, 2, $2\frac{1}{3}$, $2\frac{2}{3}$, etc. *Ans.* $\frac{1}{3}(n+5)$

14. If a body fall $16\frac{1}{12}$ feet the 1st second, 3 times as far the 2d second, 5 times as far the 3d second, and so on, how far will it fall the 20th second? *Ans.* $627\frac{1}{4}$ ft.

15. If a body fall n feet the 1st second, $3n$ feet the 2d, $5n$ feet the 3d, and so on, how far will it fall the tth second? *Ans.* $(2t-1)n$.

CASE II.

348. Given the first term, the last term, and the number of terms, to find the sum of the terms.

1. Given a, the first term; l, the last term; and n, the number of terms, to find the expression for S, the sum of the terms.

Solution.

We have the series, $S=a+(a+d)+(a+2d)+(a+3d)\dots+l.$

inverting the series, $S=l+(l-d)+(l-2d)+(l-3d)\dots+a.$

Adding the series, $2S=(a+l)+(a+l)+(a+l)+(a+l)+\dots+(a+l)$

That is, $(a+l)$ taken as many times as there are terms, or n times.

Hence, $2S=(a+l)n$;

whence, $S=\left(\frac{a+l}{2}\right)n$, or $\frac{n}{2}(a+l)$.

Rule.—*Multiply the sum of the extremes by one-half of the number of terms.*

EXAMPLES.

2. Find the sum of the arithmetical series whose first term is 2, last term 35, and number of terms 12.

SOLUTION. In this problem, $a=2$, $n=12$, and $l=35$. Substituting the values of these terms in the formula, $S=\frac{n}{2}(a+l)$, we have $S=\frac{12}{2}(2+35)$, or 6×37, which is 222.

OPERATION.
$S=\frac{n}{2}(a+l)$
$S=\frac{12}{2}(2+35)$
$S=6\times37=222$. *Ans.*

Find the sum—

3. When $a=3$, $l=40$ and $n=16$. *Ans.* 344.
4. Of 12 terms of the series $2+6+10+14$, etc. *Ans.* 288.
5. Of 16 terms of the series $3+7+11+15$, etc. *Ans.* 528.
6. Of 12 of the odd numbers $1+3+5+7$, etc. *Ans.* 144.
7. Of 12 of the even numbers $2+4+6+8$, etc. *Ans.* 156.
8. Of 18 terms of the series $\frac{1}{3}+1\frac{1}{3}+2\frac{1}{3}$, etc. *Ans.* 159.
9. Of 25 terms of the series $\frac{1}{2}+1+1\frac{1}{2}+2$, etc. *Ans.* $162\frac{1}{2}$.
10. Of 12 terms of the series $20+18+16$, etc. *Ans.* 108.
11. Of 17 terms of the series $.2+.25+.3+.35$, etc. *Ans.* 10.2.
12. Of n terms of the series $1+3+5+7$, etc. *Ans.* n^2.
13. Of n terms of the series $2+4+6+8$, etc. *Ans.* n^2+n.
14. Of n terms of $a+3a+5a+7a+$, etc. *Ans.* an^2.
15. Of 6 terms of $(a-5b)+(a-3b)+(a-b)+$, etc. *Ans.* $6a$.

16. If a body fall $16\frac{1}{12}$ feet the 1st second, 3 times as far the 2d second, 5 times as far the 3d second, and so on, how far will it fall in 20 seconds? *Ans.* $6433\frac{1}{3}$ ft.

17. If a body fall n feet the 1st second, $3n$ feet the 2d, $5n$ feet the 3d, and so on, how far will it fall in t seconds? *Ans.* t^2n ft.

CASE III.

349. Given any three of the five quantities to find either of the others.

350. The **Fundamental Formulas** of arithmetical progression are—

1. $l=a+(n-1)d$;
2. $S=\frac{n}{2}(a+l)$.

These are called Fundamental because by means of them we can solve all the problems which may arise.

351. There are three classes of problems, since the four quantities may all be in the 1st formula, or all in the 2d formula, or part in the first and part in the 2d.

352. CLASS I. When the four quantities are all contained in the first fundamental formula.

1. Find the formula for d, having given a, n and l.

SOLUTION. The formula $l=a+(n-1)d$ contains all the four quantities; we will therefore find the value of d from this formula in terms of the other quantities. Transposing, we have equation (2); dividing by the coefficient of d and transposing, we have equation (3), which is the value of d required.

OPERATION.

$$l=a+(n-1)d \quad (1)$$
$$l-a=(n-1)d \quad (2)$$
$$d=\frac{l-a}{n-1} \quad (3)$$

EXAMPLES.

2. Find the value of a, having given n, d and l.

3. Find the value of n, having given a, d and l.

4. The first term is 90, the last term 34, and the common difference 4; required the number of terms. *Ans.* 15.

5. What term of the series 2, 5, 8, etc., is 35? What term of the series 29, 27, 25, etc., is 3? *Ans.* 12th; 14th.

6. The nth term of a series whose common difference is 2, is $2n$; what is the first term? *Ans.* 2.

7. The nth term of a series whose first term is 1, is $2n-1$, required the first four terms of the series. *Ans.* 1, 3, 5, 7.

353. Class II. When the four quantities are all contained in the second fundamental formula.

1. Given a, l and S, to find n.

Solution. The formula $S=\frac{n}{2}(a+l)$ contains all the four quantities; we can therefore find the value of n from this formula in terms of a, l and S. Clearing of fractions, we have equation (2); dividing by $a+l$ and transposing, we have equation (3), which is the value of n required.

OPERATION

$$S=\tfrac{n}{2}(a+l) \quad (1)$$
$$2S=n(a+l) \quad (2)$$
$$n=\frac{2S}{a+l} \quad (3)$$

EXAMPLES.

2. Find a, having given n, l and S.
3. Find l, having given a, n and S.
4. The first term is 2, the last term 35, and the sum of the terms 222; required the number of terms. *Ans.* 12.
5. The last term is 27, the number of terms 12, and the sum of the terms 180; required the first term. *Ans.* 3.
6. Required the last term of the series whose first term is 1, number of terms n, and sum of terms n^2. *Ans.* $2n-1$.
7. Required the first four terms of the series whose first term is 1, number of terms n, and sum n^2. *Ans.* 1, 3, 5, 7.
8. Required the last term of the series whose first term is a, number of terms n, and sum of the terms an^2. *Ans.* $a(2n-1)$.

354. Class III. When part of the quantities are in the first and part in the second fundamental formula.

1. Given d, n, l, to find S.

Solution. The two fundamental formulas contain one quantity, namely, a, not involved in this problem; hence we may combine these formulas by eliminating a, and obtain an equation containing the four quantities, d, n, l and S, from which we can find the value of S.

From the first formula we find equation (3); substituting this value in equation (2), we have equation (4); reducing, we have equation (5), which expresses the value of S in terms of d, n and l

OPERATION.

$$l=a+(n-1)d \quad (1)$$
$$S=\tfrac{n}{2}(a+l) \quad (2)$$
$$a=l-(n-1)d \quad (3)$$
$$S=\tfrac{n}{2}\{l-(n-1)d+l\} \quad (4)$$
$$S=\tfrac{n}{2}\{2l-(n-1)d\} \quad (5)$$

Note.—The superfluous quantity may be eliminated by comparison or substitution, as is most convenient. For examples, see the table.

TABLE OF FORMULAS.

355. Since there are five quantities, any three of which being given a fourth may be found, there are twenty cases in all. These cases are presented in the following table:

No.	Given.	To Find.	Formulas.	
1	a, d, n	l, S	$l=a+(n-1)d$;	$S=\frac{1}{2}n[2a+(n-1)d]$.
2	l, d, n	a, S	$a=l-(n-1)d$;	$S=\frac{1}{2}n[2l-(n-1)d]$.
3	a, n, l	d, S	$d=\frac{l-a}{n-1}$;	$S=\frac{1}{2}n(a+l)$.
4	d, n, S	a, l	$a=\frac{2S-n(n-1)d}{2n}$;	$l=\frac{2S+n(n-1)d}{2n}$.
5	a, n, S	d, l	$d=\frac{2(S-an)}{n(n-1)}$;	$l=\frac{2S}{n}-a$.
6	l, n, S	d, a	$d=\frac{2(nl-S)}{n(n-1)}$;	$a=\frac{2S}{n}-l$.
7	a, d, l	n, S	$n=\frac{l-a}{d}+1$;	$S=\frac{(l+a)(l-a+d)}{2d}$.
8	a, l, S	n, d	$n=\frac{2S}{a+l}$;	$d=\frac{(l+a)(l-a)}{2S-(l+a)}$.
9	a, d, S	l	$l=-\frac{1}{2}d\pm\sqrt{2dS+(a-\frac{1}{2}d)^2}$.	
		n	$n=\frac{\pm\sqrt{(2a-d)^2+8dS}-2a+d}{2d}$.	
10	l, d, S	a	$a=\frac{1}{2}d\pm\sqrt{(l+\frac{1}{2}d)^2-2dS}$.	
		n	$n=\frac{2l+d\pm\sqrt{(2l+d)^2-8dS}}{2d}$.	

CASE IV.

356. Given two terms, to insert any number of arithmetical means between them.

1. Insert 3 arithmetical means between 4 and 16.

SOLUTION. If we insert m terms between 4 and 16, the entire series will consist of $m+2$ terms; hence in the formula for d, n will equal $m+2$. Substituting $m+2$ for n in the formula for d, and reducing, we have $d=\frac{l-a}{m+1}$. Substituting in this formula $l=16$, $a=4$ and $m=3$, we have $d=3$; hence the series will be 4, 7, 10, 13, 16.

OPERATION.

$$d=\frac{l-a}{n-1}$$

$$d=\frac{l-a}{m+2-1}=\frac{l-a}{m+1}$$

$$d=\frac{16-4}{3+1}=3$$

Series = 4, 7, 10, 13, 16

EXAMPLES.

2. Insert 4 arithmetical means between 5 and 20. *Ans.* 8, 11, 14, 17.

3. Required the arithmetical mean between 4 and 18. *Ans.* 11.

4. Insert 2 arithmetical means between $\frac{1}{2}$ and $\frac{1}{3}$. *Ans.* $\frac{7}{18}$; $\frac{4}{9}$.

5. Required the arithmetical mean between a and b. *Ans.* $\frac{a+b}{2}$.

6. Insert 2 arithmetical means between a and b. *Ans.* $\frac{2a+b}{3}$; $\frac{a+2b}{3}$.

7. Insert 3 arithmetical means between a and b. *Ans.* $\frac{3a+b}{4}$; $\frac{a+b}{2}$; $\frac{a+3b}{4}$.

8 If the arithmetical mean between two numbers, one of which is 5, is 19, what are the numbers? *Ans.* 5 and 33

PROBLEMS

IN ARITHMETICAL PROGRESSION.

357. In Arithmetical Progression problems arise in which the terms are not directly given, but are implied in the given conditions.

358. In solving these problems we may represent the unknown terms and form equations by means of the principles of arithmetical progression.

359. An arithmetical series in which x represents the first term and y the common difference is generally represented thus:

$$x,\ x+y,\ x+2y,\ x+3y,\ \text{etc.}$$

360. When there are *three terms*, it will often be found convenient to represent them thus:

$$x-y,\ x,\ x+y.$$

361. When there are *four terms*, the following method will often facilitate the solution:

$$x-3y,\ x-y,\ x+y,\ x+3y.$$

362. When there are *five terms*, the following method will often facilitate the solution:

$$x-2y,\ x-y,\ x,\ x+y,\ x+2y.$$

NOTE.—The advantage of these methods is, that the sum of the series, or the sum or difference of the extremes, or of any two terms equally distant from the extremes, will contain but one quantity.

1. Find the series where the nth term is $2n-1$.

SOLUTION. Since n represents any term, the formula $2n-1$ is true for any value of n. When $n=1$, $2n-1=1$, hence the first term is 1; when $n=2$, $2n-1=3$, hence the second term is 3; when $n=3$, $2n-1=5$, etc. Hence the series is 1, 3, 5, 7, 9, etc.

2. The sum of three numbers in arithmetical progression is 12, and the sum of their squares is 66; find the numbers.

SOLUTION.

Let $x-y=$ first term,
$x=$ second term,
$x+y=$ third term.

By the 1st condition,	$3x=12;$	(1)
by the 2d condition,	$3x^2+2y^2=66;$	(2)
from equation (1) we have,	$x=4;$	(3)
substituting in equation (2),	$48+2y^2=66;$	(4)
from which we have,	$y=3;$	(5)
hence we have,	$x-y=1$, first term,	
and	$x+y=7$, third term.	

Therefore the numbers are 1, 4 and 7.

EXAMPLES.

3. The nth term of an arithmetical progression is $\frac{1}{3}(n+5)$; required the series. *Ans.* 2; $2\frac{1}{3}$; $2\frac{2}{3}$, etc.

4. The nth term of an arithmetical progression is $\frac{1}{6}(3n-1)$; find the first term, the common difference and the sum of n terms. *Ans.* $\frac{1}{3}$; $\frac{1}{2}$; $\frac{n}{12}(3n+1)$.

5. The nth term of an arithmetical progression is $(2n-1)a$; find the series and its sum. *Ans.* a, $3a$, $5a$, etc.; sum, an^2.

6. Three numbers are in arithmetical progression; their sum is 12 and their product 48; required the numbers. *Ans.* 2, 4, 6.

7. The sum of three numbers in arithmetical progression is 18; the product of the first and second is 24; what are the numbers? *Ans.* 4, 6, 8.

8. Find three numbers in arithmetical progression such that their sum shall be 15 and the sum of their cubes 645. *Ans.* 2, 5, 8.

9. There are four numbers in arithmetical progression; the sum of the two extremes is 8, and the product of the two means is 15; what are they? *Ans.* 1, 3, 5, 7.

10. There are four numbers in arithmetical progression; the product of the first and fourth is 22, and of the second and third, 40; what are the numbers? *Ans.* 2, 5, 8, 11.

11. The sum of four numbers in arithmetical progression is 30, and the sum of the cubes of the two means is 945; what are the numbers? *Ans.* 3, 6, 9, 12.

12. The sum of four numbers in arithmetical progression is 22, and their continued product is 280; what are the numbers? *Ans.* 1, 4, 7, 10.

13. Find four numbers in arithmetical progression such that the sum of the squares of the first and fourth shall be 200, and of the second and third, 136. *Ans.* 2, 6, 10, 14.

14. There are four numbers in arithmetical progression; the product of the first and fourth is 45, and of the second and third 77; what are the numbers? *Ans.* 3, 7, 11, 15.

15. Find four numbers in arithmetical progression such that the sum of the first and fourth shall be 17, and the difference of the squares of the two means shall be 51. *Ans.* 4, 7, 10, 13.

16. There are four numbers in arithmetical progression such that the sum of the squares of the means is 164, and the sum of the squares of the extremes is 180; what are they? *Ans.* 6, 8, 10, 12.

17. There are five numbers in arithmetical progression; their sum is 40, and the sum of their squares 410; what are the numbers? *Ans.* 2, 5, 8, 11, 14.

18. There are seven numbers in arithmetical progression such that the sum of the first and fifth shall be 16, and the product of the fourth and seventh 160; required the numbers. *Ans.* 4, 6, 8, 10, 12, 14, 16.

19. If the sum of n terms of an arithmetical progression is always equal to n^2, find the first term and the common difference. *Ans.* First term, 1; com. dif. $=2$.

20. If the sum of n terms of an arithmetical progression is always equal to $\frac{1}{2}n(n+1)$, find the series. *Ans.* 1, 2, 3, 4, etc.

NOTE.—In the 19th Example take $S=\frac{1}{2}n\{2a+(n-1)d\}=n^2$; then find the first term by supposing $n=1$ and $d=0$, etc.

GEOMETRICAL PROGRESSION.

363. A **Geometrical Progression** is a series of quantities which vary by a constant multiplier.

364. The **Ratio** or rate of the progression is the constant multiplier by which the terms vary; thus, in 1, 2, 4, 8 the ratio is 2.

365. An **Ascending Progression** is one that increases from left to right; as 2, 4, 8, 16, etc.

366. A **Descending Progression** is one that decreases from left to right; as 32, 16, 8, $\frac{1}{2}$, etc.

367. The **Terms** considered are five, any three of which being given, the other two may be found.

THE FIVE TERMS.

1. The first term, a;
2. The last term, l;
3. The number of terms, n;
4. The ratio, r;
5. The sum of the terms, S.

368. Principle.—*The ratio is greater than a unit in an ascending series, and less than a unit in a descending series.*

CASE I.

369. Given the first term, the ratio and the number of terms, to find the last term.

1. Given a, the first term; r, the ratio; and n, the number of terms, to find an expression for l, the last term.

Solution. The *first* term is a; the *second* term equals $a \times r$, or ar; the *third* term equals $ar \times r$, or ar^2; the *fourth* term equals $ar^2 \times r$, or ar^3, etc. Examining these terms, we see that each term equals the *first term into r raised to a power one less than the number of the term;* hence the nth term will equal ar^{n-1}; and since l represents the nth or last term, we have $l = ar^{n-1}$.

Operation.

1st term $= a$
2d term $= ar$
3d term $= ar^2$
4th term $= ar^3$
nth term $= ar^{n-1}$
$l = ar^{n-1}$

Rule.—*Multiply the first term by the ratio raised to a power whose index is one less than the number of terms.*

Note.—An ascending series of n terms may be written as follows:

$$a,\ ar,\ ar^2,\ ar^3 \ldots .\ ar^{n-2},\ ar^{n-1}.$$

EXAMPLES.

2. Find the 8th term of the series 2, 4, 8, etc. *Ans.* 256.
3. The first term is 3 and ratio 4; what is the 7th term? *Ans.* 12288.
4. The first term is 729, the ratio $\frac{1}{3}$; required the 12th term. *Ans.* $\frac{1}{243}$.
5. Find the nth term of the series 1, 2, 4, 8, etc. *Ans.* 2^{n-1}.
6. Find the nth term of the series $2a$, $4a^2$, $8a^3$, etc. *Ans.* $(2a)^n$.
7. Find the nth term of the series 2, $4a$, $8a^2$, etc. *Ans.* $2^n a^{n-1}$.
8. If a merchant doubles his capital every 4 years, and begins with \$4000, how much has he at the end of 20 years? *Ans.* \$128000.

CASE II.

370. Given the first term, the last term and the number of terms, to find the sum of the terms.

1. Given a, the first term; l, the last term; and n, the number of terms, to find an expression for S, the sum of the terms.

Solution.

We have $S = a + ar + ar^2 + ar^3 + \ldots .\ ar^{n-1};$ (1)

multiplying (1) by r, $rS = ar + ar^2 + ar^3 + \ldots .\ ar^{n-1} + ar^n.$ (2)

Subtracting (1) from (2), $rS - S = ar^n - a;$ (3)

factoring, $S(r-1) = ar^n - a;$ (4)

whence $S = \dfrac{ar^n - a}{r-1}.$ (5)

This may be put in another form by substituting a value for ar^n.

From Case I. we have $l = ar^{n-1}$;
multiplying by r, $rl = ar^n$;
substituting in (5), $S = \frac{rl - a}{r - 1}$.

Rule.—*Multiply the last term by the ratio, subtract the first term, and divide the remainder by the ratio less one.*

EXAMPLES.

Find the sum of the series—

2. When $a = 2$, $l = 256$, and $r = 2$. *Ans.* 510.
3. When $a = 3$, $l = 12288$, and $r = 4$. *Ans.* 16383.
4. Of 9 terms of the series 2, 4, 8, 16, etc. *Ans.* 1022.
5. Of 12 terms of the series 1, 2, 4, 8, etc. *Ans.* 4095.
6. Of 10 terms of the series 1, 3, 9, 27, etc. *Ans.* 29524.
7. Of n terms of the series $1+2+4+8$, etc. *Ans.* $2^n - 1$.
8. Of n terms of the series 1, 3, 9, 27, etc. *Ans.* $\frac{1}{2}(3^n - 1)$.
9. Of n terms of the series $a + 2a + 4a + 8a$, etc. *Ans.* $a(2^n - 1)$.
10. Of n terms of the series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}$, etc. *Ans.* $\frac{2^n - 1}{2^{n-1}}$.
11. Of n terms of the series $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27}$, etc. *Ans.* $\frac{1}{2}\left(\frac{3^n - 1}{3^{n-1}}\right)$.
12. Of n terms of the series $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8}$, etc. *Ans.* $\frac{1}{3}\left(\frac{2^n - 1}{2^{n-1}}\right)$, or $\frac{1}{3}\left(\frac{2^n + 1}{2^{n-1}}\right)$.

13. A laborer agreed to work one year at the rate of $1 for January, $2 for February, $4 for March, and so on; how much did he receive in the year? *Ans.* $4095.

14. A servant-girl saved $160 one year. Now, if it were possible for her to save half as much again every year as the previous year for 8 years, how much would she save? *Ans.* $11981.87½.

INFINITE SERIES.

371. An **Infinite Series** is a series in which the number of terms is infinite; as, 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, etc.

372. The **Sum** of a decreasing geometrical series to infinity is the limit toward which the series approaches as the number of terms increases.

1. Find the sum of a decreasing geometrical series to infinity.

OPERATION.

$$S = \frac{rl - a}{r - 1} \quad (1)$$

$$S = \frac{a - rl}{1 - r} \quad (2)$$

When $rl = 0$,

$$S = \frac{a}{1 - r} \quad (3)$$

SOLUTION. In a decreasing series, r is less than 1; hence, for a decreasing series we change formula (1) to formula (2), that the denominator may be positive.

Now, as the number of terms increases, the value of l decreases; hence, when the number of terms is *infinite*, l must become infinitely small; that is, 0; hence, $rl = 0$, and the formula for S becomes a divided by $1 - r$.

Rule.—*Divide the first term by* 1 *minus the ratio.*

EXAMPLES.

Find the sum of the infinite—

2. Series $1 + \frac{1}{2} + \frac{1}{4} +$, etc. *Ans.* 2.
3. Series $\frac{1}{2} + \frac{1}{3} + \frac{2}{9} +$, etc. *Ans.* $1\frac{1}{2}$.
4. Series $\frac{1}{3} + \frac{2}{9} + \frac{4}{27}$, etc. *Ans.* 1.
5. Series $1 - \frac{1}{2} + \frac{1}{4} -$, etc. *Ans.* $\frac{2}{3}$.
6. Series $1 - \frac{2}{5} + \frac{4}{25} -$, etc. *Ans.* $\frac{5}{7}$.
7. Of the circulate .333, etc. ($= \frac{3}{10} + \frac{3}{100} +$, etc.). *Ans.* $\frac{1}{3}$.
8. Of the circulate .22727, etc. *Ans.* $\frac{5}{22}$.
9. Series $\frac{1}{a} + \frac{1}{a^2} + \frac{1}{a^3}$, etc. *Ans.* $\frac{1}{a - 1}$.
10. Series $1 + x^{-2} + x^{-4} +$, etc. *Ans.* $\frac{x^2}{x^2 - 1}$.
11. Series $a - b + \frac{b^2}{a} - \frac{b^3}{a^2} + \frac{b^4}{a^3}$, etc. *Ans.* $\frac{a^2}{a + b}$.

12. Suppose a body move 12 feet the first second, 6 feet the next second, 3 feet the next second, and so on until it stops; what is the entire distance it can reach? *Ans.* 24 ft.

13. If an ivory ball falls 12 feet to the floor and bounds back 6 feet, then, falling, bounds back 3 feet, and so on, how far will it move before it comes to rest? *Ans.* 36 ft.

14. A dog and rabbit, 20 rods apart, run so that when the dog runs the distance between them the rabbit will run $\frac{1}{10}$ of that distance; how far will the dog run to catch the rabbit? *Ans.* $22\frac{2}{9}$ rods.

CASE III.

373. Given any three of the five quantities, to find either of the others.

374. The Fundamental Formulas of geometrical progression are—

$$1.\ l=ar^{n-1};\qquad 2.\ S=\frac{rl-a}{r-1}.$$

By means of these we are enabled to solve all the cases which arise. As in arithmetical progression, there are three classes of problems.

375. Class I. When the four quantities are all contained in the first fundamental formula.

EXAMPLES.

1. Find a, given l, r and n. *Ans.* $a=\frac{l}{r^{n-1}}$.

2. Find r, given a, l and n. *Ans.* $r=\sqrt[n-1]{\frac{l}{a}}$.

3. A person agreed to labor for wages doubling every month what did he receive the first month if he received $512 the tenth month? *Ans.* $1.

4. If a man saves $6.40 the first year, and increases his savings each year in geometrical proportion, what is the rate of increase if he saves $109.35 the eighth year? *Ans.* $1\frac{1}{2}$.

376. Class II. When the four quantities are all contained in the second fundamental formula.

EXAMPLES.

1. Find a, given r, l and S. *Ans.* $a=rl-(r-1)S$.

2. Find l, given a, r and S. *Ans.* $l=\frac{a+(r-1)S}{r}$.

3. Find r, given a, l and S. *Ans.* $r=\frac{S-a}{S-l}$.

4. If a person agrees to labor for wages doubling every month, and receives $4095 in a year, how much did he receive the first month? *Ans.* $1.

5. If I discharge a debt in 10 months by monthly payments in geometrical progression, allowing the first payment to be $1 and the last $512, what will be the ratio? *Ans.* 2.

377. Class III. When some of the quantities are in the first and some in the second fundamental formula.

Note.—The four formulas for n require a knowledge of logarithms Four others, when n exceeds 2, require a knowledge of higher equations.

EXAMPLES.

1. Find S, given l, r and n. *Ans.* $S=\frac{lr^n-l}{r^n-r^{n-1}}$.

2. Find l, given r, n and S. *Ans.* $l=\frac{(r-1)Sr^{n-1}}{r^n-1}$.

3. A man bought 10 yards of cloth for $295.24, giving three times as much for each yard as for the preceding yard; what did he pay for the first yard? *Ans.* 1 cent.

TABLE OF FORMULAS.

378. Since there are five quantities, any three of which being given a fourth may be found, there are twenty cases in all. These cases are presented in the following table:

No.	Given.	Required.	FORMULAS.
1	a, r, n	l, S	$l=ar^{n-1}$; $S=\frac{ar^n-a}{r-1}$.
2	l, r, n	a, S	$a=\frac{l}{r^{n-1}}$; $S=\frac{lr^n-l}{r^n-r^{n-1}}$.
3	r, n, S	a, l	$a=\frac{(r-1)S}{r^n-1}$; $l=\frac{(r-1)Sr^{n-1}}{r^n-1}$.
4	a, n, l	r, S	$r=\sqrt[n-1]{\frac{l}{a}}$; $S=\frac{\sqrt[n-1]{l^n}-\sqrt[n-1]{a^n}}{\sqrt[n-1]{l}-\sqrt[n-1]{a}}$.
5	a, n, S	r, l	$ar^n-rS=a-S$; $l(S-l)^{n-1}=a(S-a)^{n-1}$.
6	l, n, S	r, a	$(S-l)r^n-Sr^{n-1}=-l$; $a(S-a)^{n-1}=l(S-l)^{n-1}$.
7	a, r, l	n, S	$n=\frac{\log l-\log a}{\log r}+1$; $S=\frac{lr-a}{r-1}$.
8	a, l, S	n, r	$n=\frac{\log l-\log a}{\log(S-a)-\log(S-l)}+1$; $r=\frac{S-a}{S-l}$.
9	a, r, S	n, l	$n=\frac{\log[a+(r-1)S]-\log a}{\log r}$; $l=\frac{a+(r-1)S}{r}$.
10	l, r, S	n, a	$n=\frac{\log l-\log[lr-(r-1)S]}{\log r}+1$; $a=lr-(r-1)S$.

NOTE.—Pupils who have the time will be interested in deriving the formulas of the above table. The formulas for the values of n can be derived after completing the subject of logarithms.

PROBLEMS

IN GEOMETRICAL PROGRESSION.

379. In **Geometrical Progression** problems arise in which the terms are not directly given, but are implied in the conditions.

380. In solving these problems we may represent the unknown terms and form equations by means of the principles of Geometrical Progression.

381. A geometrical series, where x represents the first term and y the ratio, is generally represented thus:

$$x,\ xy,\ xy^2,\ xy^3,\ \text{etc.}$$

382. When *three* terms are considered in the problem, they may be represented thus:

$$x,\ \sqrt{xy},\ y;$$
$$\text{or } x^2,\ xy,\ y^2.$$

383. When *four* terms are considered in the problem, they may be represented thus:

$$\frac{x^2}{y},\ x,\ y,\ \frac{y^2}{x}.$$

NOTE.—The method most convenient in any case will depend upon the nature of the problem.

EXAMPLES.

1. Find the series whose nth term is 2^{n-1}.

SOLUTION. Since n represents any term, the formula 2^{n-1} is true for any value of n. When $n=1$, $2^{n-1}=2^{1-1}=2^0$, or 1; hence the first term of the series is 1. When $n=2$, $2^{n-1}=2^{2-1}=2$; when $n=3$, $2^{n-1}=2^{3-1}=2^2$ or 4, etc.; hence the series is 1, 2, 4, 8, etc.

2. The sum of three numbers in geometrical progression is 14, and the sum of their squares is 84; what are the numbers?

SOLUTION.

Let x, $\sqrt{xy}$, and y represent the series. Then

By 1st condition,	$x+\sqrt{xy}+y=14;$	(1)
by 2d condition,	$x^2+xy+y^2=84.$	(2)
Dividing (2) by (1),	$x-\sqrt{xy}+y=6;$	(3)
adding (1) and (3),	$2x+2y=20;$	(4)
dividing by 2,	$x+y=10;$	(5)
subtracting (5) from (1),	$\sqrt{xy}=4.$	

3. Find the series whose nth term is 6^{n-1}. *Ans.* 1, 6, 36, etc.

4. Find the series whose nth term is 3^n. *Ans.* 3, 9, 27, etc.

5. Find the series whose nth term is $(2a)^n$. *Ans.* $2a$, $4a^2$, $8a^3$, etc.

6. Find the series in which the sum of n terms is $\frac{1}{2}(3^n-1)$. *Ans.* 1, 3, 9, 27, etc.

7. Find the series in which the sum of n terms is $a(2^n-1)$. *Ans.* a, $2a$, $4a$, etc.

8. Find the series in which the sum of n terms is $\frac{2^n-1}{2^{n-1}}$. *Ans.* 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, etc.

9. Find three numbers in geometrical progression such that their sum shall be 28 and the sum of their squares 336. *Ans.* 4, 8, 16.

10. The product of three numbers in geometrical progression is 216, and the sum of their squares is 364; required the numbers. *Ans.* 2, 6, 18.

11. The sum of the first and third of three numbers in geometrical progression is 10, and the sum of the cubes of the first and third is 520; required the numbers. *Ans.* 2, 4, 8.

12 There are three numbers in geometrical progression; the sum of the first and second is 32, and the sum of the second and third is 96; what are the numbers? *Ans.* 8, 24, 72.

13. The product of three numbers in geometrical progression is 216, and the sum of the squares of the extremes is 153; required the numbers. *Ans.* 3, 6, 12.

14. The sum of three numbers in geometrical progression is 39, and the sum of the extremes multiplied by the mean is 270; what are the numbers? *Ans.* 3, 9, 27.

15. There are three numbers in geometrical progression whose sum is 52, and the sum of their squares is 1456; what are the numbers? *Ans.* 4, 12, 36.

16. It is required to find three numbers in geometrical progression such that the sum of the first and last is 30, and the square of the mean is 144. *Ans.* 6, 12, 24.

17. Of four numbers in geometrical progression the sum of the first and third is 20, and the sum of the second and fourth is 60; what are the numbers? *Ans.* 2, 6, 18, 54.

18. Required to find four numbers in geometrical progression such that the sum of the first two is 10, and of the last two is 160. *Ans.* 2, 8, 32, 128.

In the 15th Example divide the 2d equation by the first. In the 18th, let x, xy, xy^2, xy^3 represent the numbers.

REVIEW QUESTIONS.

Define Progression. Arithmetical Progression. The Terms. Extremes. Means. Ascending Progression. Descending Progression. How many terms? State the four cases. The rule for each case. The formula for each case. How many cases are possible?

Define Geometrical Progression. What is the value of the ratio in an ascending progression? In a descending progression? State the three cases. Give the rule and formula for Case I. and Case II. Define an Infinite Series. State the rule for the sum of the terms of an infinite series. How many cases are possible?

MISCELLANEOUS EXAMPLES.

1. Add $(a+2b)x^n$ and $(2a-b)x^n$. *Ans.* $(3a+b)x^n$.

2. Subtract $3(m-an)$ from $3a(m-n)$. *Ans.* $3m(a-1)$.

3 Multiply a^{n-2} by $3a^{m+2}$. *Ans.* $3a^{m+n}$.

4. Multiply $a^{\frac{2}{3}}-b^{\frac{2}{3}}$ by $a^{\frac{2}{3}}+b^{\frac{2}{3}}$. *Ans.* $a^{\frac{4}{3}}-b^{\frac{4}{3}}$.

5. Divide a^nb^{m-n} by $a^{n-m}b^{-n}$. *Ans.* a^mb^m.

6. Divide $a^{\frac{3}{4}}-z^{\frac{3}{4}}$ by $a^{\frac{1}{4}}-z^{\frac{1}{4}}$. *Ans.* $a^{\frac{1}{2}}+a^{\frac{1}{4}}z^{\frac{1}{4}}+z^{\frac{1}{2}}$.

7. Divide a^n+b^n by $a^{\frac{n}{3}}+b^{\frac{n}{3}}$. *Ans.* $a^{\frac{2n}{3}}-a^{\frac{n}{3}}b^{\frac{n}{3}}+b^{\frac{2n}{3}}$.

8. Multiply $a^2+2ax-x^2$ by $a^2+2ax+x^2$.
Ans. $a^4+4a^3x+4a^2x^2-x^4$.

9. Divide $2a^4+27ab^3-81b^4$ by $a+3b$.
Ans. $2a^3-6a^2b+18ab^2-27b^3$.

10. Divide a^6-2a^3+1 by a^2-2a+1.
Ans. $a^4+2a^3+3a^2+2a+1$.

11. Value of $a-\{2b-(3c+2b-a)\}$. *Ans.* $3c$.

12. Value of $16-\{5-2x-[1-(3-x)]\}$. *Ans.* $9+3x$.

13. Value of $15x-\{4-[3-5x-(3x-7)]\}$. *Ans.* $7x+6$.

14. Value of $2x-[3y-\{4x-(5y-6x+7y)\}]$.
Ans. $12x-15y$.

15. Value of $a-[5b-\{a-(5c-\overline{2c-b}-4b)+2a-(a-\overline{2b+c})\}]$. *Ans.* $3a-2c$.

16. Prove $(a-b)^3+b^3-a^3=3ab(b-a)$.

17. Prove $(a^2+ab+b^2)^2-(a^2-ab+b^2)^2=4ab(a^2+b^2)$.

18. Prove $(a+b+c)^3-(a^3+b^3+c^3)=3(a+b)(b+c)(a+c)$.

19. Expand $(a^n-2)(a^n+2)(a^n+3)(a^n-3)$.
Ans $a^{4n}-13a^{2n}+36$.

20. Factor a^4+b^4; a^5-b^5; a^5+b^5; $a^{4n}-b^{4n}$.

21. Factor a^6-b^6; a^6+b^6; a^8-b^8; a^8+b^8; $a^{8n}-b^{8n}$.

22. Factor $a^2+9ab+20b^2$; $a^nc^n-b^nc^n+a^nd^n-b^nd^n$

23. Find the greatest common divisor of $a^2+8a+15$ and $a^2+9a+20$. *Ans.* $a+5$.

24. Find the greatest common divisor of $5(x^2-x+1)$, $4(x^6-1)$ and $2(x^3+1)$. *Ans.* x^2-x+1.

25. Find the least common multiple of x^3+1, x^3-1 and x^2-x+1. *Ans.* x^6-1.

26. Find the least common multiple of a^2-1, a^2+1, a^4+1 and a^8-1. *Ans.* a^8-1.

27. Value of $\dfrac{x^2+3x+2}{x^2+6x+5}$; $\dfrac{x^2+10x+21}{x^2-2x-15}$. *Ans.* $\dfrac{x+2}{x+5}$; $\dfrac{x+7}{x-5}$.

28. Value of $\dfrac{x^2+(a+b)x+ab}{x^2+(a+c)x+ac}$; $\dfrac{x^4+a^2x^2+a^4}{x^6-a^6}$.
Ans. $\dfrac{x+b}{x+c}$; $\dfrac{1}{x^2-a^2}$.

29. Value of $\dfrac{a-b}{b}+\dfrac{2a}{a-b}-\dfrac{a^3+a^2b}{a^2b-b^3}$. *Ans.* $\dfrac{b}{a-b}$.

30. Value of $\dfrac{a^2}{(a-b)(a-c)}+\dfrac{b^2}{(b-a)(b-c)}+\dfrac{c^2}{(c-a)(c-b)}$.
Ans. 1.

31. Value of $\left(b^n+\dfrac{a^{2n}}{b^n}\right)\left(a^n-\dfrac{b^{2n}}{a^n}\right)$. *Ans.* $\dfrac{a^{4n}-b^{4n}}{a^nb^n}$.

32. Value of $\dfrac{x^2+xy}{x^2+y^2}\times\left(\dfrac{x}{x-y}-\dfrac{y}{x+y}\right)$. *Ans.* $\dfrac{x}{x-y}$.

33. Value of $\left(\dfrac{x}{a}-\dfrac{a}{x}+\dfrac{y}{b}-\dfrac{b}{y}\right)\times\left(\dfrac{x}{a}-\dfrac{a}{x}-\dfrac{y}{b}+\dfrac{b}{y}\right)$.
Ans. $\dfrac{x^2}{a^2}+\dfrac{a^2}{x^2}-\dfrac{y^2}{b^2}-\dfrac{b^2}{y^2}$.

34. Value of $\dfrac{x^2+(a+c)x+ac}{x^2+(b+c)x+bc}\div\dfrac{x^2-a^2}{x^2-b^2}$. *Ans.* $\dfrac{x-b}{x-a}$.

35. Value of $\left(1+\dfrac{x}{y}\right)\left(1-\dfrac{x}{y}\right)\div\dfrac{y}{x^2+y^2}$. *Ans.* $\dfrac{y^4-x^4}{y^3}$.

36 Value of $\left(a^3-\dfrac{1}{a^3}\right)\div\left(a-\dfrac{1}{a}\right)$. *Ans.* $\dfrac{a^4+a^2+1}{a^2}$.

37. Value of $\left(\frac{a^2}{x^2}+1+\frac{x^2}{a^2}\right)\div\left(\frac{a}{x}-1+\frac{x}{a}\right)$. *Ans.* $\frac{a^2+ax+x^2}{ax}$.

38. Value of $\dfrac{x-1+\dfrac{6}{x-6}}{x-2+\dfrac{3}{x-6}}$. *Ans.* $\frac{x-4}{x-5}$.

39. Value of $1+\dfrac{x}{1+x+\dfrac{2x^2}{1-x}}$. *Ans.* $\frac{1+x}{1+x^2}$.

40. Value of $\dfrac{1}{1+\dfrac{a}{1+a+\dfrac{2a^2}{1-a}}}$. *Ans.* $\frac{1+a^2}{1+a}$.

41. Value of $\frac{3x-9}{x^2-7x+12}$, when $x=3$. *Ans.* -3.

42. Value of $\frac{a^2-x^2}{(a-x)^2}$, when $x=a$. *Ans.* ∞.

43. Value of $\frac{ax^2+ac^2-2acx}{bx^2-2bcx+bc^2}$, when $x=c$. *Ans.* $\frac{a}{b}$.

44. Given $\frac{x+1}{3}-\frac{3x-1}{5}=x-2$, to find x. *Ans.* $x=2$.

45. Given $\frac{5x-7}{2}-\frac{2x+7}{3}=3x-14$, to find x. *Ans.* $x=7$.

46. Given $\frac{3x-4}{2}-\frac{6x-5}{8}=\frac{3x-1}{16}$, to find x. *Ans.* $x=2\frac{1}{2}$.

47. Given $5x-[8x-3\{16-6x-(4-5x)\}]=6$. *Ans.* $x=5$.

48. Given $\frac{x+3}{2}-\frac{x-2}{3}=\frac{3x-5}{12}+\frac{1}{4}$, to find x. *Ans.* $x=28$.

49. Given $\frac{3+x}{3-x}-\frac{2+x}{2-x}-\frac{1+x}{1-x}=1$, to find x. *Ans.* $x=1\frac{1}{3}$.

50 Given $\frac{x^2-x+1}{x-1}+\frac{x^2+x+1}{x+1}=2x$, to find x. *Ans.* $x=0$.

51. Given $\frac{a(a-x)}{b}-\frac{b(b+x)}{a}=x$, to find x. *Ans.* $x=a-b$.

52. Given $\frac{a(x-a)}{b}+\frac{b(x-b)}{a}=x$, to find x. *Ans.* $x=a+b$.

53. Given $\frac{1}{x-a}-\frac{1}{x-b}=\frac{a-b}{x^2-ab}$, to find x. *Ans.* $x=\frac{2ab}{a+b}$.

54. Given $\left\{\begin{array}{l}2x+\frac{y-2}{5}=21\\ 4y+\frac{x-4}{6}=29\end{array}\right\}$. *Ans.* $\left\{\begin{array}{l}x=10,\\ y=7.\end{array}\right.$

55. Given $\left\{\begin{array}{l}\frac{x+y}{3}+\frac{y-x}{2}=9\\ \frac{x}{2}+\frac{x+y}{9}=5\end{array}\right\}$. *Ans.* $\left\{\begin{array}{l}x=6,\\ y=12.\end{array}\right.$

56. Given $\left\{\begin{array}{l}3x+9y=2.4\\ .21x-.06y=.03\end{array}\right\}$. *Ans.* $\left\{\begin{array}{l}x=.2,\\ y=.2.\end{array}\right.$

57. Given $\left\{\begin{array}{l}.3x+.125y=x-6\\ 3x-.5y=28-.25y\end{array}\right\}$. *Ans.* $\left\{\begin{array}{l}x=10,\\ y=8.\end{array}\right.$

58. Given $\left\{\begin{array}{l}x+y=a+b\\ bx+ay=2ab\end{array}\right\}$. *Ans.* $\left\{\begin{array}{l}x=a,\\ y=b.\end{array}\right.$

59. Given $\left\{\begin{array}{l}\frac{x}{a}+\frac{y}{b}=c\\ \frac{x}{b}-\frac{y}{a}=0\end{array}\right\}$. *Ans.* $\left\{\begin{array}{l}x=\frac{ab^2c}{a^2+b^2},\\ y=\frac{a^2bc}{a^2+b^2}.\end{array}\right.$

60. Given $\left\{\begin{array}{l}a(x+y)+b(x-y)=1\\ a(x-y)+b(x+y)=1\end{array}\right\}$. *Ans.* $\left\{\begin{array}{l}x=\frac{1}{a+b},\\ y=0.\end{array}\right.$

61. Given $\left\{\begin{array}{l}4x-5y+z=6\\ 7x-11y+2z=9\\ x+y+3z=12\end{array}\right\}$. *Ans.* $\left\{\begin{array}{l}x=2,\\ y=1,\\ z=3.\end{array}\right.$

62. Given $\left\{\begin{array}{l}y+z=a\\ x+z=b\\ x+y=c\end{array}\right\}$. *Ans.* $\left\{\begin{array}{l}x=\frac{1}{2}(b+c-a),\\ y=\frac{1}{2}(a-b+c),\\ z=\frac{1}{2}(a+b-c).\end{array}\right.$

63. Given $\left\{\begin{array}{l}y+z-x=a\\ x+z-y=b\\ x+y-z=c\end{array}\right\}$. *Ans.* $\left\{\begin{array}{l}x=\frac{1}{2}(b+c),\\ y=\frac{1}{2}(a+c),\\ z=\frac{1}{2}(a+b).\end{array}\right.$

64. Given $\left\{\begin{array}{l}\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1\\ \frac{x}{a}+\frac{y}{c}+\frac{z}{b}=1\\ \frac{x}{b}+\frac{y}{a}+\frac{z}{c}=1\end{array}\right\}$. *Ans.* $\left\{\begin{array}{l}x\\y\\z\end{array}\right\}=\frac{abc}{ab+bc+ac}$.

65. Given $\left\{\begin{array}{l}\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=3\\ \frac{a}{x}+\frac{b}{y}-\frac{c}{z}=1\\ \frac{2a}{x}-\frac{b}{y}-\frac{c}{z}=0\end{array}\right\}$. *Ans.* $\left\{\begin{array}{l}x=a,\\y=b,\\z=c.\end{array}\right.$

66. Value $\sqrt{\left(\frac{a^{m+n}}{a^{m-n}}\right)}$; 5th root of $\frac{a}{2}\sqrt[4]{\frac{a}{2}}$. *Ans.* a^n; $\frac{1}{2}\sqrt[4]{8a}$.

67. Value $(ax+by)^3+(ax-by)^3$. *Ans.* $2a^3x^3+6axb^2y^2$.

68. Value $(a+b+c+d)^2-(a-b+c-d)^2$. *Ans.* $4(a+c)(b+d)$.

69. Square root of $a^6-12a^5+60a^4-160a^3+240a^2-192a+64$. *Ans.* $a^3-6a^2+12a-8$.

70. Fourth root of $16a^4-96a^3y+216a^2y^2-216ay^3+81y^4$. *Ans.* $2a-3y$.

71. Cube root of $x^6+3x^5+6x^4+7x^3+6x^2+3x+1$. *Ans.* x^2+x+1.

72. Sixth root of $1+12a+60a^2+160a^3+240a^4+192a^5+64a^6$. *Ans.* $1+2a$.

73. Simplify $\{-(x^3)^{\frac{1}{2}}\}^{-\frac{1}{3}}\times\{-(-x)^{-3}\}^{\frac{1}{2}}$. *Ans.* $-x^{-2}$.

74. If $x=\frac{\sqrt{3}+1}{\sqrt{3}-1}$, and $y=\frac{\sqrt{3}-1}{\sqrt{3}+1}$, find the value of the expression x^2+xy+y^2. *Ans.* 15.

75. Given $\sqrt{(x-a)}=\frac{a}{\sqrt{(x+a)}}$, to find x. *Ans.* $x=\pm a\sqrt{2}$.

76. Given $\frac{x-4}{2x+1}=\frac{2x-1}{x+4}$, to find x. *Ans.* $x=\pm\sqrt{(-5)}$.

77. Given $\frac{\sqrt{x}-2}{\sqrt{x}+10}=\frac{\sqrt{x}-1}{\sqrt{x}+23}$, to find x. *Ans.* $x=9$.

78. Given $4x-\frac{12-x}{x-3}=22$, to find x. *Ans.* $x=6$, or $2\frac{1}{4}$.

79. Given $\frac{2x+11}{x}=5-\frac{x-5}{3}$, to find x. *Ans.* $x=11$, or 3.

80 Given $\frac{x-2}{x-3}-\frac{x-4}{x-1}=\frac{14}{15}$, to find x. *Ans.* $x=6$, or $2\frac{2}{7}$.

81. Given $\frac{x-3}{x-2}-\frac{x-1}{x-4}=-\frac{6}{5}$, to find x. *Ans.* $x=7$, or $2\frac{1}{3}$.

82. Given $\frac{x-1}{x-4}-\frac{x-3}{x-2}=\frac{11}{12}$, to find x. *Ans.* $x=8$, or $2\frac{4}{11}$.

83. Given $\frac{x+1}{x+2}+\frac{x-1}{x-2}=\frac{2x-1}{x-1}$. *Ans.* $x=4$, or 0.

84. Given $\frac{x-2}{x+2}+\frac{x+2}{x-2}=\frac{2(x+3)}{x-3}$. *Ans.* $x=1\frac{1}{3}$, or 0.

85. Given $\frac{a-\sqrt{a^2-x^2}}{a+\sqrt{a^2-x^2}}=b$. *Ans.* $x=\pm\frac{2a\sqrt{b}}{1+b}$.

86. Given $\left\{\begin{array}{l}x+y=4(x-y)\\xy=15\end{array}\right\}$. *Ans.* $\left\{\begin{array}{l}x=\pm 5,\\y=\pm 3.\end{array}\right.$

87. Given $\left\{\begin{array}{l}x^4+y^4=97\\4x^2=9y^2\end{array}\right\}$. *Ans.* $\left\{\begin{array}{l}x=\pm 3,\\y=\pm 2.\end{array}\right.$

88. Given $\left\{\begin{array}{l}x+y=3(x-y)\\x^3-y^3=56\end{array}\right\}$. *Ans.* $\left\{\begin{array}{l}x=4,\\y=2.\end{array}\right.$

89. Given $\frac{x+a}{x-a}-\frac{x-a}{x+a}=\frac{b+x}{b-x}-\frac{b-x}{b+x}$. *Ans.* $x=\pm\sqrt{(ab)}$.

90 Given $\left\{\begin{array}{l}x-y=1\\x^2-xy+y^2=21\end{array}\right\}$. *Ans.* $\left\{\begin{array}{l}x=5, \text{ or } -4,\\y=4, \text{ or } -5.\end{array}\right.$

91. Given $\left\{\begin{array}{l}3x+2y=5xy\\15x-4y=4xy\end{array}\right\}$. *Ans.* $\left\{\begin{array}{l}x=\frac{2}{3}, \text{ or } 0,\\y=\frac{2}{3}, \text{ or } 0.\end{array}\right.$

92. Given $\left\{\begin{array}{l} x+y=xy \\ ax=by \end{array}\right\}$. *Ans.* $\left\{\begin{array}{l} x=\dfrac{a+b}{a}, \text{ or } 0, \\ y=\dfrac{a+b}{b}, \text{ or } 0. \end{array}\right.$

93. Given $\left\{\begin{array}{l} \dfrac{x}{a}+\dfrac{y}{b}=2 \\ x^2+y^2=ax+by \end{array}\right\}$. *Ans.* $\left\{\begin{array}{l} x=a, \text{ or } \dfrac{2ab^2}{a^2+b^2}, \\ y=b, \text{ or } \dfrac{2a^2b}{a^2+b^2}. \end{array}\right.$

94. Given $\left\{\begin{array}{l} x^2+xy=28 \\ xy-y^2=3 \end{array}\right\}$. *Ans.* $\left\{\begin{array}{l} x=\pm 4, \text{ or } \pm\frac{7}{2}\sqrt{2}, \\ y=\pm 3, \text{ or } \pm\frac{1}{2}\sqrt{2}. \end{array}\right.$

95. Given $\left\{\begin{array}{l} x+y+\sqrt{x+y}=12 \\ x^2+y^2=41 \end{array}\right\}$. *Ans.* $\left\{\begin{array}{l} x=5, \text{ or } 4, \\ y=4, \text{ or } 5. \end{array}\right.$

96. Given $\left\{\begin{array}{l} x^2+y^2+2x+2y=23 \\ xy=6 \end{array}\right\}$. *Ans.* $\left\{\begin{array}{l} x=3, \text{ or } 2, \\ y=2, \text{ or } 3. \end{array}\right.$

97. Given $\left\{\begin{array}{l} x^2-y^2 : x-y :: 6:1 \\ xy=8 \end{array}\right\}$. *Ans.* $\left\{\begin{array}{l} x=4, \\ y=2. \end{array}\right.$

98. Given $\left\{\begin{array}{l} x^3-y^3 : (x-y)^3 :: 61:1 \\ xy=20 \end{array}\right\}$. *Ans.* $\left\{\begin{array}{l} x=5, \\ y=4. \end{array}\right.$

99. Given $\left\{\begin{array}{l} x^3+y^3 : x^3-y^3 :: 210:114 \\ xy^3=24 \end{array}\right\}$. *Ans.* $\left\{\begin{array}{l} x=3, \\ y=2. \end{array}\right.$

100. Given $\left\{\begin{array}{l} x^2+y^2=34 \\ x^2-y^2+\sqrt{(x^2-y^2)}=20 \end{array}\right\}$. *Ans.* $\left\{\begin{array}{l} x=\pm 5, \\ y=\pm 3. \end{array}\right.$

MISCELLANEOUS PROBLEMS.

1. A child was born in November, and on the tenth day of December he is as many days old as the month was on the day of his birth; when was he born? *Ans.* Nov. 20th.

2. After A has received \$10 from B he has as much money as B and \$6 more; and between them they have \$40; how much money had each at first? *Ans.* A, \$13; B, \$27.

3. A father has 6 sons, each of whom is 4 years older than his next younger brother, and the eldest is 3 times as old as the youngest; find their respective ages.

Ans. 10 yrs., 14 yrs., 18 yrs., 22 yrs., 26 yrs., 30 yrs.

4. Four men club to buy a set of ten-pins, but by clubbing with 2 more the expense of each is diminished \$1.75; what did the set cost? *Ans.* \$21.

5. A pudding consists of 2 parts of flour, 3 parts of raisins and 4 parts of suet; flour costs 3d. a pound, raisins 6d., and suet 8d.; find the cost of the several ingredients of the pudding when the whole cost is 2s. 4d. *Ans.* 3d., 9d., 1s. 4d.

6. A market-woman being asked what she paid for eggs, replied, "Six dozen eggs cost as many pence as you can buy eggs for eightpence." What was the price a dozen? *Ans.* 4d.

7. The tens' digit of a number is less by 2 than the units digit, and if the digits are inverted the new number is to the former as 7 is to 4; find the number. *Ans.* 24.

8. In paying two bills, one of which exceeded the other by $\frac{1}{3}$ of the less, the change out of a five-dollar note was $\frac{1}{2}$ the difference of the bills; find the amount of each bill. *Ans.* \$2; \$2$\frac{2}{3}$.

9. Two persons, A and B, own together 175 shares in a railway company: they agree to divide, and A takes 85 shares, while B takes 90 shares and pays \$250 to A; find the value of a share. *Ans.* \$100.

10. Find the fraction such that if you quadruple the numerator and add 3 to the denominator, the fraction is doubled; but if you add 2 to the numerator and quadruple the denominator, the fraction is halved. *Ans.* $\frac{2}{3}$.

11. How many sheep must a person buy at \$35 each that, after paying \$1.50 a score for folding them at night, he may gain \$394 by selling them at \$40 each? *Ans.* 80.

12. A colonel, on attempting to draw up his regiment in the form of a solid square, finds that he has 31 men over, and that he would require 24 men more in his regiment in order to increase the side of the square by 1 man; how many men were there in the regiment? *Ans.* 760.

13. In a certain weight of gunpowder the saltpetre com-

posed 6 pounds more than $\frac{1}{2}$ of the weight, the sulphur 5 pounds less than $\frac{1}{3}$, and the charcoal 3 pounds less than $\frac{1}{4}$; how many pounds were there of each of the three ingredients?

Ans. 18; 3; 3.

14. A cistern could be filled in 12 minutes by 2 pipes which run into it; and it could be filled in 20 minutes by 1 alone; in what time could it be filled by the other alone?

Ans. 30 min.

15. Divide the number 88 into 4 parts, such that the first increased by 2, the second diminished by 3, the third multiplied by 4, and the fourth divided by 5, may all be equal.

Ans. 10, 15, 3, 60.

16. A and B began to play together with equal sums of money: A first won $100, but afterward lost $\frac{1}{2}$ of all he then had, and then his money was $\frac{1}{2}$ as much as that of B; what money had each at first? *Ans.* $300.

17. A and B shoot by turns at a target: A puts 7 bullets out of 12 into the bull's-eye, and B puts 9 out of 12; between them they put in 32 bullets; how many shots did each fire?

Ans. 24.

18. A person buys a piece of land at $150 an acre, and by selling it in lots finds the value increased threefold, so that he clears $750, and retains 25 acres for himself; how many acres were there? *Ans.* 40.

19. A and B play at a game, agreeing that the loser shall always pay to the winner $1 less than $\frac{1}{2}$ the money the loser has: they commence with equal sums of money, and after B has lost the first game and won the second he has $2 more than A; how much had each at the commencement?

Ans. $6.

20. It is between 11 and 12 o'clock, and it is observed that the number of minute-spaces between the hands is $\frac{2}{3}$ of what it was ten minutes previously; find the time.

Ans. 20 min. of 12 o'clock.

21. A clock has two hands turning on the same centre: the swifter makes a revolution every 12 hours, and the slower every

16 hours; in what time will the swifter gain one complete revolution on the slower? *Ans.* 48 hrs.

22. An officer can form his men into a hollow square 4 deep, and also into a hollow square 8 deep: the front in the latter formation contains 16 men fewer than in the former formation; find the number of men. *Ans.* 640.

23. A certain number of 2 digits is equal to 4 times the sum of its digits; and if 18 be added to the number the digits are reversed; find the number. *Ans.* 24.

24. Prove that if 2 numbers differ by a, the difference of their squares equals $2a$ times their arithmetical mean.

25. If two integers differ by 2, show that the difference of their squares equals 4 times the integer between them.

26. The two digits which form a number change places on the addition of 9, and the sum of the original and the resulting number is 33; find the digits. *Ans.* 1 and 2.

27. Which is the greater, and how much—the square of a number, or the product of the number a unit less than it by the number a unit greater than it?

Ans. The square is 1 greater.

28. If a certain rectangular floor had been 2 feet broader and 3 feet longer, it would have been 64 square feet larger; but if it had been 3 feet broader and 2 feet longer, it would have been 68 square feet larger; find the length and breadth.

Ans. Length, 14 ft.; breadth, 10 ft.

29. When a certain number of 2 digits is doubled and increased by 36, the result is the same as if the number had been reversed and doubled, and then diminished by 36; also the number itself exceeds 4 times the sum of its digits by 3; find the number. *Ans.* 59.

30. A sets out from M to N, going $3\frac{1}{2}$ miles an hour; forty minutes afterward B sets out from N to M, going $4\frac{1}{2}$ miles an hour, and he goes half a mile beyond the middle point before he meets A; find the distance between M and N. *Ans.* 29 mi.

31. A person walked a certain distance at the rate of $3\frac{1}{2}$

miles an hour, and then ran part of the way back at the rate of 7 miles an hour, walking the remaining distance in 5 minutes: he was gone 25 minutes; how far did he run?

Ans. $\frac{7}{12}$ mi.

32. Two trains, 92 feet long and 84 feet long respectively, are moving with uniform velocities on parallel rails: when they move in opposite directions they are observed to pass each other in $1\frac{1}{2}$ seconds, but when they move in the same direction the faster train is observed to pass the other in 6 seconds; find the rate at which each train moves.

Ans. 30 mi. and 50 mi. per hour.

33. If the sum of 2 fractions is unity, show that the first, together with the square of the second, is equal to the second, together with the square of the first.

34. A man has a rectangular field containing 4 acres whose length is to its breadth as 8 : 5; required the length and breadth of the field. *Ans.* 32 rods; 20 rods.

35. The sum of two numbers is 17, and the less divided by the greater is to the greater divided by the less as 64 : 81; what are the numbers? *Ans.* 8 and 9.

36. There are two quantities whose product is a and quotient b; required the quantities. *Ans.* $\pm\sqrt{ab}$ and $\pm\sqrt{\frac{a}{b}}$.

37. A father gave to each of his children on New Year's day as many books as he had children: for each book he gave 12 times as many cents as there were children, and the cost of the whole was $15; how many children had he? *Ans.* 5.

38. A man bought a field whose length was to its breadth as 3 to 2; the price per acre was equal to the number of rods in the length of the field, and $4\frac{1}{2}$ times the distance around the field equaled the number of dollars that it cost; what was the length and breadth of the field? *Ans.* 60 rods; 40 rods.

39. A and B carried 100 eggs to market, and each received the same sum: if A had carried as many as B, he would have received 36 cents for them, and if B had carried only as many

as A, he would have received only 16 cents for them, how many eggs had each? *Ans.* A, 40; B, 60.

40. Several gentlemen made an excursion, each taking $484: each had as many servants as there were gentlemen, and the number of dollars which each had was 4 times the number of all the servants; how many gentlemen were there?

Ans. 11.

41. There is a rectangular field whose breadth is $\frac{4}{5}$ of the length. After laying out $\frac{1}{5}$ of the whole ground for a garden, it was found that there were left 400 square rods for mowing; required the length and breadth of the field.

Ans. 25 rods; 20 rods.

42. There are two square grass-plats, a side of one of which is 10 yards longer than a side of the other, and their areas are as 25 to 9; what are the lengths of the sides?

Ans. 25 yds.; 15 yds.

43. A farmer bought a number of sheep for $80: if he had bought 4 more for the same money, he would have paid $1 less for each; how many did he buy? *Ans.* 16 sheep.

44. A man divided 110 bushels of coal among a number of poor persons: if each had received 1 bushel more, he would have received as many bushels as there were persons; find the number of persons. *Ans.* 11.

45. A person bought a certain number of yards of cloth for $36, which he sold again at $4 per yard, and gained as much on the whole as 4 yards cost; find the number of yards.

Ans. 12.

46. A cistern can be supplied with water by two pipes, one of which would fill it 6 hours sooner than the other, and they both would fill it in 4 hours; how long will it take each pipe alone to fill it? *Ans.* 6 hrs.: 12 hrs.

47. The side of a square is 110 inches long; find the dimensions of a rectangle which shall have its perimeter 4 inches longer than that of the square, and its area 4 square inches less than that of the square.

Ans. Length, 126 in.; breadth, 96 in.

48. The third term of an arithmetical progression is 4 times the first term, and the sixth term is 17; required the first six terms of the series. *Ans.* 2, 5, 8, etc.

49. A square tract of land contains $\frac{1}{8}$ as many acres as there are rods in the fence enclosing it; required the length of the fence. *Ans.* 320 rods.

50. An English woman sells eggs at such a price that 10 more in half a crown's worth lowers the price threepence per score; required the price per score. *Ans.* 15d.

51. An English woman sells eggs at such a price that 10 fewer in half a crown's worth raises the price threepence per score; required the price per score. *Ans.* 12d

52. Find the side of a cube which shall contain 4 times as many solid units as there are linear units in the distance between its two opposite corners. *Ans.* $2\sqrt{3}$.

53. A grass-plat, 18 yards long and 12 wide, is surrounded by a border of flowers of uniform width; the areas of the grass-plat and border are equal; what is the width of the border? *Ans.* 3 yds.

54. A and B set out to meet each other from two places 320 miles apart: A traveled 8 miles a day more than B, and the number of days in which they met was equal to half the number of miles B went in a day; how far did each travel before they met? *Ans.* A, 192 mi.; B, 128 mi.

55. Two clerks, A and B, sent ventures in a ship bound to India: A gained $120, and at this rate he would have gained as many dollars on a hundred as B sent out. B gained $36, which was but one-fourth as much per cent. as A gained; how much money was sent out by each? *Ans.* A, $100; B, $120.

56. In a collection containing 27 coins, each silver coin is worth as many cents as there are copper coins; and each copper coin is worth as many cents as there are silver coins; and the whole is worth $1; how many coins are there of each sort? *Ans.* 2 of silver; 25 of copper.

57. The third term of an arithmetical progression is 18, and the seventh term is 30; find the sum of 17 terms. *Ans.* 612.

58. A man by selling a horse for 264 dollars gains as much per cent. as the horse cost him; required the cost of the horse. *Ans.* $120.

59 Two numbers are in the ratio of 4 to 5, but if one is increased and the other diminished by 10, the ratio of the resulting numbers is inverted; required the numbers. *Ans.* 40 and 50.

60. Show that the difference between the square of a number consisting of 2 digits, and the square of the number formed by changing the places of the digits, is divisible by 99.

61. A laborer, having built 105 rods of fence, found that, had he built 2 rods less a day, he would have been 6 days longer in completing the job; how many rods did he build per day? *Ans.* 7.

62. The common difference in an arithmetical progression is equal to 2, and the number of terms is equal to the second term; what is the first term if the sum is 35? *Ans.* 3.

63. If the sum of two fractions is unity, show that the first, together with the square of the second, is equal to the second together with the square of the first.

64. A man having a garden 12 rods long and 10 rods wide, wishes to make a gravel-walk half-way around it; what will be the width of the walk if it takes up $\frac{1}{10}$ of the garden? *Ans.* 9.234 ft.

65. If 8 gold coins and 9 silver ones are worth as much as 6 gold coins and 19 silver ones, find the ratio of the value of a silver coin to that of a gold coin. *Ans.* 1 : 5.

66. A rectangular picture is surrounded by a narrow frame which measures altogether 10 linear feet, and costs, at 12 cents per foot, 20 times as many cents as there are square feet in the picture; required the length and breadth of the picture. *Ans.* Length, 3 ft.; breadth, 2 ft.

67. A person walked to the top of a mountain at the rate of $2\frac{1}{3}$ miles an hour, and down the same way at the rate of $3\frac{1}{2}$ miles an hour, and was gone 5 hours; how far did he walk altogether? *Ans.* 14 mi.

68. A and B hired a pasture, into which A put 4 horses, and B as many as cost him 18 shillings a week. Afterward, B put in 2 additional horses, and found that he must pay 20 shillings a week; at what rate was the pasture hired?

Ans. 30s. a week.

69. Two partners, A and B, gained $700 by trade: A's money was in trade 3 months, and his gain was $300 less than his stock; and B's money, which was $250 more than A's, was in trade 5 months; required A's stock. *Ans.* $500.

70. A sets out for a certain place, and travels 1 mile the first day, 2 the second, 3 the third, and so on; five days afterward B sets out from the same place, and travels 12 miles a day; how long will A travel before he is overtaken by B?

Ans. 8 or 15 days.

71. A certain number of students go on an excursion: if there were five more, and each should pay $1 more, the expense would be $$61\frac{1}{2}$ more; but if there were 3 less, and each should pay $$1\frac{1}{2}$ less, the expense would be $42 less; required the number of students and the fare of each.

Ans. Number, 14; fare, $$8\frac{1}{2}$.

72. A traveler set out from a certain place, and went 1 mile the first day, 3 the second, 5 the third, and so on; after he had been gone three days a second traveler set out, and went 12 miles the first day, 13 the second, and so on; in how many days will the second overtake the first? *Ans.* In 2 or 9 days.

73. A set out from C toward D, and traveled 7 miles a day: after he had gone 32 miles, B set out from D toward C, and went every day $\frac{1}{19}$ of the whole journey; and after he had traveled as many days as he went miles in 1 day, he met A; required the distance between C and D.

Ans. 76 mi., or 152 mi.

74. Two trains start at the same time from two towns, and

each proceeds at a uniform rate toward the other town: when they meet it is found that one train has run 108 miles more than the other, and that if they continue to run at the same rate they will finish the journey in 9 and 16 hours respectively; required the distance between the towns and the rates of the trains.

Ans. Distance, 756 mi.; rate, 1st, 36 mi.; 2d, 27 mi. an hour.

75. A criminal having escaped from prison, traveled 10 hours before his escape was known; he was then pursued so as to be gained upon 3 miles an hour; after his pursuers had traveled 8 hours, they met an express going at the same rate as themselves, who had met the criminal 2 hours and 24 minutes before; in what time from the commencement of the pursuit will they overtake him? *Ans.* 20 hrs.

SUPPLEMENT.

SECTION X.

INEQUALITIES, INDETERMINATE AND HIGHER EQUATIONS.

INEQUALITIES.

384. An **Inequality** is an expression signifying that one quantity is greater or less than another; as $ax-b>c$.

385. The **First Member** of an inequality is the part on the left of the sign; the *second member* is the part on the right.

386. In treating inequalities the terms *greater* and *less* must be understood in their algebraic sense; thus,

1. *A negative quantity is regarded as less than* zero.

2. *Of two negative quantities, the greater is the one which has the less number of units.*

387. Two inequalities are said to exist *in the same sense* when the first member is greater in both or less in both; thus, $4>3$ and $6>5$.

388. Two inequalities are said to exist *in a contrary sense* when the first member is greater in one and less in the other; thus, $4>1$ and $3<5$.

389. The following examples will be readily solved by the student.

EXAMPLES.

1. Given $\frac{x}{2}+\frac{2x}{3}>\frac{3x}{4}+\frac{5}{3}$, to find a limit of x.

Solution.—Clearing of fractions, we have $6x+8x>9x+20$; transposing, etc., we have $5x>20$; hence, $x>4$.

2. Given $5x>\frac{3x}{2}+14$; find the limit of x. *Ans.* $x>4$.

3. Given $\frac{2x}{5}-\frac{2x}{3}<\frac{x}{4}-\frac{31}{12}$; find a limit of x. *Ans.* $x>5$.

4. Given $\frac{2x}{5}-\frac{2x}{3}>\frac{2x}{5}-2$; find a limit of x. *Ans.* $x<3$.

5. Given $4+\frac{x}{3}<7+\frac{x}{4}$; find a limit of x. *Ans.* $x<36$.

6. Given $2x+5y>16$ and $2x+y=12$; find the limits of x and y. *Ans.* $x<5\frac{1}{2}$; $y>1$.

7. Given $3x-5<2x+1$ and $4x+1>13+x$; find the value of x if integral. *Ans.* $x=5$.

8. Given $x+2y>18$ and $2x+3y=34$; find limits of x and y. *Ans.* $x<14$; $y>2$.

9. Twice an integer, plus 5, is less than 3 times the integer, plus 3, and 4 times the integer, less 4, is greater than 6 times the integer, minus 12; required the integer. *Ans.* 3.

10. Twice a number, plus 7, is not greater than 19; and three times the number, minus 5, is not less than 13; what is the number? *Ans.* 6.

THEOREMS IN INEQUALITIES.

1. Prove that the sum of the squares of two unequal quantities, a and b, is greater than twice their product.

For, $(a-b)^2$ is positive whatever the values of a and b;

hence, $(a-b)^2>0$;

or, $a^2-2ab+b^2>0$.

Hence, $a^2+b^2>2ab$.

2. Prove that $a^2+b^2+c^2>ab+ac+bc$.

By Theo. 1, $a^2+b^2>2ab$,
$a^2+c^2>2ac$,
$b^2+c^2>2bc$.

Hence, adding, $2a^2+2b^2+2c^2>2ab+2ac+2bc$.

Whence, $a^2+b^2+c^2>ab+ac+bc$.

3. Prove that $a+b>2\sqrt{ab}$, unless $a=b$.

4. Prove that $a^3b+ab^3>2a^2b^2$, unless $a=b$.

5. Prove that $3a^2+b^2>2a(a+b)$, unless $a=b$.

6. Prove that $a^3+1>a^2+a$, unless $a=1$.

7. Prove that the sum of any fraction and its reciprocal is greater than 2.

8. Prove that $\frac{a}{b^2}+\frac{b}{a^2}>\frac{1}{a}+\frac{1}{b}$, unless $a=b$.

9. Prove that $a-b>(\sqrt{a}-\sqrt{b})^2$, when $a>b$.

10. Prove that the ratio of a^2+b^2 to a^3+b^3 is less than the ratio of $a+b$ to a^2+b^2.

11. If $x^2=a^2+b^2$, and $y^2=c^2+d^2$, which is greater, xy or $ac+bd$, and xy or $ad+bc$? *Ans.* xy.

INDETERMINATE EQUATIONS.

390. An **Indeterminate Equation** is an equation in which the values of the unknown quantities are unlimited.

391. Thus, in the equation $2x+3y=35$, x and y may have different values; and if *any* value be assigned to one of the quantities, a *corresponding* value may be found of the other.

392. The *solution* of indeterminate equations, though the number of corresponding values is unlimited, is usually limited to finding *positive integral values.*

393. Of the several interesting cases that may arise we shall consider only two.

NOTE.—The treatment of indeterminate equations is usually called *Indeterminate Analysis.*

CASE I.

394. To find positive integral values of the unknown quantities in the equation.

1. Given $2x+3y=35$, to find positive integral values for x and y.

SOLUTION.

Given, $2x+3y=35$. (1)

Transposing, $2x=35-3y$. (2)

Whence, $x=\frac{35-3y}{2}=17-y+\frac{1-y}{2}$. (3)

Since y is an integer, $17-y$ is an *integer*; and since also x is an *integer*, $\frac{1-y}{2}$ is also an integer.

Let m represent this integer.

Then, $\frac{1-y}{2}=m$, (5)

and $1-y=2m$. (6)

Whence, $y=1-2m$. (7)

Sub. in (1), $2x+3-6m=35$. (8)

Whence, $x=16+3m$. (9)

In equation (7), for y to be *integral* and *positive*, m may be 0, or *negative*, but cannot be *positive*. In equation (9), for x to be *integral* and *positive*, m can be 0, or *positive*, or *negative* while less than 5. Hence m may be 0, -1, -2, -3, or -4.

Substituting these values of m in (7) and (9), we have

$x=16,\ 13,\ 10,\ 7,\ 4,\ 1.$

$y=1,\ 3,\ 5,\ 7,\ 9,\ 11.$

NOTE.—We shall use *Int.* to mean *an integer.*

2. Given $7x+9y=23$, to find positive integral values for x and y.

SOLUTION.

Here, $x=\frac{23-9y}{7}=3-y+\frac{2-2y}{7}$. (1)

Now, $\frac{2-2y}{7}$ must be an integer.

If we put $\frac{2-2y}{7}=m$, then $y=\frac{2-7m}{2}$, a *fractional* expression; but we wished to obtain an *integral* expression for the value of y. To avoid this difficulty, it is necessary to operate on $\frac{2-2y}{7}$, so as to make the coefficient of y a *unit.*

Since $\frac{2-2y}{7}$ is integral, any multiple of $\frac{2-2y}{7}$ is integral. Multiply, then, by some number that will make the coefficient of y contain the denominator with a *remainder of* 1.

Multiplying by 4, we have

$$\frac{2-2y}{7}\times 4=\frac{8-8y}{7}=1-y+\frac{1-y}{7}=Int.$$

Hence, $\frac{1-y}{7}=Int.=m$, and $y=1-7m$.

Sub. in (1), $x=2+9m$.

Here, for x and y to be positive integers, m can be only 0.

Substituting $m=0$, we have $x=2$ and $y=1$.

NOTE.—Other methods of reducing besides multiplying may be used, as may be seen in the following solution, the object being to obtain an integral form for the value of y.

3. Given $19x-14y=11$, to find integral values of x and y.

SOLUTION.

Here, $x=\frac{14y+11}{19}=Int.$; also $\frac{19y}{19}=Int.$

Subtracting, $\frac{19y}{19}-\frac{14y+11}{19}=\frac{5y-11}{19}=Int.$

Also, $\frac{5y-11}{19}\times 4=\frac{20y-44}{19}=y-2+\frac{y-6}{19}=Int.$

Hence, $\frac{y-6}{19}=m$, and $y=19m+6$;

and, $x=14m-5$.

Taking $m=0,\ 1,\ 2,\ 3$, etc., we have

$x=5,\ 19,\ 33,\ 47$, etc.,

$y=6,\ 25,\ 44,\ 58$, etc.

NOTES.—1. If the equation is in the form $ax+by=c$, the number of answers will be always limited, and in some cases a solution is impossible. The form $ax-by=\pm c$ will admit of an infinite number of answers.

2. If in $ax\pm by=c$, a and b have a common factor not common to c, there can be no integral solution.

4. How can 78 cents be paid with 5-cent and 3-cent pieces, and in how many ways?

SOLUTION. Let x = the number of 5-cent pieces, and y = the number of 3-cent pieces; then $5x+3y=78$, from which, by the method explained above, we find $x=15, 12, 9, 6, 3, 0$; and $y=1, 6, 11, 16, 21, 26$. Hence it can be paid in 5 ways when both kinds of pieces are used.

5. Given $2x+3y=25$, to find positive integral values for x and y. *Ans.* $x=2, 5, 8, 11$; $y=7, 5, 3, 1$.

6. Given $3x-8y=-16$, to find positive integral values for x and y. *Ans.* $x=8, 16$, etc.; $y=5, 8$, etc.

7. Given $8x+11y=49$, to find positive integral values for x and y. *Ans.* $x=2$; $y=3$.

8. Given $14x=5y+17$, to find the least positive integral values for x and y. *Ans.* $x=3$; $y=5$.

9. Given $19x-13y=17$, to find the least positive integral values for x and y. *Ans.* $x=5$; $y=6$.

10. Divide 100 into two such parts that one may be divided by 7, and the other by 11. *Ans.* 56 and 44.

11. In how many different ways may I pay a debt of £20 in half-guineas and half-crowns? *Ans.* 7 ways.

12. In how many ways can £100 be paid in guineas and crowns? *Ans.* 19.

13. What is the simplest way for a person who has only guineas to pay 10s. 6d. to another who has only half-crowns? *Ans.* In 3 guineas, receiving 21 half-crowns.

NOTE.—The *crown* equals 5 shillings, and the *guinea* equals 21 shillings.

CASE II.

395. To find the least integer which, divided by given numbers, shall leave given remainders.

1. Find the least integer which, being divided by 17, leaves a remainder of 7, and being divided by 26 leaves a remainder of 13.

SOLUTION.

Let x = the required integer.

Then, $\frac{x-7}{17}$ and $\frac{x-13}{26}$ = integers.

Let $\frac{x-7}{17}=m$; then, $x=17m+7$; substitute this in the second fraction,

$$\frac{17m+7-13}{26}=\frac{17m-6}{26}=Int.$$

Hence, $\frac{26m}{26}-\frac{17m-6}{26}$, or $\frac{9m+6}{26}=Int.$

And $\frac{9m+6}{26}\times 3=\frac{27m+18}{26}=m+\frac{m+18}{26}=Int.$

Whence, $\frac{m+18}{26}=Int.$, which we represent by n.

Then, $\frac{m+18}{26}=n$; hence $m=26n-18$.

Now, if $n=1$, we shall have $m=8$.

Hence, $x=17m+7=17\times 8+7=143$, the number.

ANOTHER SOLUTION. Let N = the number.

Then $\frac{N-7}{17}=x$ (1), and $\frac{N-13}{26}=y$ (2).

Whence, $N=17x+7$ (3), and $N=26y+13$ (4);

and $17x+7=26y+13$,

or, $17x-26y=6$.

Then find x and y, as in Case I., and substitute in (3) and (4).

2. Find the least number which, being divided by 3, 4, and 5, shall leave respectively the remainders 2, 3, and 4.

SOLUTION. Let x = the integer, then $\frac{x-2}{3}=Int.=m$; whence $x=3m+2$. Also, $\frac{x-3}{4}=Int.$; by substitution, $=\frac{3m-1}{4}=n$; whence, $m=n+\frac{n+1}{3}$. Placing $\frac{n+1}{3}=p$, we have $n=3p-1$; $m=4p-1$, and $x=12p-1$.

But, $\frac{x-4}{5}=Int.=\frac{12p-5}{5}=2p-1+\frac{2p}{5}$. Now, $\frac{2p}{5}=Int.$; hence, $\frac{2p}{5}\times 3=\frac{6p}{5}=Int.=p+\frac{p}{5}$. Put, $\frac{p}{5}=q$; then, $p=5q$, and $x=60q-1$. Now if $q=1$, $x=59$; if $q=2$, $x=119$, etc.

3. Find the least integer which, being divided by 6, shall leave the remainder 2, and divided by 13 shall leave the remainder 3. *Ans.* 68.

4. Find the least number which, being divided by 17 and 26, shall leave for remainders 7 and 13 respectively. *Ans.* 143.

5. What is the least integral number which, being divided by 3, 5, and 6, shall leave the respective remainders 1, 3, and 4? *Ans.* 28.

6. A man buys cows and colts for \$1000, giving \$19 for each cow and \$29 for each colt; how many did he buy of each? *Ans.* 45 cows and 5 colts, or 16 cows and 24 colts.

7. A farmer bought 100 animals for \$100: geese at 50 cents, pigs at \$3, and calves at \$10; how many were there of each kind? *Ans.* 94, 1, 5.

8. A farmer buys oxen, sheep, and ducks, 100 in all, for £100; required the number of each if the oxen cost £5, the sheep £1, and the ducks 1 shilling each. *Ans.* 19, 1, 80.

9. A lady bought 10 books of three different kinds for \$30; the first kind cost $\$4\frac{1}{4}$ each, the second $\$2\frac{1}{2}$ each, the third \$2 each; required the number of each kind. *Ans.* 4, 2, 4.

10. A market-woman finds by counting her eggs by threes she has 2 over, and counting by fives has 4 over; how many had she if the number is between 40 and 60? *Ans.* 44 or 59.

11. A boy has between 100 and 200 marbles; when he counts them by 12s, 10 remain, but when he counts them by 15s, 4 remain; how many marbles had he? *Ans.* 154.

12. A person wishes to purchase 20 animals for £20: sheep at 31 shillings, pigs at 11s., and rabbits at 1s. each; how many of each kind can he buy?

Ans. Sheep, 10, 11, 12. Pigs, 8, 5, 2. Rabbits, 2, 4, 6.

NOTE.—The solution of indeterminate equations of a higher degree is called *Diophantine Analysis.*

HIGHER EQUATIONS.

396. A **Cubic Equation** is an equation in which the highest power of the unknown quantity is the third power; as $x^3+4x^2+5x=10$.

397. A **Biquadratic Equation** is an equation in which the highest power of the unknown quantity is the fourth power; as $x^4+3x^3+4x^2+5x=13$.

398. The general form of a higher equation is $x^n+ax^{n-1}+bx^{n-2}+ \ldots tx+u=0$.

399. No general method of solving equations above the second degree, that is practicable, has yet been discovered.

NOTE.—Cardan's method for cubics and Ferrari's method for biquadratics fail in so many cases as not to be practically general. Abel has shown that a general solution above the fourth degree is impossible.

400. The following PRINCIPLES, which are demonstrated in higher algebra, may often be used in finding the roots of an equation.

PRIN. 1. *If a is a root of an equation (the unknown quantity being x), the equation is divisible by $x-a$.*

Thus, if 2 is a root, the equation is divisible by $x-2$; if -2 is a root, the equation is divisible by $x-(-2)$, or $x+2$.

PRIN. 2. *The coefficient of the second term, ax^{n-1}, is the sum of the roots, with their signs changed.*

PRIN. 3. *The term independent of x, when in the first member, is the product of the roots, with their signs changed.*

Thus, a cubic equation, in which a, b, and c are the roots, is equivalent to $(x-a)(x-b)(x-c)=0$; and when developed is $x^3-(a+b+c)x^2+(ab+ac+bc)x-(abc)=0$.

NOTE.—Principles 2 and 3 will often enable us to conjecture the roots of an equation; and Prin. 1 will enable us to test any supposed root.

SOLUTION OF CUBIC EQUATIONS.

401. Cubic Equations can often be solved by special artifices, a few of which will be given.

CASE I.

402. Solution by inspection and the application of the above principles.

1. Given $x^3-6x^2+11x=6$, to find x.

SOLUTION. The factors of 6 are 1, 2, and 3; and their sum equals the coefficient of the 2d term; hence we may suppose one of these factors, as 3, to be a root of the equation. Transposing 6 to the first member, and dividing by $x-3$, we have $x^2-3x+2=0$; therefore, 3 is one root, and solving $x^2-3x=-2$, we find 1 and 2 to be the other roots.

2. Solve $x^3-9x^2+26x=24$. *Ans.* $x=2$, 3, and 4.
3. Solve $x^3-11x^2+38x=40$. *Ans.* $x=2$, 4, 5.
4. Solve $x^3-3x^2-10x=-24$. *Ans.* $x=2$, -3, 4.
5. Solve $x^3+4x^2+x=6$. *Ans.* $x=1$, -2, -3.
6. Solve $x^3-4x^2-7x=-10$. *Ans.* $x=1$, -2, 5.
7. Solve $x^3-2x^2+4x=8$. *Ans.* $x=2$, $2\sqrt{-1}$, $-2\sqrt{-1}$.
8. Solve $x^3-7x^2+16x=10$. *Ans.* $x=1$, $3+\sqrt{-1}$, $3-\sqrt{-1}$.

CASE II.

403. Solution by making both members a perfect cube.

1. Given $x^3+3x^2+3x=7$.

SOLUTION.

Given $\quad x^3+3x^2+3x=7. \quad (1)$

Adding 1, $\quad x^3+3x^2+3x+1=8. \quad (2)$

Whence $\quad x+1=2$, or $x=1$.

Dividing Eq. 1 by $x-1$, we have $x^2+4x+7=0$.

Whence, $x=-2+\sqrt{-3}$ and $-2-\sqrt{-3}$.

2. Solve $x^3-3x^2+3x=9$. *Ans.* $x=3, +\sqrt{-3}, -\sqrt{-3}$.

3. Solve $x^3+6x^2+12x=-16$.
Ans. $x=-4, -1+\sqrt{-3}, -1-\sqrt{-3}$.

4. Solve $x^3+9x^2+27x=-35$.
Ans. $x=-5, -2+\sqrt{-3}, -2-\sqrt{-3}$.

5. Solve $x^3-9x^2+27x=91$.
Ans. $x=7, 1+2\sqrt{-3}, 1-2\sqrt{-3}$.

CASE III.

404. Solution when the equation is readily factored.

1. Given $x^3-6x=-4$.

SOLUTION.

Given	$x^3-6x=-4$.
Whence	$x^3-4x=2x-4$.
And	$x(x^2-4)=2(x-2)$.

Since this is divisible by $x-2$, $x-2=0$, or $x=2$.

Dividing by $x-2$, we have $x(x+2)=2$ or $x^2+2x=2$,

whence, $x=-1\pm\sqrt{3}$.

2. Solve $x^3-3x=-2$. *Ans.* $x=1, 1, -2$.
3. Solve $x^3-3x=2$. *Ans.* $x=-1, -1, 2$.
4. Solve $x^3-7x=-6$. *Ans.* $x=1, 2, -3$.
5. Solve $x^3-a^2x-x=-a$. *Ans.* $x=a, \frac{1}{2}(-a\pm\sqrt{a^2+4})$.
6. Solve $x^3-x^2-2x=-2$. *Ans.* $x=1, +\sqrt{2}, -\sqrt{2}$.
7. Solve $x^3-2x=\sqrt{3}$. *Ans.* $x=\sqrt{3}, \frac{1}{2}(-\sqrt{3}\pm\sqrt{-1})$.
8. Solve $3x^3-7x^2-7x=-3$. *Ans.* $x=\frac{1}{3}, 3, -1$.

CASE IV.

405. Solution by reducing to a biquadratic, and then changing to make both sides squares.

1. Given $x^3-7x=-6$, to find x.

SOLUTION.

Given	$x^3-7x=-6$.	(1)
Whence	$x^4-7x^2=-6x$.	(2)
Add $4x^2$	$x^4-3x^2=4x^2-6x$.	
Complete square,	$x^4-3x^2+\left(\frac{3}{2}\right)^2=4x^2-6x+\left(\frac{3}{2}\right)^2$	
Hence	$x^2-\frac{3}{2}=2x-\frac{3}{2}$, or $-2x+\frac{3}{2}$.	
Whence	$x=2$, or $x^2+2x=3$.	
	$\therefore x=1$, or -3.	

2. Solve $x^3+3x=14$. *Ans.* $x=2, -1\pm\sqrt{-6}$.
3. Solve $x^3-12x=16$. *Ans.* $x=4, -2, -2$.
4. Solve $x^3+5x=6$. *Ans.* $x=1, \frac{1}{2}(-1\pm\sqrt{-23})$.
5. Solve $x^3-4x=48$. *Ans.* $x=4, 2(-1\pm\sqrt{-2})$.
6. Solve $x^3-13x=-12$. *Ans.* $x=1, 3, -4$.
7. Solve $x^3+6x=20$. *Ans.* $x=2, -1\pm3\sqrt{-1}$.

SOLUTION OF BIQUADRATIC EQUATIONS.

406. Biquadratic Equations may often be solved by special artifices, a few of which we present.

CASE I.

407. Solution by inspection and applying the principles of equations.

1. Given $x^4-10x^3+35x^2-50x=-24$, to find x.

SOLUTION. We notice that $24=1\times2\times3\times4$, and $10=1+2+3+4$; hence, we presume that some of these factors are roots of the equation. Dividing by $x-1$, we see that the equation is divisible by $x-1$; hence 1 is a root; and in a similar way we find that 2, 3, and 4 are roots.

2. Solve $x^4-5x^3+5x^2+5x=6$. *Ans.* $x=1, -1, 2, 3$.
3. Solve $x^4+x^3-7x^2-x=-6$. *Ans.* $x=1, -1, 2, -3$.
4. Solve $x^4-6x^3-x^2+54x=72$. *Ans.* $x=2, 3, -3, +4$.
5. Solve $x^4-4x^3-9x^2+16x=-20$.
Ans. $x=-1, 2, -2, 5$.
6. Solve $x^4-4x^3-8x^2+4x=-7$. *Ans.* $1, -1, 2\pm\sqrt{11}$.

CASE II.

408. Solution by factoring when the factors can be readily obtained.

1. Given $x^4+4x^3-8x=32$, to find x.

SOLUTION.

Given	$x^4+4x^3-8x=32.$
Transposing,	$x^4+4x^3=8x+32.$
Factoring.	$x^3(x+4)=8(x+4),$
or,	$(x^3-8)(x+4)=0.$
Whence,	$x^3-8=0$ and $x+4=0,$
and	$x=-4$ or $2.$
Dividing,	x^3-8 by $x-2$, we obtain a quadratic,
from which	$x=-1\pm\sqrt{-3}.$

2. Solve $x^4-2x^3-x=-2$. *Ans.* $x=1,\ 2,\ \frac{1}{2}(-1\pm\sqrt{-3})$.
3. Solve $x^4-3x^3-8x=-24$. *Ans.* $x=2,\ 3,\ -1\pm\sqrt{-3}$.
4. Solve $x^4-ax^3-n^3x=-an^3$.

Ans. $x=a,\ n,\ \frac{n}{2}(-1\pm\sqrt{-3})$.

5. Solve $x^4+3x^3-3x=9$.

Ans. $x=-3;\ \sqrt[3]{3},\ \frac{\sqrt[3]{3}}{2}(-1\pm\sqrt{-3})$.

CASE III.

409. Solution by reducing to a quadratic form.

1. Given $x^4+2x^3-3x^2-4x=5$, to find x.

SOLUTION.

Given	$x^4+2x^3-3x^2-4x=5.$
Whence,	$(x^2+x)^2-4(x^2+x)+4=9,$
or	$x^2+x-2=\pm 3.$

From which x can be found.

2. Solve $x^4-4x^3+8x^2-8x=12$. *Ans.* $1\pm\sqrt{3},\ 1\pm\sqrt{-5}$.
3. Solve $x^4+2x^3-7x^2-8x=-12$. *Ans.* $1,\ 2,\ -2,\ -3$.
4. Solve $x^4+2x^3-3x^2-4x=-4$. *Ans.* $1,\ 1,\ -2,\ -2$.
5. Solve $x^4-6x^3+11x^2-6x=8$.

Ans. $\frac{1}{2}(3\pm\sqrt{17}),\ \frac{1}{2}(3\pm\sqrt{-7})$.

CASE IV.

410. Solution by reducing both members to a binomial square.

1. Solve $x^4-6x^3+12x^2-10x=-3$.

SOLUTION.

Given	$x^4-6x^3+12x^2-10x=-3.$
Whence,	$(x^2-3x)^2+3x^2-10x=-3,$
	$+4x^2-12x=x^2+2x-3,$
	$+4(x^2-3x)+4=x^2+2x+1;$
or	$(x^2-3x)^2+4(x^2-3x)+4=x^2+2x+1.$
Whence,	$(x^2-3x)+2=x+1.$
Whence,	$x=1;\ 1;\ 1;\ 3.$

2. Solve $x^4-4x^3-19x^2+46x=-120$.

Ans. $x=-2,\ -3,\ 4,\ 5$.

3. Solve $x^4+4x^3-x^2-16x=12$.

Ans. $x=-1,\ -2,\ 2,\ -3$.

4. Solve $x^4-9x^3+30x^2-46x=-24$.

Ans. $x=1,\ 4,\ 2\pm\sqrt{-2}$.

5. Solve $x^4+4x^3-6x^2+4x=7$. *Ans.* $x=\pm\sqrt{-1},-2\pm\sqrt{11}$.
6. Solve $x^4-12x^3+48x^2-68x=-15$. *Ans.* $x=3,\ 5,\ 2\pm\sqrt{3}$.

NOTE.—Cubics and biquadratics, when any of their roots are integral, can usually be solved by artifices similar to those we have explained.

The more general methods of finding approximate roots of numerical equations are those of *Double Position*, *Newton's Method of Approximation*, and *Horner's Method*, for an explanation of which the student is referred to works on Higher Algebra.

RECIPROCAL EQUATIONS.

411. A **Reciprocal Equation** is one in which the reciprocal of x may be substituted for x without altering the equation.

412. Thus, in $x^4-3x^3+4x^2-3x+1=0$, if we substitute $\frac{1}{x}$ for x, we shall obtain the same equation.

NOTES.—1. Such equations are also called *recurring equations*, because the coefficients recur in the same order.

2. It can be shown that a reciprocal equation of an *odd* degree is divisible by $x-1$ or $x+1$, according as the last term is *positive* or *negative*.

3. Also, a reciprocal equation of an *even* degree is divisible by x^2-1 when its last term is *positive*.

EXAMPLES.

1. Solve $x^4-3x^3+4x^2-3x+1=0$.

SOLUTION.

Given $x^4-3x^3+4x^2-3x+1=0.$

Divide by x^2, $x^2-3x+4-\frac{3}{x}+\frac{1}{x^2}=0.$

Whence $x^2+\frac{1}{x^2}-3\left(x+\frac{1}{x}\right)+4=0.$

Or $\left(x+\frac{1}{x}\right)^2-3\left(x+\frac{1}{x}\right)=-2.$

Complete the sq., $\left(x+\frac{1}{x}\right)^2-3\left(x+\frac{1}{x}\right)+\frac{9}{4}=\frac{1}{4}.$

Extract the root $\left(x+\frac{1}{x}\right)-\frac{3}{2}=\pm\frac{1}{2}.$

Whence $x=1,\ 1,\ \frac{1}{2}(1\pm\sqrt{-3}.$

NOTE.—It is sometimes simpler to substitute some other quantity, as z, for $x+\frac{1}{x}$, and find the value of x from that of z.

2. Solve $x^4+x^3+x+1=0$. *Ans.* $x=-1,\ -1,\ \frac{1}{2}(1\pm\sqrt{-3}$.

3. Solve $x^4-5x^3+6x^2-5x+1=0$.
Ans. $x=2\pm\sqrt{3},\ \frac{1}{2}(1\pm\sqrt{-3})$.

4. Solve $x^4-10x^3+26x^2-10x+1=0$.
Ans. $x=3\pm2\sqrt{2},\ 2\pm\sqrt{3}$.

5. Solve $x^4-3x^3+3x+1=0$. *Ans.* $x=1\pm\sqrt{2},\ \frac{1}{2}(1\pm\sqrt{5})$.

6. Solve $x^4-\frac{5}{2}x^3+2x^2-\frac{5}{2}x+1=0$. *Ans.* $x=2,\ \frac{1}{2},\ \pm\sqrt{-1}$.

7. Solve $x^4-3x^3+3x-1=0$. *Ans.* $x=\pm1,\ \frac{1}{2}(3\pm\sqrt{5})$.

8. Solve $x^2+x^{-2}+x+x^{-1}-4=0$.
Ans. $x=1,\ 1,\ \frac{1}{2}(-3\pm\sqrt{5})$.

NOTES.—1. In equation 7 divide by x^2-1, and then reduce and find the values of x.

2. It may be readily shown that any two corresponding pair of roots are reciprocals of one another.

SECTION XI.

EXPONENTS AND LOGARITHMS.

THEORY OF EXPONENTS.

413. AN **Exponent** denotes the power of a quantity or the number of times it is used as a factor.

Thus a^3 means $a\times a\times a$, or a used as a factor three times; and a^n means $a\times a\times a\times\ .\ .\ .\ .$ to n factors, or a used as a factor n times.

414. By the original conception of a *power* the exponent n could be conceived only as a *positive integer*, and the rules for multiplication, division, etc. were all based on this conception.

415. Subsequently it was seen that division gave rise to *negative exponents* and evolution to *fractional exponents*, and that these could be used the same as positive integral exponents.

NOTE.—We shall now give a complete logical discussion of the subject, assuming only the definition of an exponent and the rules of addition and subtraction.

POSITIVE EXPONENTS

PRIN. 1. When m and n are positive integers, $a^m\times a^n=a^{m+n}$.

For $a^m=a\times a\times a\times\ .\ .\ .\ .$ to m factors. (Def.)

And $a^n=a\times a\times a\times\ .\ .\ .\ .$ to n factors. (Def.)

Hence $a^m\times a^n=a\times a\times\ .\ .\ .\ .\ \times a\times a\times a\ .\ .\ .\ .$ to $m+n$ factors, which by the definition equals a^{m+n}.

PRIN. 2. When m and n are positive integers, and m is greater than n, $a^m\div a^n=a^{m-n}$.

EXAMPLES.

1. Solve $x^4-3x^3+4x^2-3x+1=0$.

SOLUTION.

Given $x^4-3x^3+4x^2-3x+1=0$.

Divide by x^2, $x^2-3x+4-\frac{3}{x}+\frac{1}{x^2}=0$.

Whence $x^2+\frac{1}{x^2}-3\left(x+\frac{1}{x}\right)+4=0$.

Or $\left(x+\frac{1}{x}\right)^2-3\left(x+\frac{1}{x}\right)=-2$.

Complete the sq., $\left(x+\frac{1}{x}\right)^2-3\left(x+\frac{1}{x}\right)+\frac{9}{4}=\frac{1}{4}$.

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Whence $x=1, 1, \frac{1}{2}(1\pm\sqrt{-3}$.

NOTE.—It is sometimes simpler to substitute some other quantity, as z, for $x+\frac{1}{x}$, and find the value of x from that of z.

2. Solve $x^4+x^3+x+1=0$. *Ans.* $x=-1, -1, \frac{1}{2}(1\pm\sqrt{-3}$.
3. Solve $x^4-5x^3+6x^2-5x+1=0$. *Ans.* $x=2\pm\sqrt{3}, \frac{1}{2}(1\pm\sqrt{-3})$.
4. Solve $x^4-10x^3+26x^2-10x+1=0$. *Ans.* $x=3\pm 2\sqrt{2}, 2\pm\sqrt{3}$.
5. Solve $x^4-3x^3+3x+1=0$. *Ans.* $x=1\pm\sqrt{2}, \frac{1}{2}(1\pm\sqrt{5})$.
6. Solve $x^4-\frac{5}{2}x^3+2x^2-\frac{5}{2}x+1=0$. *Ans.* $x=2, \frac{1}{2}, \pm\sqrt{-1}$.
7. Solve $x^4-3x^3+3x-1=0$. *Ans.* $x=\pm 1, \frac{1}{2}(3\pm\sqrt{5})$.
8. Solve $x^2+x^{-2}+x+x^{-1}-4=0$. *Ans.* $x=1, 1, \frac{1}{2}(-3\pm\sqrt{5})$.

NOTES.—1. In equation 7 divide by x^2-1, and then reduce and find the values of x.

2. It may be readily shown that any two corresponding pair of roots are reciprocals of one another.

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Hence $a^m\times a^n=a\times a\times \ldots \times a\times a\times a \ldots$ to $m+n$ factors, which by the definition equals a^{m+n}.

PRIN. 2. When m and n are positive integers, and m is greater than n, $a^m\div a^n=a^{m-n}$.

For $a^{m-n} \times a^n = a^{m-n+n}$. (Prin. 1.)

Reducing, $a^{m-n} \times a^n = a^m$.

Dividing by a^n, $\frac{a^m}{a^n} = a^{m-n}$.

PRIN. 3. When m and n are positive integers, $(a^m)^n$ equals a^{mn}.

For $(a^m)^n = a^m \times a^m \times a^m \times \ldots$ to n factors;

But $a^m = a \times a \times a \times \ldots$ to m factors.

Hence $(a^m)^n = a \times a \times a \times \ldots$ to $m \times n$ factors,

which is indicated thus, a^{mn}.

PRIN. 4. When m and n are positive integers, $\sqrt[n]{a^m} = a^{\frac{m}{n}}$.

For, since in raising a quantity to the nth power we multiply the exponent by n, in extracting the nth root we must divide the exponent by n; whence the nth root of a^m is $a^{m \div n}$, or $a^{\frac{m}{n}}$.

NEGATIVE EXPONENTS.

416. The **Negative Exponent** arises from division when the exponent of the divisor is greater than the exponent of the dividend.

PRIN. 5. Prove that $a^{-n} = \frac{1}{a^n}$.

For $a^{m-n} = \frac{a^m}{a^n}$. Prin. 2.

Dividing by a^m, $a^{-n} = \frac{1}{a^n}$.

This may also be shown as follows:

Now, $a^m \times a^{-n} = a^{m-n} = \frac{a^m}{a^n}$. Prin. 2.

Hence, $a^{-n} = \frac{1}{a^n}$. Div. by a^m.

PRIN. 6. Prove that $a^m \div a^n = a^{m-n}$ when m is less than n.

Now, $a^m \div a^n = \frac{a^m}{a^n} = \frac{1}{a^{n-m}}$.

But, $\frac{1}{a^{n-m}} = a^{-(n-m)} = a^{m-n}$.

PRIN. 7. Prove that $a^m \times a^n = a^{m+n}$ when one or both exponents are negative.

First, suppose either exponent, as n, is negative. Let $n = -s$.

Then $a^m \times a^n = a^m \times a^{-s} = a^m \times \frac{1}{a^s} = \frac{a^m}{a^s} = a^{m-s}$.

Substituting n for $-s$, $a^{m-s} = a^{m+n}$; hence $a^m \times a^n = a^{m+n}$.

Second, suppose both exponents are negative. Let $m = -r$ and $n = -s$.

Then, $a^m \times a^n = a^{-r} \times a^{-s} = \frac{1}{a^r} \times \frac{1}{a^s} = \frac{1}{a^{r+s}} = a^{-(r+s)} = a^{-r-s}$.

Substituting, $a^{-r-s} = a^{m+n}$; hence $a^m \times a^n = a^{m+n}$.

PRIN. 8. Prove that $a^m \div a^n = a^{m-n}$ when either or both exponents are negative.

First, suppose either exponent, as n, is negative. Let $n = -s$.

Then, $a^m \div a^n = a^m \div a^{-s} = a^m \div \frac{1}{a^s} = a^m \times \frac{a^s}{1} = a^m a^s = a^{m+s}$.

Substituting, $a^{m+s} = a^{m-n}$.

Second, suppose both exponents are negative. Let $m = -r$ and $n = -s$.

Then, $a^m \div a^n = a^{-r} \div a^{-s} = \frac{1}{a^r} \div \frac{1}{a^s} = \frac{1}{a^r} \times a^s = \frac{a^s}{a^r} = a^{s-r}$.

Substituting, $a^{s-r} = a^{-n-(-m)} = a^{m-n}$.

PRIN. 9. Prove that $(a^m)^n = a^{mn}$ when one or both exponents are negative.

First, suppose m is negative, and let $m = -r$.

Then, $(a^m)^n = (a^{-r})^n = \left(\frac{1}{a^r}\right)^n = \frac{1}{a^{rn}} = a^{-rn} = a^{mn}$.

Second, suppose n is negative, and let $n = -p$.

Then, $(a^m)^n = (a^m)^{-p} = \frac{1}{(a^m)^p} = \frac{1}{a^{mp}} = a^{-mp} = a^{mn}$.

Third, suppose m and n are both negative, and let $m = -r$ and $n = -p$.

Then, $(a^m)^n = (a^{-r})^{-p} = \frac{1}{(a^{-r})^p} = \frac{1}{a^{-rp}} = a^{rp} = a^{-m \times -n} = a^{mn}$.

PRIN. 10. Prove that $\sqrt[n]{a^m}=a^{\frac{m}{n}}$, when either m or n, or both, are negative.

First, suppose m is negative, and let $m=-r$.

Then $\sqrt[n]{a^m}=\sqrt[n]{a^{-r}}=\sqrt[n]{\frac{1}{a^r}}=\frac{1}{a^{\frac{r}{n}}}=a^{-\frac{r}{n}}=a^{\frac{m}{n}}$.

Second, suppose n is negative, and $n=-r$. Let $x=\sqrt[n]{a^m}$.

Then $x^n=a^m$, and $x^{-r}=a^m$, or $\frac{1}{x^r}=a^m$; $x^r=\frac{1}{a^m}$; $x=\frac{1}{a^{\frac{m}{r}}}=a^{-\frac{m}{r}}=a^{\frac{m}{n}}$.

Third, suppose both m and n are negative; let $m=-p$, $n=-r$, and $x=\sqrt[n]{a^m}$.

Then $x^n=a^m$, $x^{-r}=a^{-p}$, $\frac{1}{x^r}=\frac{1}{a^p}$, $x^r=a^p$, $x=a^{\frac{p}{r}}=a^{\frac{m}{n}}$.

NOTE.—No practical significance is attached to a *negative index* of a root; but the form is a possible one, and the above demonstration proves the principle to be general.

417. Thus we see that whether m and n are positive or negative integers, we have the following:

I. $a^m\times a^n=a^{m+n}$. III. $(a^m)^n=a^{mn}$.

II. $a^m\div a^n=a^{m-n}$. IV. $\sqrt[n]{a^m}=a^{\frac{m}{n}}$.

FRACTIONAL EXPONENTS.

418. A **Fractional Exponent** arises from evolution by dividing the exponent of the power by the index of the root, when the former is not a multiple of the latter.

419. We shall show the meaning of the fractional exponent and prove a few principles to be used in showing the universality of the fundamental rules.

PRIN. 11. Prove that $(a^{\frac{m}{n}})^p=a^{\frac{mp}{n}}$ when $\frac{m}{n}$ and p are positive or negative.

First, suppose p is positive, $\frac{m}{n}$ being either positive or negative.

Raising $a^{\frac{m}{n}}$ to the pth power, we have $a^{\frac{m}{n}}\times a^{\frac{m}{n}}\times a^{\frac{m}{n}}\times\ldots$ to p factors, or $a^{\frac{m}{n}+\frac{m}{n}+\frac{m}{n}+\ldots\text{ to } p \text{ terms}}=a^{\frac{m}{n}\times p}=a^{\frac{mp}{n}}$.

Second, suppose p is negative, and let $p=-s$.

Then $(a^{\frac{m}{n}})^p=(a^{\frac{m}{n}})^{-s}=\frac{1}{(a^{\frac{m}{n}})^s}=\frac{1}{a^{\frac{ms}{n}}}=a^{-\frac{ms}{n}}=a^{-\frac{m}{n}\times -p}=a^{\frac{mp}{n}}$.

PRIN. 12. Prove that $a^{\frac{m}{n}}=\sqrt[n]{a^m}$, m and n being either positive or negative.

First, suppose m is positive or negative, n being positive.

By Prin. 11, $(a^{\frac{m}{n}})^n=a^m$; extract nth root, $a^{\frac{m}{n}}=\sqrt[n]{a^m}$.

Second, suppose n is negative, and let $n=-s$.

Then, $(a^{\frac{m}{n}})^n=(a^{-\frac{m}{s}})^{-s}=\frac{1}{(a^{-\frac{m}{s}})^s}=\frac{1}{a^{-m}}=a^m$.

Hence, $(a^{\frac{m}{n}})^n=a^m$, and $a^{\frac{m}{n}}=\sqrt[n]{a^m}$.

COR. Hence, $a^{\frac{1}{n}}=\sqrt[n]{a}$, or $\sqrt[n]{a}=a^{\frac{1}{n}}$; also $(a^{\frac{1}{n}})^n=a^{\frac{n}{n}}$, or a.

NOTE.—This principle was derived under the previous article, but is here proved by another process of reasoning.

PRIN. 13. Prove that $\sqrt[n]{a^m}=(a^m)^{\frac{1}{n}}$.

Let $a^m=x$; then $\sqrt[n]{a^m}=\sqrt[n]{x}=x^{\frac{1}{n}}$. (Prin. 12, Cor.)

But since $x=a^m$, $x^{\frac{1}{n}}=(a^m)^{\frac{1}{n}}$; hence $\sqrt[n]{a^m}=(a^m)^{\frac{1}{n}}$.

COR. Hence $a^{\frac{m}{n}}=(a^m)^{\frac{1}{n}}$.

PRIN. 14. Prove that $a^{\frac{1}{n}}\times b^{\frac{1}{n}}=(ab)^{\frac{1}{n}}$.

Let $x=x^{\frac{1}{n}}\times b^{\frac{1}{n}}$.

Then, $x^n=(a^{\frac{1}{n}}\times b^{\frac{1}{n}})^n=(a^{\frac{1}{n}})^n\times(b^{\frac{1}{n}})^n=ab$.

Hence, $x^n=(ab)$; therefore, $x=(ab)^{\frac{1}{n}}$. (Prin. 12, Cor.)

COR. 1. In the same way it may be shown that

$$a^{\frac{1}{n}} \div b^{\frac{1}{n}} = \left(\frac{a}{b}\right)^{\frac{1}{n}}.$$

COR. 2. Hence also $a^{\frac{1}{n}} \times b^{\frac{1}{n}} \times c^{\frac{1}{n}} = (abc)^{\frac{1}{n}}$.

PRIN. 15. Prove that $a^{\frac{m}{n}} \times b^{\frac{m}{n}} = (ab)^{\frac{m}{n}}$.

Let $x = a^{\frac{m}{n}} \times b^{\frac{m}{n}}$.

Then, $x^n = (a^{\frac{m}{n}} \times b^{\frac{m}{n}})^n = (a^{\frac{m}{n}})^n \times (b^{\frac{m}{n}})^n = a^m \times b^m = (ab)^m$.

Hence, $x^n = (ab)^m$; and $x = (ab)^{\frac{m}{n}}$; therefore, $a^{\frac{m}{n}} \times b^{\frac{m}{n}} = (ab)^{\frac{m}{n}}$.

PRIN. 16. Prove that $(a^{\frac{1}{m}})^n = (a^n)^{\frac{1}{m}}$.

By Prin. 11, $(a^{\frac{1}{m}})^n = a^{\frac{n}{m}}$; let $x = a^{\frac{n}{m}}$.

Then, $x^m = a^n$, and $x = (a^n)^{\frac{1}{m}}$. Prin. 12.

COR. In a similar way it may be shown that $(a^{\frac{1}{m}})^{\frac{1}{n}} = (a^{\frac{1}{n}})^{\frac{1}{m}}$.

PRIN. 17. Prove that $(a^m)^{\frac{1}{n}} \times (a^p)^{\frac{1}{n}} = (a^m \times a^p)^{\frac{1}{n}}$.

Let $x = (a^m)^{\frac{1}{n}} \times (a^p)^{\frac{1}{n}}$; then $x^n = a^m \times a^p$;

Hence, $x = (a^m \times a^p)^{\frac{1}{n}}$.

PRIN. 18. Prove that $a^{\frac{m}{n}} = a^{\frac{mp}{np}}$.

Let $x = a^{\frac{m}{n}}$; then $x^n = a^m$, and $x^{np} = a^{np}$.

Hence, $x = a^{\frac{mp}{np}}$; therefore, $a^{\frac{m}{n}} = a^{\frac{mp}{np}}$.

420. We shall now proceed to show that the rules for multiplication, division, involution, and evolution apply to fractional exponents as well as integral.

PRIN. 19. Prove that $a^{\frac{p}{q}} \times a^{\frac{r}{s}} = a^{\frac{p}{q}+\frac{r}{s}}$.

$$a^{\frac{p}{q}} \times a^{\frac{r}{s}} = a^{\frac{ps}{qs}} \times a^{\frac{qr}{qs}}. \qquad \text{Prin. 18.}$$
$$= (a^{ps})^{\frac{1}{qs}} \times (a^{qr})^{\frac{1}{qs}}. \qquad \text{Prin. 13, Cor.}$$
$$= (a^{ps} \times a^{qr})^{\frac{1}{qs}}. \qquad \text{Prin. 17.}$$
$$= (a^{ps+qr})^{\frac{1}{qs}}.$$
$$= a^{\frac{ps}{qs}+\frac{qr}{qs}} = a^{\frac{p}{q}+\frac{r}{s}}.$$

COR. In the same way it may be shown that

$$a^{\frac{p}{q}} \div a^{\frac{r}{s}} = a^{\frac{p}{q}-\frac{r}{s}}.$$

PRIN. 20. Prove that $(a^{\frac{p}{q}})^{\frac{r}{s}} = a^{\frac{pr}{qs}}$.

Let $x = (a^{\frac{p}{q}})^{\frac{r}{s}}$; then $x^s = (a^{\frac{p}{q}})^r = a^{\frac{pr}{q}}$.

Hence $x^{qs} = a^{pr}$; and $x = a^{\frac{pr}{qs}}$.

COR. In the same way it may be shown that

$$\sqrt[\frac{r}{s}]{a^{\frac{p}{q}}} = a^{\frac{p}{q} \div \frac{r}{s}} = a^{\frac{p}{q} \times \frac{s}{r}} = a^{\frac{ps}{qr}}.$$

SCHOLIUM. When the exponents in Prin. 19 and 20 are negative, we can let m and n represent the fractions, and since the principles are true for m and n, they are true for negative fractions. Or we can prove them by the methods used for negative integral exponents.

421. It is thus shown that the following relations are universal, m and n being positive or negative, integral or fractional.

I. $a^m \times a^n = a^{m+n}$. III. $(a^m)^n = a^{mn}$.

II. $a^m \div a^n = a^{m-n}$. IV. $\sqrt[n]{a^m} = a^{\frac{m}{n}}$.

NOTE.—The student will be interested in noticing that this general discussion has introduced two forms of expression that are not usually employed in algebra—viz. *negative indices* and *fractional indices* of roots. Thus, since n is general, $\sqrt[n]{a}$ gives the forms $\sqrt[-2]{a}$; $\sqrt[\frac{2}{3}]{a}$; $\sqrt[-\frac{2}{3}]{a}$.

EXAMPLES.

1. Prove $a^{m-n}=\frac{1}{a^{n-m}}$.

2. Prove $(a)^{-\frac{1}{n}}=\frac{1}{a^{\frac{1}{n}}}$.

3. Prove $\sqrt[-n]{a^m}=\frac{1}{a^{\frac{m}{n}}}$.

4. Show the meaning of the negative exponent, a^{-n}.

5. Show the meaning of the fractional exponent, $a^{\frac{3}{4}}$ or $a^{\frac{n}{m}}$.

6. Show the meaning of a fractional index, $\sqrt[\frac{1}{2}]{a}$ or $\sqrt[\frac{n}{m}]{a}$.

7. Show the meaning of a negative index, $\sqrt[-2]{a}$ or $\sqrt[-n]{a}$.

8. Show the meaning of a negative fractional index, $\sqrt[-\frac{2}{3}]{a}$ or $\sqrt[-\frac{n}{m}]{a}$.

EXAMPLES IN REDUCTION.

1. $9^{-\frac{1}{2}}$. *Ans.* $\frac{1}{3}$.
2. $4^{-\frac{3}{2}}$. *Ans.* $\frac{1}{8}$.
3. $64^{-\frac{1}{3}}$. *Ans.* $\frac{1}{4}$.
4. $\frac{1}{81^{-\frac{3}{4}}}$. *Ans.* 27.
5. $(a^{-2})^{\frac{1}{3}}$ *Ans.* $\frac{1}{a^{\frac{2}{3}}}$.
6. $\frac{1}{(a^2)^{-\frac{1}{3}}}$. *Ans.* $a^{\frac{2}{3}}$.
7. $(x^{-2})^{-3}$. *Ans.* x^6.
8. $\sqrt{a^{-4}}$. *Ans.* $\frac{1}{a^2}$.
9. $(m^{-\frac{2}{3}})^{-\frac{3}{4}}$. *Ans.* $m^{\frac{1}{2}}$.
10. $(x^{-\frac{m}{n}})^{\frac{n}{m}}$. *Ans.* $\frac{1}{x}$.
11. $(4a^{-\frac{2}{3}})^{-\frac{3}{2}}$. *Ans.* $\frac{a}{8}$.
12. $(64n^{-3})^{-\frac{2}{3}}$. *Ans.* $\left(\frac{n}{4}\right)^2$.
13. $(32a^{-15})^{\frac{2}{5}}$. *Ans.* $\frac{4}{a^6}$.
14. $(5^{\frac{3}{4}}a^{-6})^{-\frac{2}{3}}$. *Ans.* $\frac{a^4}{\sqrt{5}}$.
15. $\sqrt[3]{a^{-3}}$. *Ans.* $\frac{1}{a}$.
16. $\sqrt[-2]{\frac{a^{-2}}{16}}$. *Ans.* $4a$.
17. $\sqrt[\frac{2}{3}]{4a^2}$. *Ans.* $8a^3$.
18. $\sqrt[-\frac{3}{4}]{8a^{-3}}$. *Ans.* $\left(\frac{a}{2}\right)^4$.

19. $\sqrt[-3]{\frac{a^6}{64}}$. *Ans.* $\left(\frac{2}{a}\right)^2$.
20. $\sqrt[-3]{\frac{a^{-9}}{27}}$. *Ans.* $3a^3$.
21. $\sqrt[\frac{2}{3}]{\frac{a^4}{c^2}}$. *Ans.* $\left(\frac{a^2}{c}\right)^3$.
22. $\sqrt[-\frac{4}{5}]{a^{-8}b^4}$. *Ans.* $\left(\frac{a^2}{b}\right)^5$.
23. $\sqrt[\frac{1}{n}]{a^{\frac{1}{n}}b^{\frac{m}{n}}}$. *Ans.* ab^m.
24. $\sqrt[-\frac{1}{n}]{a^{-2n}b^{4n}}$. *Ans.* $a^{2n^2}b^{-4n^2}$.
25. $\sqrt[-\frac{m}{n}]{a^{-\frac{1}{n}}b^{\frac{1}{n}}}$. *Ans.* $\left(\frac{a}{b}\right)^{\frac{1}{m}}$.
26. $\left(\sqrt[-\frac{m}{n}]{a^{\frac{m}{n}}c^{\frac{n}{m}}}\right)^{-\frac{m}{n}}$. *Ans.* $a^{\frac{m}{n}}c^{\frac{n}{m}}$.

EXAMPLES IN MULTIPLICATION.

1. $a^{\frac{1}{3}}\times a^{-\frac{1}{6}}$. *Ans.* $a^{\frac{1}{6}}$.
2. $a^{-2}\times\sqrt{a}$. *Ans.* $\left(\frac{1}{a}\right)^{\frac{3}{2}}$.
3. $a^{\frac{n}{2}}\times a^{\frac{1}{2}(m-n)}$. *Ans.* $\sqrt{a^m}$.
4. $a^2c^0\times a^{-2}c^{2n}$. *Ans.* c^{2n}.
5. $x^{\frac{1}{2}}\times\sqrt{x^{\frac{1}{3}}}$. *Ans.* $x^{\frac{2}{3}}$.
6. $m^{-\frac{1}{3}}\times\sqrt{m^{\frac{1}{2}}}$. *Ans.* $\left(\frac{1}{m}\right)^{\frac{1}{12}}$.
7. $a^{\frac{1}{n}}\times a^{\frac{1}{m}}$. *Ans.* $a^{\frac{m+n}{n}}$.
8. $y^{\frac{3}{4}}\times\sqrt{y^{-1}}$. *Ans.* $y^{\frac{1}{4}}$.
9. $(4^m a)^{m-n+p}\times(4^m a)^{n-m-p}$. *Ans.* 1.
10. $a^{m-n}\times 3a^{3m-2}\times 4a^{n+5}c$. *Ans.* $12a^{4m+3}c$.
11. $(a^{-m}b^p\times a^n b^{-2}\times a^{m+n}b^4$. *Ans.* $a^{2n}b^{p+2}$.
12. $(a+b)^{m+n}c^n\times(a+b)^{m-n}c^{-n}$. *Ans.* $(a+b)^{2m}$.

Multiply the following:

13. $a^{\frac{m}{2}}b^{-\frac{n}{3}}\times a^{m+n}\times(a^{\frac{1}{2}}b^{\frac{1}{3}})^{2m-2n}$. *Ans.* $a^{\frac{5m}{2}}b^{\frac{2m}{3}-n}$.
14. $(8^{-2}a^2x)^{m+3n}\times(8^3a^{-2}x^2)^{m+2n}$. *Ans.* $8^m a^{2n}x^{3m+7n}$.
15. $(a+b)^{-3}c^4\times(a+b)^{n+3}c^{-n}\times(a+b)^{1+n}c^{-2}$. *Ans.* $(a-b)^{2n+1}c^{2-n}$.
16. $(a^{\frac{2}{3}}+b^{\frac{2}{3}})$ by $a^{\frac{2}{3}}-b^{\frac{2}{3}}$. *Ans.* $a^{\frac{4}{3}}-b^{\frac{4}{3}}$.
17. $a^{m+n}+b^{m-n}$ by $a^{m+n}-b^{m-n}$. *Ans.* $a^{2(m+n)}-b^{2(m-2)}$.
18. x^n-y^{mn-n} by $x^{mn-n}y^n$. *Ans.* $x^{mn}y^n-x^{mn-n}y^{mn}$.

19. $a^{\frac{m}{n}}+b^{\frac{m}{n}}$ by $a^{\frac{m}{n}}-b^{\frac{m}{n}}$. *Ans.* $a^{\frac{2m}{n}}-b^{\frac{2m}{n}}$.

20. $x+x^{\frac{1}{2}}+2$ by $x+x^{\frac{1}{2}}-2$. *Ans.* $x^2+2x^{\frac{3}{2}}+x-4$.

21. $x^{\frac{2}{3}}+x^{\frac{1}{3}}y^{\frac{1}{3}}+y^{\frac{2}{3}}$ by $x^{\frac{1}{3}}-y^{\frac{1}{3}}$. *Ans.* $x-y$.

22. a^4+a^2+1 by $a^{-4}-a^{-2}+1$. *Ans.* a^4+1+a^{-4}.

23. $a^{-\frac{2}{3}}+a^{-\frac{1}{3}}+1$ by $a^{-\frac{1}{3}}-1$. *Ans.* $a^{-1}-1$.

24. $a^{\frac{4}{3}}-2+a^{-\frac{4}{3}}$ by $a^{\frac{2}{3}}-a^{-\frac{2}{3}}$. *Ans.* $a^2-3a^{\frac{2}{3}}+3a^{-\frac{2}{3}}-a^{-2}$.

25. $-3a^{-5}+2a^{-4}b^{-1}$ by $-2a^{-3}-3a^{-4}b$.
Ans. $-4a^{-7}b^{-1}+9a^{-9}b$.

26. $a^{\frac{7}{2}}-a^3+a^{\frac{5}{2}}-a^2+a^{\frac{3}{2}}-a+a^{\frac{1}{2}}-1$ by $a^{\frac{1}{2}}+1$. *Ans.* a^4-1.

EXAMPLES IN DIVISION.

1. $a^{\frac{1}{2}} \div a^{-\frac{1}{3}}$. *Ans.* $a^{\frac{5}{6}}$.
2. $a^{-\frac{2}{3}} \div a^{\frac{3}{4}}$. *Ans.* $a^{-\frac{17}{12}}$.
3. $a^m \div a^{m-n}$. *Ans.* a^n.
4. $b^{\frac{1}{6}} \div b^{-\frac{1}{3}}$. *Ans.* $b^{\frac{1}{2}}$.
5. $a^{-n} \div a^{-2n+1}$. *Ans.* a^{n-1}.
6. $a^{m-n} \div a^{n+m}$. *Ans.* a^{-2n}.
7. $b^0 \div b^{-n}$. *Ans.* b^n.
8. $a^{\frac{1}{2}} \div \sqrt{a^{\frac{2}{3}}}$. *Ans.* $a^{\frac{1}{6}}$.
9. $\sqrt{a^{-2n}} \div (a^{-2})^n$. *Ans.* a^n.
10. $x^n \div \sqrt{x^{-4n}}$. *Ans.* x^{3n}.
11. $a^{\frac{n}{2}} \div a^{\frac{2n}{3}}$. *Ans.* $\left(\frac{1}{a^n}\right)^{\frac{1}{6}}$.
12. $(a+b)^{\frac{1}{2}} \div (a+b)^{\frac{1}{3}}$.
Ans. $(a+b)^{\frac{1}{6}}$.
13. $(a-x)^{-\frac{2}{3}} \div (a-x)^{\frac{1}{3}}$.
Ans. $(a-x)^{-1}$.
14. $6a^{\frac{2}{3}}b^{\frac{1}{3}} \div 3a^{-\frac{1}{3}}b^{\frac{2}{3}}$. *Ans.* $2ab^{-\frac{1}{3}}$.
15. $4a^{-\frac{3}{4}}b^{\frac{1}{3}} \div 2a^{\frac{1}{3}}b^{-\frac{3}{4}}$.
Ans. $2\left(\frac{b}{a}\right)^{\frac{13}{12}}$.
16. $(a-b)^{\frac{m}{2}+n} \div (a-b)^{2n-\frac{m}{2}}$.
Ans. $(a-b)^{m-n}$.
17. $a^n b^{-\frac{m}{2}} c^{-2} \div a^{-n} b^{\frac{m}{2}} c^{n-4}$.
Ans. $a^{2n}b^{-m}c^{2-n}$.
18. $(a^{3n}x^{9n})^{\frac{1}{3}} \div \sqrt[\frac{1}{n}]{a^3x}$.
Ans. $\left(\frac{x}{a}\right)^{2n}$.

Divide the following:

19. $a^{\frac{3}{4}}-b^{\frac{3}{4}}$ by $a^{\frac{1}{4}}-b^{\frac{1}{4}}$. *Ans.* $a^{\frac{1}{2}}+a^{\frac{1}{4}}b^{\frac{1}{4}}+b^{\frac{1}{2}}$.

20. $a^{\frac{2}{3}}-b^{\frac{2}{3}}$ by $a^{\frac{1}{6}}-b^{\frac{1}{6}}$. *Ans.* $a^{\frac{1}{2}}+a^{\frac{1}{3}}b^{\frac{1}{6}}+a^{\frac{1}{6}}b^{\frac{1}{3}}+b^{\frac{1}{2}}$.

21. $a-x$ by $a^{\frac{1}{3}}-x^{\frac{1}{3}}$. *Ans.* $a^{\frac{2}{3}}+a^{\frac{1}{3}}x^{\frac{1}{3}}+x^{\frac{2}{3}}$.

22. a^n-b^m by $a^{\frac{n}{4}}-b^{\frac{m}{4}}$. *Ans.* $a^{\frac{3n}{4}}+a^{\frac{2n}{4}}b^{\frac{m}{4}}+a^{\frac{n}{4}}b^{\frac{2m}{4}}+b^{\frac{3m}{4}}$.

23. $a^{2n}-b^{-3m}$ by $a^{\frac{n}{2}}-b^{-\frac{3m}{4}}$. *Ans.* $a^{\frac{3n}{2}}+a^nb^{-\frac{3m}{4}}+a^{\frac{n}{2}}b^{-\frac{3m}{2}}+b^{-\frac{9m}{4}}$.

24. $a^{m-2}b^{2-n}c^{m-n}$ by $a^{m-n}b^{n+2}c^{2-n}$. *Ans.* $a^{n-2}b^{-2n}c^{m-2}$.

25. $a^2b^{-2}+2+a^{-2}b^2$ by $ab^{-1}+a^{-1}b$. *Ans.* $ab^{-1}+a^{-1}b$

26. $8x^{-1}+27y^{-2}$ by $2x^{-\frac{1}{3}}+3y^{-\frac{2}{3}}$. *Ans.* $4x^{-\frac{2}{3}}-6x^{-\frac{1}{3}}y^{-\frac{2}{3}}+9y^{-\frac{4}{3}}$.

27. $a^{\frac{3n}{2}}-a^{-\frac{3n}{2}}$ by $a^{\frac{n}{2}}-a^{-\frac{n}{2}}$. *Ans.* a^n+1+a^{-n}.

28. $x^{\frac{3}{2}}-xy^{\frac{1}{2}}+x^{\frac{1}{2}}y-y^{\frac{3}{2}}$ by $x^{\frac{1}{2}}-y^{\frac{1}{2}}$. *Ans.* $x+y$.

29. $a^{\frac{4}{3}}+a^{\frac{2}{3}}b^{\frac{2}{3}}+b^{\frac{4}{3}}$ by $a^{\frac{2}{3}}+a^{\frac{1}{3}}b^{\frac{1}{3}}+b^{\frac{2}{3}}$. *Ans.* $a^{\frac{2}{3}}-a^{\frac{1}{3}}b^{\frac{1}{3}}+b^{\frac{2}{3}}$.

30. $x^{-4}+x^{-2}y^{-2}+y^{-4}$ by $x^{-2}-x^{-1}y^{-1}+y^{-2}$.
Ans. $x^{-2}+x^{-1}y^{-1}+y^{-2}$.

31. $a^{\frac{2}{3}}+b^{\frac{2}{3}}-c^{\frac{2}{3}}+2a^{\frac{1}{3}}b^{\frac{1}{3}}$ by $a^{\frac{1}{3}}+b^{\frac{1}{3}}+c^{\frac{1}{3}}$. *Ans.* $a^{\frac{1}{3}}+b^{\frac{1}{3}}-c^{\frac{1}{3}}$.

32. Find the value of $\{(a^{2m})^3 \times (a^{-n})^2\}^{\frac{1}{3m-n}}$. *Ans.* a^2.

33. Find the value of $[\{(a-b)^n\}^{n-\frac{1}{n}}]^{\frac{1}{n+1}}$. *Ans.* $(a-b)^{n-1}$.

34. Find the value of $\{(b^{m-n})^{-2} \times (b^{-m+2n})^{-3}\}^{\frac{a}{m-4n}}$. *Ans.* b^a.

35. Find the value of $[\{(a^{\frac{m}{n}})^{\frac{n}{m}}\}^{\frac{p}{q}})^{\frac{q}{p}} \times [\{(a^{\frac{n}{m}})^{\frac{m}{n}}\}^{\frac{q}{p}}]^{\frac{p}{q}}$.
Ans. a^2.

36. Find the value of $[\{(a^m)^{-\frac{1}{n}}\}^p]^{-\frac{1}{q}} \div [\{(a^{-m})^{\frac{1}{n}}\}^{-p}]^{\frac{1}{q}}$. *Ans.* 1.

LOGARITHMS.

422. The **Logarithm** of a number is the exponent denoting the power to which a fixed number must be raised to produce the first number.

Thus, if $B^x = N$, then x is called the logarithm of N.

423. The **Base** of the system is the *fixed number* which is raised to the different powers to produce the numbers.

Thus, in $B^x = N$, x is the logarithm of N to the base B; so in $4^3 = 64$, 3 is the logarithm of 64 to the base 4.

424. The term *logarithm*, for convenience, is usually written *log.* The expressions above may be written $\log N = x$ and $\log 64 = 3$.

425. In the **Common System** of logarithms the base is 10, and the nature of logarithms is readily seen with this base; thus,

$10^2 = 100$; hence $\log 100 = 2$.
$10^3 = 1000$; hence $\log 1000 = 3$.
$10^4 = 10{,}000$; hence $\log 10{,}000 = 4$.
$10^{2.369} = 234$; hence $\log 234 = 2.369$.

426. We shall first derive the general principles of logarithms, the base being *any number*, and then explain the common numerical system.

PRINCIPLES.

PRIN. 1. *The logarithm of* 1 *is* 0, *whatever the base.*

For, let B represent any base, then $B^0 = 1$; hence by the definition of a logarithm, 0 is the log. of 1, or $\log 1 = 0$.

PRIN. 2. *The logarithm of the base of a system of logarithms is unity.*

For, let B represent any base, then $B^1 = B$; hence 1 is the log. of B, or $\log B = 1$.

PRIN. 3. *The logarithm of the product of two or more numbers is equal to the sum of the logarithms of those numbers.*

For, let $m = \log M$, and $n = \log N$.
Then, $B^m = M$, $B^n = N$.
Multiplying, $B^{m+n} = M \times N$.
Hence $m + n = \log (M \times N)$.
Or, $\log (M \times N) = \log M + \log N$.

PRIN. 4. *The logarithm of the quotient of two numbers is equal to the logarithm of the dividend minus the logarithm of the divisor.*

For, let $m = \log M$, and $n = \log N$.
Then, $B^m = M$, $B^n = N$.
Dividing, $B^{m-n} = M \div N$.
Hence, $\log (M \div N) = m - n$.
Or $\log (M \div N) = \log M - \log N$.

PRIN. 5. *The logarithm of any power of a number is equal to the logarithm of the number multiplied by the exponent of the power.*

For, let $m = \log M$.
Then, $B^m = M$.
Raising to nth power, $B^{n \times m} = M^n$.
Whence, $\log M^n = n \times m$.
Or $\log M^n = n \times \log M$.

PRIN. 6. *The logarithm of the root of a number is equal to the logarithm of the number divided by the index of the root.*

For, let $m = \log M$.
Then, $B^m = M$.
Taking nth root, $B^{\frac{m}{n}} = M^{\frac{1}{n}}$.
Whence, $\log M^{\frac{1}{n}} = \frac{m}{n}$.
Or $\log M^{\frac{1}{n}} = \frac{\log M}{n}$.

427. These principles are illustrated by the following examples, which the pupil will work.

EXAMPLES.

1. $\text{Log}\ (a.b.c) = \log a + \log b + \log c.$
2. $\text{Log}\left(\frac{ab}{c}\right) = \log a + \log b - \log c.$
3. $\text{Log}\ a^n = n \log a.$
4. $\text{Log}\ (a^x b^y) = x \log a + y \log b.$
5. $\text{Log}\ \frac{a^x b^y}{c^z} = x \log a + y \log b - z \log c.$
6. $\text{Log}\ \sqrt{ab} = \frac{1}{2} \log a + \frac{1}{2} \log b.$
7. $\text{Log}\ (a^2 - x^2) = \log (a+x) + \log (a-x).$
8. $\text{Log}\ \sqrt{a^2 - x^2} = \frac{1}{2} \log (a+x) + \frac{1}{2} \log (a-x).$
9. $\text{Log}\ a^2 \sqrt[3]{a^{-2}} = \frac{4}{3} \log a.$
10. $\text{Log}\ \frac{\sqrt{a^2 - x^2}}{(a+x)^2} = \frac{1}{2}\ \{\log (a-x) - 3 \log (a+x)\}.$

COMMON LOGARITHMS.

428. The **Base** of the common system of logarithms is 10. This base is most convenient for numerical calculations, because our numerical system is decimal.

429. In this system every number is conceived to be some power of 10, and by the use of fractional exponents may be thus, approximately, expressed.

430. Raising 10 to different powers, we have

$$10^0 = 1; \quad \text{hence } 0 = \log 1.$$
$$10^1 = 10; \quad \text{hence } 1 = \log 10.$$
$$10^2 = 100; \quad \text{hence } 2 = \log 100.$$
$$10^3 = 1000; \quad \text{hence } 3 = \log 1000.$$
etc. etc.

$$\text{Also, } 10^{-1} = .1; \quad \text{hence } -1 = \log .1.$$
$$10^{-2} = .01; \quad \text{hence } -2 = \log .01.$$
$$10^{-3} = .001; \quad \text{hence } -3 = \log .001.$$

431. Hence the logarithms of all numbers

between 1 and 10 will be $0 +$ a fraction;
between 10 and 100 will be $1 +$ a fraction;
between 100 and 1000 will be $2 +$ a fraction;
between 1 and .1 will be $-1 +$ a fraction;
between .1 and .01 will be $-2 +$ a fraction;
between .01 and .001 will be $-3 +$ a fraction.

432. Thus it has been found that the log. of 76 is 1.8808, and the log. of 458 is 2.6608. This means that

$$10^{1.8808} = 76, \text{ and } 10^{2.6608} = 458.$$

433. When the logarithm consists of an integer and a decimal, the integer is called the *characteristic*, and the decimal part the *mantissa*. Thus, in 2.660865, 2 is the *characteristic*, and .660865 is the *mantissa*.

PRINCIPLES OF COMMON LOGARITHMS.

PRIN. 1. *The characteristic of a logarithm of a number is one less than the number of integral places in the number.*

For, from Art. 430, $\log 1 = 0$ and $\log 10 = 1$; hence the logarithm of numbers from 1 to 10 (which consist of *one* integral place) will have 0 for the characteristic. Since $\log 10 = 1$ and $\log 100 = 2$, the logarithm of numbers from 10 to 100 (which consist of *two* integral places) will have *one* for the characteristic, and so on; hence *the characteristic is always one less than the number of integral places.*

PRIN. 2. *The characteristic of the logarithm of a decimal is negative, and is equal to the number of the place occupied by the first significant figure of the decimal.*

For, from Art. 430, $\log .1 = -1$, $\log .01 = -2$, $\log .001 = -3$; hence the logarithms of numbers from .1 to 1 will have -1 for a characteristic; the logarithms of numbers between .01 and .1 will have -2 for a characteristic, and so on; hence *the characteristic of a decimal is always negative, and equal to the number of the place of the first significant figure of the decimal.*

PRIN. 3. *The logarithm of the product of any number multiplied by* 10 *is equal to the logarithm of the number increased by* 1.

For, suppose $\log M = m$; then, by Prin. 3, Art. 426,

$$\log (M \times 10) = \log M + \log 10; \text{ but } \log 10 = 1;$$

Hence $\log (M \times 10) = m + 1$.

Thus, $\log (76 \times 10) = 1.880814 + 1$; or $\log 760 = 2.880814$.

PRIN. 4. *The logarithm of the quotient of any number divided by 10 is equal to the logarithm of the number diminished by* 1.

For, suppose $\log M = m$; then, by Prin. 4, Art. 426,

$$\log (M \div 10) = \log M - \log 10;$$

Hence, $\log (M \div 10) = m - 1$.

Thus, $\log (458 \div 10) = 2.660865 - 1$; or $\log 45.8 = 1.660865$.

PRIN. 5. *In changing the decimal point of a number we change the characteristic, but do not change the mantissa of its logarithm.*

This follows from Principles 3 and 4. To illustrate:

$\log 234 = 2.369216$. $\qquad \log .234 = \bar{1}.369216$.

$\log 23.4 = 1.369216$. $\qquad \log .0234 = \bar{2}.369216$.

$\log 2.34 = 0.369216$.

Thus we see that the *characteristic becomes negative, but not the mantissa.* The minus sign is written *over* the characteristic to show that it only is negative.

EXERCISES ON LOGARITHMS.

434. Common Logarithms are used to facilitate the operations of multiplying, dividing, etc. Tables of logarithms are constructed and used for this purpose.

435. We shall give the logarithms of a few prime numbers to four decimal places, and show how they are used.

$\log 2 = 0.3010$	$\log 7 = 0.8451$	$\log 17 = 1.2304$
$\log 3 = 0.4771$	$\log 11 = 1.0414$	$\log 19 = 1.2787$
$\log 5 = 0.6990$	$\log 13 = 1.1139$	$\log 23 = 1.3617$

MULTIPLICATION WITH LOGARITHMS.

436. Numbers are multiplied by means of logarithms by taking the sum of their logarithms. (See Art. 426.)

1. Find the logarithm of 2×5.

SOLUTION. From Prin. 3, Art. 426, the log. of 2×5 equals the log. of 2 plus the log. of 5; log $2 = 0.3010$, log $5 = 0.6990$; their sum is 1.0000.

OPERATION.
$$\begin{aligned} \log 2 &= 0.3010 \\ \log 5 &= 0.6990 \\ \hline \log 10 &= 1.0000 \end{aligned}$$

Find by the use of the logarithms given in Art. 435 the following:

2. $\log (3 \times 7)$.	5. $\log 5 \times 10$.	8. $\log 2 \times 7 \times 13$.
3. $\log (5 \times 7)$.	6. $\log 7 \times 10$.	9. $\log (5 \times 17 \times 23)$.
4. $\log (7 \times 11)$.	7. $\log 13 \times 10$.	10. $\log (7 \times 19 \times 23)$.

NOTE.—In actual practice with a table we find the number corresponding to the logarithm of the product, and thus obtain the product of the numbers.

437. The logarithms above given will enable us to find the logarithms of many numbers of which the prime numbers are factors. Find the following:

1. log 4.	5. log 20.	9. log 56.	13. log 1.15.
2. log 6.	6. log 26.	10. log 85.	14. log .230.
3. log 10.	7. log 30.	11. log 8.5.	15. log .380.
4. log 15.	8. log 42.	12. log 115.	16. log .0035.

DIVISION WITH LOGARITHMS.

438. Numbers are divided by means of logarithms by subtracting the logarithm of the divisor from the logarithm of the dividend.

1. Find the log. of $5 \div 2$.

SOLUTION. From Prin. 4, Art. 426, the log. of the quotient of 5 divided by 2 equals log 5 minus log 2; log $5 = 0.6990$, etc.

OPERATION.
$$\begin{aligned} \log 5 &= 0.6990 \\ \log 2 &= 0.3010 \\ \hline \log (5 \div 2) &= 0.3980 \end{aligned}$$

Find by the logs. given in Art. 435 the logs. of the following:

2. $\frac{3}{7}$.	6. $\frac{3}{5}$.	10. $\frac{.02}{11}$.	14. $\frac{10}{17}$.
3. $\frac{5}{7}$.	7. $\frac{1.7}{3}$.	11. $\frac{.03}{5}$.	15. $\frac{1.7}{.10}$.
4. $\frac{7}{2}$.	8. $\frac{15}{7}$.	12. $\frac{23}{17}$.	16. $\frac{3\times .5}{2\times .7}$.
5. $\frac{11}{3}$.	9. $\frac{21}{5}$.	13. $\frac{.05}{.003}$.	17. $\frac{2\times .3\times .05}{.7\times .11+22}$.

NOTE.—For an explanation of the nature and use of the *arithmetical complement* see Trigonometry.

POWERS AND ROOTS WITH LOGARITHMS.

439. The powers or roots of numbers are readily obtained by logarithms, according to Prin. 5, Art. 422.

1. Find the log. of 7^3.

SOLUTION. By Prin. 5, Art. 426, log 7^3 equals log 7 multiplied by 3; log 7 = 0.8451; multiplying by 3, we have 2.5353; hence log 7^3, or log 343 = 2.5353.

OPERATION.

$$\begin{array}{r} \log 7 = 0.8451 \\ 3 \\ \hline \log 7^3 \text{ or } 343 = 2.5353 \end{array}$$

Find the logarithms of the following:

2. 3^2.	6. 13^5.	10. $3^{\frac{1}{2}}$.	14. $2^3\times 3^2$.
3. 5^3.	7. 15^4.	11. $7^{\frac{2}{3}}$.	15. $3^2\div .05^3$.
4. 7^4.	8. 17^3.	12. $.07^{\frac{3}{4}}$.	16. $.07^3\times .014^2$.
5. 11^5.	9. 19^2.	13. $.01^{\frac{1}{4}}$.	17. $.09^{\frac{1}{2}}\div .021^{\frac{1}{3}}$.

NOTE.—Teachers who wish to give their pupils a knowledge of the use of the tables and numerical computation with logarithms will find the subject presented in my Geometry and Trigonometry.

EXPONENTIAL EQUATIONS.

440. An **Exponential Equation** is an equation in which the unknown quantity is an exponent; as,

$$a^x = b, \qquad x^x = a, \qquad b^{ax} = c, \text{ etc.}$$

441. Such equations are most readily solved by means of logarithms.

1. Given $a^x = b$, to find x.

SOLUTION.

Given, $a^x = b$.

Taking log. of members, $x \log a = \log b$.

Whence, $x = \frac{\log b}{\log a}$.

2. Given $5^x = 10$, to find x.

SOLUTION.

Given, $5^x = 10$.

Taking log., $x \log 5 = \log 10$.

Whence, $x = \frac{\log 10}{\log 5} = \frac{1.000}{0.6990} = 1.4306$.

3. Given $5^{\frac{2}{x}} = \frac{7}{3}$, to find x.

SOLUTION.

Given, $5^{\frac{2}{x}} = \frac{7}{3}$.

Raising to x power, $5^2 = \frac{7^x}{3^x}$.

Taking log., $2 \log 5 = x \log 7 - x \log 3$.

Whence, $x = \frac{2 \log 5}{\log 7 - \log 3}$.

Or, $x = \frac{2\times .6990}{.8451 - .4771} = 3.7989$.

EXAMPLES.

4. Given $5^x = 8$, to find x. *Ans.* $x = 1.2918$.
5. Given $4^{2x} = 8^{\frac{2}{3}}$, to find x. *Ans.* $x = 0.5$.
6. Given $a^x = bc$, to find x. *Ans.* $x = \frac{\log b + \log c}{\log a}$.
7. Given $a^x = b^2c^3$, to find x. *Ans.* $x = \frac{2 \log b + 3 \log c}{\log a}$.
8. Given $5^{\frac{2}{x}} = 30$, to find x. *Ans.* $x = 0.9464$.

9. Given $\frac{ab^x - c}{n} = d$, to find x. *Ans.* $x = \frac{\log(nd+c) - \log a}{\log b}$.

10. Given $ab^{\frac{1}{x}} = c$, to find x. *Ans.* $x = \frac{\log b}{\log c - \log a}$.

11. Given $a^{mx+n} = b$, to find x. *Ans.* $x = \frac{\log b - n \log a}{m \log a}$.

12. Given $m^{ax}n^{bx} = p$, to find x. *Ans.* $x = \frac{\log p}{a \log m + b \log n}$.

13. Given $a^{2x} - 2a^x = 63$, to find x. *Ans.* $x = \frac{2 \log 3}{\log a}$.

14. Given $3^{2x} + 3^x = 6$, to find x. *Ans.* $x = 0.6308$.

15. Given $n^x + \frac{1}{n^x} = m$, to find x. *Ans.* $x = \frac{\log \frac{1}{2}(m \pm \sqrt{m^2 - 4})}{\log n}$.

16. Given $a^x + b^y = 2m$ and $a^x - b^y = 2n$, to find x and y.

Ans. $x = \frac{\log(m+n)}{\log a}$, $y = \frac{\log(m-n)}{\log b}$.

17. Given $x^y = y^x$, and $x^2 = y^3$, to find x and y.

Ans. $x = 3\frac{3}{8}$, $y = 2\frac{1}{4}$.

18. In a geometrical progression, given a, r and s, to find n.

Ans. See page 268.

19. In a geometrical progression, given l, r and s, to find n.

Ans. See page 268.

20. In compound interest, if P represents the principal, $R = 1 + r$, the rate, A the amount, and t the time, show that $A = P \times R^t = P(1+r)^t$.

21. From the above formula derive the following formulas:

1. $\log A = \log P + t \log(1+r)$; 3. $\log(1+r) = \frac{\log A - \log P}{t}$;

2. $\log P = \log A - t \log(1+r)$; 4. $t = \frac{\log A - \log P}{\log(1+r)}$.

NOTE.—Exponential equations of the form $x^x = a$ cannot be solved by elementary algebra. Numerical forms like $x^x = 10$ may be solved by Double Position.

SECTION XII.

PERMUTATIONS, COMBINATIONS, BINOMIAL THEOREM.

PERMUTATIONS.

442. Permutations are the different orders in which a number of things can be arranged.

Thus, the permutations of a and b are ab and ba; the permutations of a, b, and c, taken two at a time, are ab, ba, ac, ca, bc, cb.

443. Things may be arranged in sets of one, of two, of three, etc. Thus, the three letters a, b, and c may be arranged in sets as follows:

Of one,	a,	b,	c.
Of two,	ab, ac;	ba, bc;	ca, cb.
Of three,	abc, acb;	bac, bca;	cab, cba.

NOTES.—1. It is convenient to let P_2 represent the number of permutations when taken two together; P_3, the number when taken three together, etc.; P_r, the number when taken r together.

2. The term *permutations* is sometimes restricted to the case where the quantities are taken *all* together, while the term *arrangements* or *variations* is given to the grouping by twos, threes, etc., the number in the group being less than the whole number of things.

PROBLEMS.

444. *To find the number of permutations or arrangements that can be formed of* n *things taken two at a time, three at a time,* etc.

9. Given $\frac{ab^x - c}{n} = d$, to find x. *Ans.* $x = \frac{\log(nd+c) - \log a}{\log b}$.

10. Given $ab^{\frac{1}{x}} = c$, to find x. *Ans.* $x = \frac{\log b}{\log c - \log a}$.

11. Given $a^{mx+n} = b$, to find x. *Ans.* $x = \frac{\log b - n \log a}{m \log a}$.

12. Given $m^{ax}n^{bx} = p$, to find x. *Ans.* $x = \frac{\log p}{a \log m + b \log n}$.

13. Given $a^{2x} - 2a^x = 63$, to find x. *Ans.* $x = \frac{2 \log 3}{\log a}$.

14. Given $3^{2x} + 3^x = 6$, to find x. *Ans.* $x = 0.6308$.

15. Given $n^x + \frac{1}{n^x} = m$, to find x. *Ans.* $x = \frac{\log \frac{1}{2}(m \pm \sqrt{m^2 - 4})}{\log n}$.

16. Given $a^x + b^y = 2m$ and $a^x - b^y = 2n$, to find x and y.

Ans. $x = \frac{\log(m+n)}{\log a}$, $y = \frac{\log(m-n)}{\log b}$.

17. Given $x^y = y^x$, and $x^2 = y^3$, to find x and y.

Ans. $x = 3\frac{3}{8}$, $y = 2\frac{1}{4}$.

18. In a geometrical progression, given a, r and s, to find n.

Ans. See page 268.

19. In a geometrical progression, given l, r and s, to find n.

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20. In compound interest, if P represents the principal, $R = 1 + r$, the rate, A the amount, and t the time, show that $A = P \times R^t = P(1+r)^t$.

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1. $\log A = \log P + t \log(1+r)$; 3. $\log(1+r) = \frac{\log A - \log P}{t}$;

2. $\log P = \log A - t \log(1+r)$; 4. $t = \frac{\log A - \log P}{\log(1+r)}$.

NOTE.—Exponential equations of the form $x^x = a$ cannot be solved by elementary algebra. Numerical forms like $x^x = 10$ may be solved by Double Position.

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Of one,	a,	b,	c.
Of two,	ab, ac;	ba, bc;	ca, cb.
Of three,	abc, acb;	bac, bca;	cab, cba.

NOTES.—1. It is convenient to let P_2 represent the number of permutations when taken two together; P_3, the number when taken three together, etc.; P_r, the number when taken r together.

2. The term *permutations* is sometimes restricted to the case where the quantities are taken *all* together, while the term *arrangements* or *variations* is given to the grouping by twos, threes, etc., the number in the group being less than the whole number of things.

PROBLEMS.

444. *To find the number of permutations or arrangements that can be formed of* n *things taken two at a time, three at a time,* etc.

Let $a, b, c, d, \ldots k$ represent n things.

First, if we reserve one of the n letters, as a, to place before each of the others, there will remain $n-1$ letters; and placing a before each of the $n-1$ letters, there will be $n-1$, in which a stands first; as ab, ac, $ad, \ldots ak$. Similarly, if we put b before each of the other letters, there will be $n-1$ arrangements in which b stands first. Similarly, there will be $n-1$ arrangements in which c stands first. Hence, since each of the n letters may be first, in all there will be $n(n-1)$ arrangements of n things taken *two* at a time.

Second, let us now find the number of arrangements of the n letters taken *three* at a time. If we reserve one letter, as a, there remain $n-1$ letters; the number of permutations of $n-1$ letters taken two together, from what has been shown, is $(n-1)(n-2)$. Putting a before each of these, there will be $(n-1)(n-2)$ permutations of n letters taken *three* together, in which a stands first. Similarly, there are $(n-1)(n-2)$ permutations, in which b stands first, and so for each of the n letters; hence the whole number of permutations of n letters taken *three* together is $n(n-1)(n-2)$.

Similarly, we find that the number of permutations of n letters taken *four* at a time is $n(n-1)(n-2)(n-3)$. From these cases it may be inferred, by analogy, that the number of permutations of n things taken r at a time is

$$n(n-1)(n-2)(n-3) \ldots (n-\overline{r-1}).$$

445. We shall now prove that the formula for the permutation of n things taken r at a time, derived above by analogy, is true.

Suppose it to be true that the number of permutations of n letters taken $r-1$ at a time is

$$n(n-1) \ldots \{n-(\overline{r-1}-1)\},$$

or $$n(n-1) \ldots (n-\overline{r-1}+1),$$

then we can show that a similar formula will give the number of permutations of the letters taken r at a time. For, out of the $n-1$ letters $b, c, d, \ldots$ we can form (Art. 444),

$$(n-1)(n-2) \ldots (n-1-\overline{r-1}-1),$$

or, $$(n-1)(n-2) \ldots (n-\overline{r-1}),$$

permutations each of $r-1$ letters, in which a stands first. Similarly, we have the same number when b stands first, and as many when c stands first, and so on. Hence, on the whole, there are

$$n(n-1)(n-2) \ldots (n-\overline{r-1})$$

permutations of n letters taken r at a time.

This proves that if the formula holds when the letters are taken $r-1$ at a time, it will hold when they are taken r at a time. But it has been shown to hold when they are taken *three* at a time; hence it holds when they are taken *four* at a time; hence also it must hold when they are taken *five* at a time, and so on; therefore the formula holds universally.

NOTE.—The method of reasoning employed in Art. 445 is called *Mathematical Induction*. It is based on the principle that *a truth is universal if when it is true in* n *cases it is true in* n + 1 *cases*. It is regarded as a valid method of demonstration, while the method by analogy or pure induction is not.

446. *To find the number of permutations of* n *things taken all together.*

In the formula just proved put $r=n$, and we have

$$P_n = n(n-1)(n-2) \ldots 1.$$

447. For the sake of brevity $n(n-1)(n-2) \ldots 1$ is often written $\lfloor n$, which is read "*factorial n.*"

Thus, $\lfloor n$ denotes $1 \times 2 \times 3 \times \ldots \times n$; that is, the product of the natural numbers from 1 to n inclusive.

448. *Any number of* r *things, or combination of* r *things, will produce* $\lfloor r$ *permutations.*

For by Article 446, the r things which make the given combination can be arranged in $\lfloor r$ different ways.

449. *To find the number of permutations of* n *things taken all together when some occur more than once.*

Let there be n letters, and suppose a occurs p times, b occurs q times, and c occurs r times, the rest, d, e, f, etc., occurring but once. Let N represent the required number of permutations.

Now, if in any of the permutations we suppose the p letters a to be changed into p new letters, different from the rest, then without changing the situation of any of the remaining letters we could, from their interchange with one another, produce $\underline{|p}$ permutations; hence if the p letters a were changed into p different letters, the whole number of permutations would be $N\times\underline{|p}$. Similarly, if the q letters b were also changed into q new letters, different from any of the rest, the whole number of permutations would be $N\times\underline{|p}\times\underline{|q}$. So also if the r letters c were also changed, the whole number of permutations would be $N\times\underline{|p}\times\underline{|q}\times\underline{|r}$. But this number would be equal to the number of permutations of n different things taken all together; that is, to $\underline{|n}$.

Thus, $$N\times\underline{|p}\times\underline{|q}\times\underline{|r}=\underline{|n}.$$

Hence $$N=\frac{\underline{|n}}{\underline{|p}\times\underline{|q}\times\underline{|r}}.$$

450. *To find the number of permutations of* n *things when each may occur once, twice, thrice, ... up to* r *times.*

Let there be n letters, a, b, c, ... Taking them one at a time, we shall have n arrangements. Taking them two at a time, a may stand before a, or before any one of the remaining letters; similarly, b may stand before b, or before any one of the remaining letters, and so on; thus there are $n\times n$, or n^2, different arrangements. Taking them three at a time, each one of the n letters may be combined with the n^2 arrangements, making $n\times n^2$, or n^3 in all. Similarly, when the letters are taken r at a time, the whole number of permutations will be n^r.

451. If they are taken n at a time, or all together, r becomes n, and the number of permutations becomes n^n.

EXAMPLES.

1. How many permutations can be formed of the letters in the word *chair*, taken three together?

SOLUTION. Here $n=5$, and $r=3$; hence substituting in the formula $P_3=n(n-1)\ . \ . \ . \ . \ (n-r-1))$, we have $5\times4\times3=60$.

2. In how many ways may the letters of the word *home* be written? *Ans.* 24.

3. In how many ways can 5 persons arrange themselves at table so as not to sit twice in the same order? *Ans.* 120.

4. In how many different ways, taken all together, can the 7 prismatic colors be arranged? *Ans.* 5040.

5. The number of permutations of a set of things taken *four* together is twice as great as the number taken *three* together; how many things in the set? *Ans.* 5.

6. In how many ways can 8 persons form a circle by joining hands? *Ans.* 5040.

7. How many permutations can be made of the letters of the word *Caraccas*, taken all together? *Ans.* 1120.

8. How many permutations can be made of the letters of the word *Mississippi*, taken all together? *Ans.* 34650.

9. The number of permutations of n things taken four together is six times the number taken three together; find the value of n. *Ans.* $n=9$.

10. The number of arrangements of 15 things, taken r together, is ten times the number taken $(r-1)$ together; find the value of r. *Ans.* $r=6$.

11. In how many ways can 2 sixes, 3 fives, and 5 twos be thrown with 10 dice? *Ans.* $\dfrac{\underline{|10}}{\underline{|2}\ \underline{|3}\ \underline{|5}}$.

12. In how many different ways can six letters be arranged when taken singly, two at a time, three at a time, and so on, until they are taken all at a time? *Ans.* 1956.

NOTE.—Find the sum of the different permutations.

COMBINATIONS.

452. The **Combinations** of a set of things are the different collections that can be formed out of them without regarding the *order* in which they are placed.

Thus, the combinations of the letters a, b, c, taken *two* together, are ab, ac, bc; ab and ba, though different permutations, form the same combination.

453. Each *combination* of things when taken two together, as ab, gives *two permutations;* and when taken *three* together, as abc, gives $3\times2\times1$ permutations.

454. *In general, each combination or collection of* **r** *things gives* $\underline{|r}$ *permutations.*

Thus the combination *abc* gives the permutations *abc*, *acb*, etc.; that is, the permutations of three things taken all together, which by Art. 446 is $\underline{|3}$, or $3\times2\times1$. Similarly, it is seen that the combination of r things gives $\underline{|r}$ permutations.

455. *To find the number of combinations that can be formed out of* **n** *things taken* **r** *at a time.*

The number of *permutations* of n things taken r at a time is $n(n-1)\ldots\ldots(n-\overline{r-1})$ (Art. 444), and each collection or combination of r things produces $\underline{|r}$ permutations (Art. 454); hence the number of combinations of n things taken r at a time equals the number of permutations divided by $\underline{|r}$; or, letting C_r represent the number of combinations, we have

$$C_r=\frac{n(n-1)(n-2)\ldots\ldots(n-\overline{r-1})}{\underline{|r}}.$$

456. If we multiply both numerator and denominator of the previous expression by $\underline{|n-r}$, we have

$$C_r=\frac{\underline{|n}}{\underline{|r}\;\underline{|n-r}}.$$

457. *The number of combinations of* **n** *things taken* **r** *at a time is the same as the number of them taken* **n − r** *at a time.*

For the number of combinations of n things taken $n-r$ at a time is

$$\frac{n(n-1)(n-2)\ldots\ldots(n-\overline{n-r-1})}{\underline{|n-r}};$$

that is,
$$\frac{n(n-1)(n-2)\ldots\ldots(r-1)}{\underline{|n-r}}.$$

Multiply both numerator and denominator by $\underline{|r}$, and we obtain $C_{n-r}=\frac{\underline{|n}}{\underline{|r}\;\underline{|n-r}}$, which, by Art. 456, is the number of combinations of n things taken r at a time.

458. Hence in finding the number of combinations taken r together, when $r>\frac{1}{2}n$, the shortest way is to find the number taken $(n-r)$ together.

459. *To find the value of* **r** *from which the number of combinations of* **n** *things taken* **r** *at a time is greatest.*

The formula $C_r=\frac{n(n-1)(n-2)\ldots\ldots(n-\overline{r-1})}{1\times2\times3\times\ldots\ldots r}$ may be written

$$C_r=\frac{n}{1}\times\frac{n-1}{2}\times\frac{n-2}{3}\ldots\ldots\frac{n-\overline{r-1}}{r}.$$

Now, it is seen that the numerators of this formula decrease from right to left by unity, and the denominators increase by unity; hence at some point in this series the factors become *less* than 1; therefore the value of C_r is greatest when the product includes *all* the factors *greater* than 1.

Now, when n is an *odd* number, the numerator and denominator of each factor will be alternately both odd and both even; so that the factor greater than 1, but nearest to 1, will be that factor whose numerator exceeds the denominator by 2. Hence, in this case the value of r must be such that

$$n-\overline{r-1}=r+2,\text{ or }r=\frac{n-1}{2}.$$

When n is *even*, the numerator of the first factor will be even and the denominator odd; the numerator of the second factor will be odd and the denominator even, and so on alternately; hence the factor greater than 1, but nearest to 1, will be the factor whose numerator exceeds the denominator by 1. Hence, in this case the value of r must be such that

$$n-\overline{r-1}=r+1,\text{ or }r=\frac{n}{2}.$$

NOTE.—For other principles of Permutations and Combinations see works on Higher Algebra.

EXAMPLES.

1. How many combinations can be formed from the letters of the word *Prague*, taken three together?

SOLUTION. Here $n=6$ and $r=3$; hence substituting in the formula, we have $C_3=\frac{n(n-1)\ldots\ldots n(n-\overline{r-1})}{\underline{|r}}=\frac{6\times5\times4}{1\times2\times3}=20.$

2. How many combinations may be made of the letters a, b, c, d, e, f, taken three together? four together? *Ans* 20; 15.

3. How many spans of horses can be selected from 20 horses? How many double spans? *Ans.* 190; 4845.

4. How many combinations of 3 or of 5 letters can be made out of 8 letters? *Ans.* 56.

5. How many combinations can be made of the letters of the word *longitude*, taken four at a time? *Ans.* 126.

6. How many combinations may be formed of 16 things taken 5 at a time? *Ans.* 4368.

7. How many different parties of 6 men can be formed out of a company of 20 men? *Ans.* 38760.

8. A guard of 5 soldiers is to be formed by lot out of 20 soldiers; in how many ways can this be done? how often will any one soldier be on guard? *Ans.* 15504; 3876.

9. In how many different ways can a class of 6 boys be placed in a line, one being denied the privilege of the head of the class? *Ans.* 600.

10. The number of the combinations of n things taken four together is to the number taken two together as 15 : 2; find the value of n. *Ans.* $n=12$.

11. The number of permutations of n things taken 5 at a time is equal to 120 times the number of combinations taken 3 at a time; find n. *Ans.* $n=8$.

12. A and B have each the same number of horses, and find that A can make twice as many different teams by taking 3 horses together as B can by taking 2 horses together; how many horses has each? *Ans.* 8.

THE BINOMIAL THEOREM.

460. The **Binomial Theorem** is derived on page 162 by induction; we shall now demonstrate it by a more rigid method of reasoning called *mathematical induction*.

461. By actual multiplication we obtain

(1) $$(x+a)(x+b)=x^2+(a+b)x+ab.$$

(2) $$(x+a)(x+b)(x+c)=x^3+(a+b+c)x^2+(ab+ac+bc)x+abc.$$

(3) $$(x+a)(x+b)(x+c)(x+d)=x^4+(a+b+c+d)x^3 \\ +(ab+ac+ad+bc+bd+cd)x^2 \\ +(abc+bcd+cda+dab)x+abcd.$$

462. Examining these results, we observe certain *laws* in the development:

1. *The number of terms is one more than the number of binomial factors involved.*

2. *The exponent of x in the first term is the same as the number of binomial factors, and decreases by unity in each succeeding term.*

3. *The coefficient of x in the first term is unity.*

The coefficient of x in the second term is the sum of the second letters, a, b, c, of the binomial factors.

The coefficient of x in the third term is the sum of the products of the second letters taken two at a time.

The coefficient of the fourth term is the sum of the products of the second letters taken three at a time, and so on.

4. *The last term is the product of all the second letters of the binomial factors.*

463. We shall now show that these laws always hold, whatever be the number of the binomial factors.

Suppose the laws to hold for $n-1$ factors, so that

$$(x+a)(x+b)\ \ldots\ (x+k)=x^{n-1}+px^{n-2}+qx^{n-3}+rx^{n-4}+\ \ldots\ u,$$

where $p=$ the sum of the letters $a, b, c, \ldots k$.

$q=$ the sum of the products of these letters taken two at a time.

$r=$ the sum of the products of these letters taken three at a time.

. . . .

$u=$ the product of all these letters.

Then multiply by another factor, $x+l$, and arrange the product according to the powers of x; thus,

$$(x+a)(x+b)(x+c)\ \ldots\ (x+k)(x+l)=x^n+(p+l)x^{n-1} \\ +(q+pl)x^{n-2}+(r+ql)x^{n-3}+\ \ldots\ +ul.$$

The laws (1) and (2) evidently hold in this expression.

Now, $p+l=a+b+c+\ldots\ldots k+l$

$=$ the sum of the letters $a, b, c, \ldots\ldots k, l$.

Also, $q+pl=q+l(a+b+c+\ldots\ldots+k)$

$=$ the sum of the products of all the letters $a, b, c, \ldots\ldots k, l$, taken two at a time.

Also, $r+ql=r+l(ab+ac+bc+\ldots\ldots)$

$=$ the sum of the products of all the letters $a, b, c, \ldots\ldots k, l$, taken three at a time.

Also, $ul=$ the product of all the letters.

Hence, if the laws hold for $n-1$ factors, they hold for n factors. But it has been shown that they hold for *four* factors; therefore they hold for *five* factors, and therefore for *six* factors, and so on. Therefore they hold universally.

464. We shall now proceed to find the general formula for the expansion of $(x+a)^n$.

Let P, Q, R etc. represent the coefficients in the above formula, and we have

$$(x+a)(x+b)\ldots\ldots(x+k)(x+l)=x^n+Px^{n-1}+Qx^{n-2}+Rx^{n-3}+\ldots\ldots+V.$$

Here $P=$ the sum of the letters $a, b, c, \ldots\ldots k, l$, which are n in number.

$Q=$ the sum of the products of these letters taken two at a time, so that there are $\frac{n(n-1)}{1\,.\,2}$ of these products, ab, ac, etc.

$R=$ the sum of the products of these letters taken three at a time, so that there are $\frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}$ of these products, abc, abd, etc., and so on.

$V=abc\ldots\ldots kl$.

Now, suppose $b, c\ldots\ldots k, l$ are each equal to a; then,

First, $(x+a)(x+b)\ldots\ldots(x+k)(x+l)$ equals $(x+a)^n$;

Second, P will equal a taken n times, or na;

Third, ab, ac, etc. will each become a^2, and Q will equal a^2 taken $\frac{n(n-1)}{1\,.\,2}$ times, or $\frac{n(n-1)}{1\,.\,2}a^2$.

Fourth, abc, abd, etc. will each become a^3, and R will equal a^3 taken $\frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}$ times, or $\frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}a^3$; and so on.

Finally, $abc\ldots\ldots kl$ will equal a taken as a factor n times, or a^n, and V will equal a^n. Therefore,

$$(x+a)^n=x^n+nax^{n-1}+\frac{n(n-1)}{1\,.\,2}a^2x^{n-2}+\frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}a^3x^{n-3}+\frac{n(n-1)(n-2)(n-3)}{1\,.\,2\,.\,3\,.\,4}a^4x^{n-4}+\ldots\ldots+a^n.$$

465. If in the formula a and x be interchanged, the development will proceed by ascending powers of x; thus,

$$(a+x)^n=a^n+na^{n-1}x+\frac{n(n-1)}{1\,.\,2}a^{n-2}x^2+\frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}a^{n-3}x^3+\frac{n(n-1)(n-2)(n-3)}{1\,.\,2\,.\,3\,.\,4}a^{n-4}x^4+\ldots\ldots+x^n.$$

466. From the examination of this formula several laws will be observed, as follows:

1. *The sum of the exponents of a and x in each term equals n.*

2. *If x is negative, every odd power of x will be negative and the even powers positive; thus,*

$$(a-x)^n=a^n-na^{n-1}x+\frac{n(n-1)}{1\,.\,2}a^{n-2}x^2-\frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}a^{n-3}x^3+\text{etc.}$$

467. The coefficient of the rth term of the development of $(a+x)^n$ is

$$\frac{n(n-1)(n-2)\ldots\ldots(n-r+2)}{1\,.\,2\,.\,3\ldots\ldots r-1}.$$

For, the coefficient of the second term is n, which is the combination of n things taken singly; the coefficient of the third term is $\frac{n(n-1)}{1\,.\,2}$, which is the combination of n things taken two at a time; and generally, the coefficient of the rth term is the number of combinations of n things taken $r-1$ at a time, which, by Art. 455, is equal to

$$\frac{n(n-1)(n-2)\ \ldots\ldots\ (n-r+2)}{1\,.\,2\,.\,3\ \ldots\ldots\ r-1}.$$

468. This is called the *general coefficient;* and by making $r=2, 3, 4$, etc., all the others can be derived from it.

Thus, suppose we wish to find the 5th term of $(a-x)^7$. Here $r=5$ and $n=7$; and the coefficient of the term required is $\frac{7\,.\,6\,.\,5\,.\,4}{1\,.\,2\,.\,3\,.\,4}=35$, and the 5th term is $35a^3x^4$.

469. In the expansion of $(a+x)^n$ the *coefficients of terms equally distant from the beginning and end are the same.*

For the coefficient of the rth term from the beginning is

$$\frac{n(n-1)(n-2)\ \ldots\ldots\ (n-r+2)}{\underline{|r-1}},$$

which, by multiplying both numerator and denominator by $\underline{|n-r+1}$ becomes

$$\frac{\underline{|n}}{\underline{|r-1}\ \underline{|n-r+1}}.$$

The rth term from the end is the $(n-r+2)$th term from the beginning, and its coefficient is

$$\frac{n(n-1)(\ \ldots\ldots\ )\{n-(n-r+2)+2\}}{\underline{|n-r+1}},\text{ or }\frac{n(n-1)\ \ldots\ldots\ r}{\underline{|n-r+1}}.$$

Multiplying both terms by $\underline{|r-1}$, this becomes also

$$\frac{\underline{|n}}{\underline{|r-1}\ \underline{|n-r+1}}.$$

470. The expansion of a binomial can always be reduced to the case in which one of the two quantities is unity.

Thus $(a+x)^n=a^n\left(1+\frac{x}{a}\right)^n=a^n(1+y)^n$, if $y=\frac{x}{a}$. We may then expand $(1+y)^n$, and multiply each term by a^n, and thus obtain the expansion of $(a+x)^n$.

471. The *sum of the coefficients* of the terms in the expansion of $(1+x)^n$ is 2^n.

For $(1+x)^n=1+nx+\frac{n(n-1)}{1\,.\,2}x^2+\ \ldots\ldots\ nx^{n-1}+x^n$. Now, since this is true for all values of x, it is true when $x=1$; whence

$$(1+1)^n=2^n=1+n+\frac{n(n-1)}{1\,.\,2}+\ \ldots\ldots\ n+1;$$

that is, the sum of the coefficients is 2^n.

472. The *sum of the coefficients* of the *odd terms* in the expansion $(1+x)^n$ *is equal* to the *sum of the coefficients* of the *even terms.*

If we let $x=-1$, the expansion of $(1+x)^n$ becomes

$$0=1-n+\frac{n(n-1)}{1\,.\,2}-\frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}+\text{etc.},$$

in which we have the sum of the *odd coefficients, minus* the sum of the *even coefficients* equal to zero; hence the two sums are equal.

473. Since the two sums are equal, each is one-half of 2^n (Art. 471), or $2^n \div 2 = 2^{n-1}$.

NOTES.—1. It may also be shown that in the expansion of $(a+x)^n$ the *middle term* will have the *greatest coefficient* when n is *even;* and the *two middle terms* will have *equal coefficients* when n is *odd,* and be the *greatest terms.*

2. This demonstration of the binomial theorem is restricted to n being a *positive integer.* The theorem is also true when n is *negative* or *fractional;* but the demonstration is too difficult for a work on elementary algebra. We shall assume that n is general, and give examples showing its application.

EXAMPLES.

1. Expand $(b+y)^{-2}$.

SOLUTION. In the general formula (Art. 465) substitute b for a, y for x, and -2 for n, and we have

$$(b+y)^{-2}=b^{-2}+-2b^{-2-1}y+\frac{-2(-2-1)}{1\,.\,2}b^{-2-2}y^2$$

$$+\frac{-2(-2-1)(-2-2)}{1\,.\,2\,.\,3}b^{-2-3}y^3+\text{etc.}$$

Reducing, we have $(b+y)^{-2}=\frac{1}{b^2}-\frac{2y}{b^3}+\frac{3y^2}{b^4}-\frac{4y^3}{b^5}+\text{etc.}$

2. Expand $(1+y)^{\frac{1}{2}}$.

SOLUTION. In the general formula put 1 for a, y for x, and $\frac{1}{2}$ for n; substitute and reduce, and we have

$$(1+y)^{\frac{1}{2}} = 1+\tfrac{1}{2}y-\tfrac{1}{8}y^2+\tfrac{1}{16}y^3-\tfrac{5}{128}y^4+\text{etc.}$$

3. Expand $(1+2x-x^2)^{\frac{1}{2}}$.

SOLUTION. Put y for $2x-x^2$; then $(1+2x-x^2)^{\frac{1}{2}}=(1+y)^{\frac{1}{2}}$.

$$(1+y)^{\frac{1}{2}} = 1+\tfrac{1}{2}y-\tfrac{1}{8}y^2+\tfrac{1}{16}y^3-\tfrac{5}{128}y^4+\text{etc.}$$
$$= 1+\tfrac{1}{2}(2x-x^2)-\tfrac{1}{8}(2x-x^2)^2+\tfrac{1}{16}(2x-x^2)^3-\tfrac{5}{128}(2x-x^2)^4+\text{etc.}$$

Reducing, $(1+2x-x^2)^{\frac{1}{2}} = 1+x-x^2+x^3-\frac{3}{2}x^4+\text{etc.}$

4. Expand $(1-x)^{-1}$. *Ans.* $1+x+x^2+x^3+x^4+x^5+\text{etc.}$

5. Expand $(1+a)^{-\frac{1}{2}}$. *Ans.* $1-\frac{1}{2}a+\frac{3}{8}a^2-\frac{5}{16}a^3+\frac{35}{128}a^4-\text{etc.}$

6. Expand $(1-x)^{\frac{1}{3}}$.

Ans. $1-\dfrac{x}{3}-\dfrac{2x^2}{3\,.\,6}-\dfrac{2\,.\,5x^3}{3\,.\,6\,.\,9}-\dfrac{2\,.\,5\,.\,8x^4}{3\,.\,6\,.\,9\,.\,12}-\text{etc.}$

7. Expand $\dfrac{a}{(1-x)^2}$. *Ans.* $a+2ax+3ax^2+4ax^3+5ax^4+\text{etc.}$

8. Expand $\sqrt{a^2-x^2}$. *Ans.* $a-\dfrac{x^2}{2a}-\dfrac{x^4}{8a^3}-\dfrac{x^6}{16a^5}-\dfrac{5x^8}{128a^7}-\text{etc.}$

9. Write the coefficient of x^r in $(1-x)^{-2}$, and coefficient of x^r in $(1-x)^{-4}$. *Ans.* $r+1$; $\dfrac{(r+1)(r+2)(r+3)}{1\,.\,2\,.\,3}$.

10. Write 4th term of $(x+2y)^n$, and 6th term of $(3x-y)^{-\frac{3}{4}}$.

Ans. $\dfrac{4n(n-1)(n-2)}{3}x^{n-3}y^3$; $-\dfrac{3\,.\,7\,.\,11\,.\,15\,.\,19}{4^5\lfloor\underline{5}}(3x)^{-\frac{3}{4}-5}y^5$.

11. Write the $(r+1)$th term of $(1-x)^{-3}$, and the 5th term of $(3x^{\frac{1}{2}}-4y^{\frac{1}{2}})^9$. *Ans.* $\dfrac{(r+1)(r+2)}{2}x^r$; $\dfrac{9\,.\,8\,.\,7\,.\,6}{\lfloor\underline{4}}3^5x^{\frac{5}{2}}4^4y^2$.

12. Write the middle term of $(a+x)^{10}$, and two middle terms of $(a+x)^9$. *Ans.* $\dfrac{\lfloor\underline{10}}{\lfloor\underline{5}\,\lfloor\underline{5}}a^5x^5$; $\dfrac{\lfloor\underline{9}}{\lfloor\underline{4}\,\lfloor\underline{5}}(a^5x^4+a^4x^5)$.

13. $(1+x-x^2)^4 = 1+4x+2x^2-8x^3-5x^4+8x^5+2x^6-4x^7+x^8$.

14. $(1+x+x^2)^{\frac{1}{2}} = 1+\dfrac{x}{2}+\dfrac{3x^2}{8}-\dfrac{3x^3}{16}$, etc.

15. If the 6th, 7th, and 8th terms of $(x+y)^n$ are respectively 112, 7, and $\frac{1}{4}$, find x, y, and n. *Ans.* $x=4$, $y=\frac{1}{2}$, $n=8$.