SOLUTION 2D. Clearing the equation of fractions and reducing partly, we have (2); transposing and uniting, we have (3); whence we have (4). This apparent value of x cannot be verified, as the pupil may see by substitution.

OPERATION 2D.

$$\frac{3x+5}{x+2}=\frac{3x-6}{x-2} \quad (1)$$
$$3x^2-x-10=3x^2-12 \quad (2)$$
$$-x=-2 \quad (3)$$
$$x=2 \quad (4)$$

2. Required a number such that its $\frac{1}{4}$, increased by its $\frac{1}{6}$, is equal to its $\frac{11}{20}$, diminished by its $\frac{2}{15}$.

Ans. $\left(x=\frac{0}{0}\right)$. Indeterminate.

3 Required a number whose $\frac{5}{6}$, diminished by 4, is equal to the sum of its $\frac{1}{2}$ and $\frac{1}{3}$, diminished by 3.

Ans. ($x=\infty$). Impossible.

4. A and B dug a ditch for $20, A receiving $2, and B $3, a day; how many days did each labor, if they did not labor the same number of days? *Ans.* Indeterminate.

5. Twenty years ago, A was 40 years old, and his son was only $\frac{1}{4}$ as old; now the son is $\frac{1}{2}$ as old as the father; gaining thus, when will the son be as old as the father? *Ans.* ($x=\infty$).

6. Find a fraction such that if 2 be subtracted from the numerator, or if 3 be added to the denominator, the resulting fractions will equal $\frac{2}{3}$. *Ans.* Indeterminate.

7. Required a number such that 4 times the number, diminished by 12, divided by the number minus 3, may equal 4 times the number, plus 9, divided by the number plus 3.

Ans. Impossible.

NOTE.—The 6th reduces to the indeterminate form, although $\frac{4}{3}$, $\frac{8}{9}$, $1\frac{2}{15}$, etc., will answer the conditions of the problem.

REVIEW QUESTIONS.

State the principles of Zero and Infinity. Define Generalization. A General Problem. A Formula. A Negative Solution. State the principles of negative solutions. Define the Discussion of a Problem. An Indeterminate Problem. An Impossible Problem. When is a problem indeterminate? When in possible?

SECTION VI.

INVOLUTION, EVOLUTION AND RADICALS

INVOLUTION.

206. Involution is the process of raising a quantity to any given power.

207. A **Power** of a quantity is the product obtained by using the quantity as a factor any number of times.

208. An **Exponent** of a quantity is a number which indicates the power to which the quantity is to be raised.

Thus, let a represent any quantity;

then, $a=a^1$, is the *first* power of a.
$aa=a^2$, is the *second* power of a.
$aaa=a^3$, is the *third* power of a.
$aaaa=a^4$, is the *fourth* power of a.

When the exponent is n, as a^n, it indicates the nth power of a.

209. The **Exponent** (called also the *Index*) indicates how many times the quantity is used as a factor.

The FIRST POWER of a quantity is the quantity itself.
The SECOND POWER of a quantity is called its *square*.
The THIRD POWER of a quantity is called its *cube*.

PRINCIPLES.

1. *The square of a quantity is the product obtained by using the quantity as a factor twice.*
2. *The cube of a quantity is the product obtained by using the quantity as a factor three times.*
3. *All the powers of a positive quantity are positive.*

For, the square of a positive quantity is positive, since it is the product of two positive quantities; and its cube is positive, since it is also the product of two positive quantities, etc.

4. *The* EVEN *powers of a* NEGATIVE *quantity are* POSITIVE, *and the* ODD *powers are* NEGATIVE.

The square is positive, since it is the product of two negative quantities; the *cube* is *negative*, since it is the square, which is *positive*, multiplied by the quantity, which is *negative*; the *fourth* power is *positive*, since it is the cube, a *negative*, multiplied by the quantity, a *negative*; etc.

CASE I.

210. To raise a monomial to a given power.

1. Raise $4a^2b$ to the third power.

SOLUTION. Multiplying $4a^2b$ by itself, we have $16a^4b^2$, and multiplying the square by $4a^2b$, we have $64a^6b^3$. Examining the result, we see we have the *cube* of the coefficient, and the letters of the given quantity with *three* times the exponents which they have in the root.

OPERATION

$$\begin{array}{r} 4a^2b \\ 4a^2b \\ \hline 16a^4b^2 \\ 4a^2b \\ \hline 64a^6b^3 \end{array}$$

Rule.—I. *Raise the coefficient to the required power, and multiply the exponent of each letter by the index of the power.*

II. *When the quantity is positive, the power will be positive; when the quantity is negative, the even powers will be positive and the odd powers negative.*

EXAMPLES.

2. Square of $3ab^2$. *Ans.* $9a^2b^4$.
3. Square of $5a^3c^2$. *Ans.* $25a^6c^4$.
4. Square of $-6a^2x^3$. *Ans.* $36a^4x^6$.
5. Cube of $3a^3x$. *Ans.* $27a^9x^3$.
6. Cube of $-4a^2c^3$. *Ans.* $-64a^6c^9$.
7. Fourth power of $3a^nb^3$. *Ans.* $81a^{4n}b^{12}$.
8. Fourth power of $-2a^mc^n$. *Ans.* $16a^{4m}c^{4n}$.
9. Fifth power of $-a^2b^3c^4$. *Ans.* $-a^{10}b^{15}c^{20}$.
10. Fifth power of $2a^{-3}b^6c^n$. *Ans.* $32a^{-15}b^{30}c^{5n}$.
11. Sixth power of $2a^3c^2$. *Ans.* $64a^{18}c^{12}$.
12. Seventh power of $-2a^3c^2$. *Ans.* $-128a^{21}c^{14}$
13. Eighth power of $-a^{2n}b^3c^{-4}d^{-n}$. *Ans.* $a^{16n}b^{24}c^{-32}d^{-8n}$

14. nth power of $a^2b^3c^4$. *Ans.* $a^{2n}b^{3n}c^{4n}$.
15. nth power of $-2x^2y^{-3}z^4$. *Ans.* $\pm 2^nx^{2n}y^{-3n}z^{4n}$.
16. Value of $(-2a^2b^3)^5$. *Ans.* $-32a^{10}b^{15}$.
17. Value of $(-a^2b^nc)^4$. *Ans.* $a^8b^{4n}c^4$.
18. Value of $(-2b^{-3}m^{-n})^7$. *Ans.* $-128b^{-21}m^{-7n}$.
19. Value of $(-a^nb^{2n})^n$. *Ans.* $\pm a^{n^2}b^{2n^2}$.
20. Value of $(-x^{2n}b^{-2n}c^m)^n$. *Ans.* $\pm x^{2n^2}b^{-2n^2}c^{mn}$.

CASE II.

211. To raise a fraction to a given power.

1. Find the third power of $\frac{a^2}{c^3}$.

SOLUTION. Using the quantity three times as a factor, we have $\frac{a^2}{c^3}\times\frac{a^2}{c^3}\times\frac{a^2}{c^3}=\frac{a^6}{c^9}$.

OPERATION.

$$\frac{a^2}{c^3}\times\frac{a^2}{c^3}\times\frac{a^2}{c^3}=\frac{a^6}{c^9}$$

Rule.—*Raise both numerator and denominator to the required power.*

EXAMPLES.

2. Square of $\frac{2a}{3c}$. *Ans.* $\frac{4a^2}{9c^2}$.
3 Square of $\frac{3a^2}{4c^2}$. *Ans.* $\frac{9a^4}{16c^4}$.
4. Cube of $\frac{a^2b}{cd^3}$. *Ans.* $\frac{a^6b^3}{c^3d^9}$.
5. Cube of $-\frac{2ac^2}{3x^2}$. *Ans.* $-\frac{8a^3c^6}{27x^6}$
6. Square of $-\frac{3a^2c^n}{4a^nc^2}$. *Ans.* $\frac{9c^{2n-4}}{16a^{2n-4}}$
7. Fourth power of $\frac{am^2}{2c^3}$. *Ans.* $\frac{a^4m^8}{16c^{12}}$
8. Fifth power of $-\frac{2xy}{ac^2}$. *Ans.* $-\frac{32x^5y^5}{a^5c^{10}}$.
9. Sixth power of $-\frac{a^2c^n}{a^nbc}$ *Ans.* $\frac{c^{6n-6}}{a^{6n-12}b^6}$

10. Second power of $\frac{2a^{-2}x}{3bz^{-3}}$. *Ans.* $\frac{4a^{-4}x^2}{9b^2z^{-6}}$.

11. Third power of $-\frac{a^3b^2c^{-n}}{a^nb^mc^{-2}}$. *Ans.* $-\frac{c^{6-3n}}{a^{3n-9}b^{3m-6}}$.

12. Find value of $\left(\frac{-ax^n}{2a^nx^2}\right)^4$. *Ans.* $\frac{x^{4n-8}}{16a^{4n-4}}$.

13. Find value of $\left(-\frac{mx^n}{a^nb^{-n}}\right)^5$. *Ans.* $-\frac{m^5x^{5n}}{a^{5n}b^{-5n}}$.

14. Find value of $\left(\frac{a^2b^{-3}}{2cd}\right)^n$. *Ans.* $\frac{a^{2n}b^{-3n}}{2^nc^nd^n}$.

15. Find value of $\left(-\frac{a^nc^2}{a^2c^m}\right)^m$, when m is even. *Ans.* $\frac{a^{m(n-2)}}{c^{m(m-2)}}$.

16. Find value of $\left(-\frac{a^nb^3c^2}{a^2b^nx^2}\right)^m$, when m is odd.
Ans. $-\frac{a^{m(n-2)}c^{2m}}{b^{m(n-3)}x^{2m}}$.

17. Find the square of the fraction $\frac{a^{2n}(x-2)}{a^n(x-3)}$.
Ans. $\frac{a^{2n}(x-2)^2}{(x-3)^2}$.

18. Find the cube of the fraction $\frac{a^3(a^2-4)}{a^3-5a^2+6a}$.
Ans. $\frac{a^6(a+2)^3}{(a-3)^3}$.

CASE III.

212. To raise a polynomial to any given power.

1. To find the square of $a+b$.

SOLUTION. $(a+b)^2=(a+b)(a+b)$, which by multiplying we find to be equal to $a^2+2ab+b^2$.

Rule.—*Find the product of the quantity taken as a factor as many times as there are units in the exponent of the power.*

EXAMPLES.

2. Square of $x-1$. *Ans.* x^2-2x+1.
3. Cube of $a-c$. *Ans.* $a^3-3a^2c+3ac^2-c^3$.
4. Square of $2a^2-3c$. *Ans.* $4a^4-12a^2c+9c^2$.
5. Cube of $1-x$. *Ans.* $1-3x+3x^2-x^3$.
6. Cube of $2a-b^2$. *Ans.* $8a^3-12a^2b^2+$, etc.
7. Fourth power of $a-b$. *Ans.* $a^4-4a^3b+6a^2b^2-$, etc.
8. Fourth power of $2a-c^2$. *Ans.* $16a^4-32a^3c^2+$, etc.
9. Square of $a-b+c$. *Ans.* $a^2-2ab+b^2+2ac-2bc+c^2$.
10. Square of a^2-2b+c^2. *Ans.* $a^4-4a^2b+4b^2+2a^2c^2-4bc^2+c^4$.
11. Cube of $a+b+c$.
Ans. $a^3+3a^2b+3ab^2+b^3+3a^2c+6abc+3b^2c+3ac^2+3bc^2+c^3$.
12. Cube of $2a-3b^2+c^3$.
Ans. $8a^3-36a^2b^2+54ab^4-27b^6+12a^2c^3-36ab^2c^3+27b^4c^3$
$+6ac^6-9b^2c^6+c^9$.

CASE IV.

213. Special methods of squaring a polynomial.

1. Find the square of $a+b+c+d$.

SOLUTION. Squaring the polynomial by actual multiplication, and arranging the terms, we shall have the square as written in the margin. Examining this, we see a certain law which may be stated as follows:

OPERATION.
$(a+b+c+d)^2=$
$a^2+b^2+c^2+d^2+2ab+2ac+$
$2ad+2bc+2bd+2cd$.

Rule.—*The square of a polynomial equals the square of each term and twice the product of the terms taken two and two.*

NOTES.—1. The rule may be briefly stated thus: *The square of every one, twice the product of every two.*

2. The pupils will readily solve the given problems mentally by means of this rule.

EXAMPLES.

2. Square of $a+b+c$. *Ans.* $a^2+b^2+c^2+2ab+2ac+2bc$.
3. Square of $a+b-c+d$.
   Ans. $a^2+b^2+c^2+d^2+2ab-2ac+2ad$, etc.
4. Square of $u-x+y-z$.
   Ans. $u^2+x^2+y^2+z^2-2ux+2uy-2uz$, etc.
5. Square of $a+b+c+d+e$.
   Ans. $a^2+b^2+c^2+d^2+e^2+2ab+2ac$, etc.
6. Square of $a+b+c+d+e+f+g$.
   Ans. $a^2+b^2+c\ +d^2+e^2+f^2+g^2$, etc.
7. Square of $2a+3b+4c+d$.
   Ans. $4a^2+9b^2+16c^2+d^2+12ab+16ac$, etc.
8. Square of $l-m+n-o+p-q+r$.
   Ans. $l^2+m^2+n^2+o^2+p^2+q^2+r^2-2lm$, etc.
9. Square of $a+b-c-d+e+f-g-h+i+j$.
   Ans. $a^2+b^2+c^2+d^2+e^2+f^2+g^2$, etc.
10. Square a polynomial consisting of the letters of the alphabet to m. *Ans.* $a^2+b^2+c^2+d^2+e^2+f^2+g^2$, etc.

214. Second Method.—Arranging the terms in another order and factoring, we have the following:

$$(a+b+c+d)^2=a^2+2ab+b^2+2(a+b)c+c^2+2(a+b+c)d+d^2.$$

Stating this in ordinary language, we have the following rule:

Rule.—*The square of a polynomial equals the square of the first term, plus twice the product of the first term into the second, plus the square of the second, plus twice the sum of the first two terms into the third, plus the square of the third*, etc.

215. Third Method.—Still another form is the following, which pupils may translate into common language:

$$(a+b+c+d)^2=a^2+(2a+b)b+[2(a+b)+c]c+[2(a+b+c)+d]d$$

CASE V.

216. Special method of cubing a polynomial.

1. Find the cube of $a+b+c$.

Solution. Cubing the polynomial by actual multiplication, we have the expression (1). Factoring a part of this expression, we have the cube in the form of expression (2). Examining the last expression, we perceive a law in its formation which may be expressed as follows:

OPERATION.

$$(a+b+c)^3=$$

$$(1)\quad a^3+3a^2b+3ab^2+b^3+3a^2c+6abc+3b^2c+3ac^2+3bc^2+c^3;$$

$$(2)\quad a^3+3a^2b+3ab^2+b^3+3(a+b)^2c+3(a+b)c^2+c^3.$$

Rule.—*The cube of a polynomial equals the* CUBE *of the* FIRST *term, plus* THREE *times the* SQUARE *of the* FIRST *into the* SECOND, *plus* THREE *times the* FIRST *into the* SQUARE *of the* SECOND, *plus the* CUBE *of the* SECOND; *plus* THREE *times the* SQUARE *of the* SUM *of the* FIRST *and* SECOND *into the* THIRD, *plus* THREE *times the* SUM *of the* FIRST *and* SECOND *into the* SQUARE *of the* THIRD, *plus the* CUBE *of the* THIRD, etc.

EXAMPLES.

2. Cube $b+c+d$. *Ans.* $b^3+3b^2c+3bc^2+c^3+3(b+c)^2d+$, etc.
3. Cube $x+y+z$. *Ans.* x^3+, etc., $+3(x+y)^2z+3(x+y)z^2+z^3$.
4. Cube $a+b+c+d$.
   Ans. a^3+, etc., $+3(a+b+c)^2d+3(a+b+c)d^2+d^3$
5. Cube $a+b+c+d+e$.
   Ans. a^3+, etc., $+3(a+b+c+d)^2e+3(a+b+c+d)e^2+e^3$.
6. Cube $a+b-c+d-e$. *Ans.* a^3+, etc., $+3(a+b-c+d)e^2-e^3$.
7. Cube x^2-2y+z^3.
   Ans. $x^6-6x^4y+12x^2y^2-8y^3+3(x^2-2y)^2z^3+$, etc.

Note.—In solving the 7th, expand $a+b+c$, and then substitute x^2 for a, $-2y$ for b, and z^3 for c, or involve it directly.

217. Another Form.—The formula for cubing a polynomial may be put in another form, which is sometimes more convenient, as follows:

$$(a+b+c)^3=a^3+(3a^2+3ab+b^2)b+[3(a+b)^2+3(a+b)c+c^2]c$$

THE BINOMIAL THEOREM.

218. The **Binomial Theorem** expresses a general method of raising a binomial to any power.

219. This theorem affords a much shorter method of raising binomials to required powers than the tedious process of multiplication.

NOTE.—The Theorem was discovered by *Sir Isaac Newton.* It was considered of so much importance that the formula expressing it was engraved upon his monument in Westminster Abbey.

220. To derive the binomial theorem, we will raise two binomials to different powers by actual multiplication, and then examine these powers to discover the law of their formation.

Let us raise $a+b$ to the 2d, 3d, 4th and 5th powers.

$$\begin{array}{rl}
 & a+b \\
 & a+b \\
\hline
 & a^2+ab \\
 & \quad ab+b^2 \\
\hline
\text{2d power,} & a^2+2ab+b^2 \\
 & a+b \\
\hline
 & a^3+2a^2b+ab^2 \\
 & \quad a^2b+2ab^2+b^3 \\
\hline
\text{3d power,} & a^3+3a^2b+3ab^2+b^3 \\
 & a+b \\
\hline
 & a^4+3a^3b+3a^2b^2+ab^3 \\
 & \quad a^3b+3a^2b^2+3ab^3+b^4 \\
\hline
\text{4th power,} & a^4+4a^3b+6a^2b^2+4ab^3+b^4 \\
 & a+b \\
\hline
 & a^5+4a^4b+6a^3b^2+4a^2b^3+ab^4 \\
 & \quad a^4b+4a^3b^2+6a^2b^3+4ab^4+b^5 \\
\hline
\text{5th power,} & a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5
\end{array}$$

Raising $a-b$ to the 2d, 3d, 4th and 5th powers, we have—

$$(a-b)^2=a^2-2ab+b^2;$$
$$(a-b)^3=a^3-3a^2b+3ab^2-b^3;$$
$$(a-b)^4=a^4-4a^3b+6a^2b^2-4ab^3+b^4;$$
$$(a-b)^5=a^5-5a^4b+10a^3b^2-10a^2b^3+5ab^4-b^5.$$

In deriving a law from these examples there are five things to be considered:

1st. The *number* of terms; 2d. The *signs* of the terms; 3d. The *letters* in the terms; 4th. The *exponents* of the letters; 5th. The *coefficients* of the terms.

I. NUMBER OF TERMS.—Examining the results in the given examples, we see that the *second* power has *three* terms, the *third* power *four* terms, the *fourth* power *five* terms, the *fifth* power *six* terms; hence we infer that

The number of terms in any power of a binomial is one greater than the exponent of the power.

II. SIGNS OF THE TERMS.—By examining the signs of the terms in the different powers, we infer the following principles:

1. *When both terms of the binomial are positive, all the terms will be positive.*

2. *When the first term is positive and the second negative, all the* ODD *terms, counting from the left, will be* POSITIVE, *and all the* EVEN *terms will be* NEGATIVE.

III. THE LETTERS.—By examining the letters in the different powers we infer the following principle:

The first letter of the binomial appears in all the terms except the last; the second letter appears in all except the first; and their product appears in all except the first and the last.

IV. THE EXPONENTS.—By examining the exponents of the terms in the different powers, we infer the following principles:

1. *The exponent of the leading letter or quantity in the first term is the same as the exponent of the power, and decreases by unity in each successive term toward the right.*

2. *The exponent of the second letter or quantity in the second term is one, and increases by unity in each successive term toward the right, until in the last term it is the same as the exponent of the power.*

3. *The sum of the exponents in any term is equal to the exponent of the power.*

V. THE COEFFICIENTS.—By examining the coefficients of the terms in the different powers, we infer the following principles:

1. *The coefficient of the first and the last term is* 1.

2. *The coefficient of the second term is the exponent of the power.*

Thus, in the *second* power it is 2; in the *third* power, 3; in the *fourth* power, 4; and in the *fifth* power, 5.

3. *The coefficient of any term, multiplied by the exponent of the leading letter in that term, and divided by the number of the term, equals the coefficient of the next term.*

Thus, in examining the powers of $a+b$ or $a-b$, we see that in the 4th power the coefficient of the second term, 4, multiplied by 3, the exponent of a in that term, and divided by 2, the number of the term, equals 6, the coefficient of the following term; in the 5th power we have 5, the coefficient of the 2d term, multiplied by 4, the exponent of a in that term, and divided by 2, the number of the term, equals 10, the coefficient of the 3d term; also 10, multiplied by 3 and divided by 3, equals 10, the coefficient of the 4th term, etc.

NOTES.—1. We see that the coefficients of the last half of the terms when *even*, or the terms after the middle when *odd*, are the same as the coefficients of the preceding terms, inversely; hence we may write the coefficients of the terms after the middle term without actual calculation.

2. The above method of deriving the Binomial Theorem is by observation and induction from particular cases. For a more rigid demonstration see Supplement, page 330.

EXAMPLES.

1. Raise $x-y$ to the fourth power.

SOLUTION.

Letters and exponents,	x^4	x^3y	x^2y^2	xy^3	y^4
Coefficients and signs,	1	-4	$+6$	-4	$+1$
Combining,	$x^4-4x^3y+6x^2y^2-4xy^3+y^4$				

NOTE.—In practice, we first write the literal part of the development, and then, commencing at the first term, insert the coefficients with their signs.

2. Develop $(a+x)^3$. *Ans.* $a^3+3a^2x+3ax^2+x^3$.

3. Develop $(a+c)^4$. *Ans.* $a^4+4a^3c+6a^2c^2+4ac^3+c^4$.

4. Develop $(a-b)^5$.
Ans. $a^5-5a^4b+10a^3b^2-10a^2b^3+5ab^4-b^5$.

5. Develop $(x-y)^6$.
Ans. $x^6-6x^5y+15x^4y^2-20x^3y^3+15x^2y^4-6xy^5+y^6$.

6. Develop $(a+b)^7$.
Ans. $a^7+7a^6b+21a^5b^2+35a^4b^3+35a^3b^4+21a^2b^5+7ab^6+b^7$.

7. Develop $(a-x)^8$.
Ans. $a^8-8a^7x+28a^6x^2-56a^5b^3+70a^4x^4-56a^3x^5+28a^2x^6-8ax^7+x^8$.

8. Develop $(1-x)^6$.
Ans. $1-6x+15x^2-20x^3+15x^4-6x^5+x^6$.

9. Develop $(a-c)^9$.
Ans. $a^9-9a^8c+36a^7c^2-84a^6c^3+126a^5c^4-126a^4c^5+84a^3c^6-36a^2c^7+9ac^8-c^9$.

10. Develop $(a+c)^{10}$.
Ans. $a^{10}+10a^9c+45a^8c^2+120a^7c^3+210a^6c^4+252a^5c^5+210a^4c^6+120a^3c^7+45a^2c^8+10ac^9+c^{10}$.

11. Develop $(a+x)^n$.
Ans. $a^n+na^{n-1}x+\frac{n(n-1)}{2}a^{n-2}x^2+\frac{n(n-1)(n-2)}{2\,3}a^{n-3}x^3+\ldots\, nax^{n-1}+x^n$.

BINOMIALS, WITH COEFFICIENTS AND EXPONENTS.

221. The **Binomial Theorem** can also be applied to binomials when one or both terms have coefficients and exponents.

1. Raise $2a^2+3b$ to the fourth power.

SOLUTION.

Let $2a^2=m$ and $3b=n$; then $m+n$ will equal $2a^2+3b$.

We have $(m+n)^4=m^4+4m^3n+6m^2n^2+4mn^3+n^4$.

Substituting, $(2a^2)^4+4(2a^2)^3 3b+6(2a^2)^2(3b)^2+4(2a^2)(3b)^3+(3b)^4$.

Reducing, $16a^8+96a^6b+216a^4b^2+216a^2b^3+81b^4$.

NOTE.—It can also be solved by writing the second expression directly and then reducing, without making the substitution.

2. Develop $(2a-3b)^3$. *Ans.* $8a^3-36a^2b+54ab^2-27b^3$.

3. Develop $(a^2+3x)^4$. *Ans.* $a^8+12a^6x+54a^4x^2+108a^2x^3+81x^4$.

4. Develop $(4a-3a^2x^3)^3$.
Ans. $64a^3-144a^4x^3+108a^5x^6-27a^6x^9$.

5. Develop $\left(x-\frac{a}{c}\right)^4$. *Ans.* $x^4-\frac{4ax^3}{c}+\frac{6a^2x^2}{c^2}-\frac{4a^3x}{c^3}+\frac{a^4}{c^4}$.

6. Develop $\left(a^2-\frac{x}{a}\right)^5$.
Ans. $a^{10}-5a^7x+10a^4x^2-10ax^3+\frac{5x^4}{a^2}-\frac{x^5}{a^5}$.

7. Develop $\left(3z-\frac{1}{z}\right)^4$. *Ans.* $81z^4-108z^2+54-\frac{12}{z^2}+\frac{1}{z^4}$.

8. Develop $(\frac{1}{2}a-\frac{2}{3}c)^5$.
Ans. $\frac{a^5}{32}-\frac{5a^4c}{24}+\frac{5a^3c^2}{9}-\frac{20a^2c^3}{27}+\frac{40ac^4}{81}-\frac{32c^5}{243}$.

9. Develop $(n^2-n^{-2})^6$.
Ans. $n^{12}-6n^8+15n^4-20+15n^{-4}-6n^{-8}+n^{-12}$.

10. Develop $(1+\frac{3}{2}x)^5$.
Ans. $1+\frac{15x}{2}+\frac{45x^2}{2}+\frac{135x^3}{4}+\frac{405x^4}{16}+\frac{243x^5}{32}$.

11. Develop $(2a^2x-4az^3)^6$.
Ans. $64a^{12}x^6-768a^{11}x^5z^3+3840a^{10}x^4z^6\cdots$, etc.

APPLIED TO POLYNOMIALS.

222. The **Binomial Theorem** may also be applied to polynomials by regarding them as binomials.

1. Find the second power of $a+b+c+d$.

SOLUTION.

Let $x=a+b$ and $y=c+d$; then $(x+y)^2=(a+b+c+d)^2$.

By the Theorem, $(x+y)^2=x^2+2xy+y^2$.

Substituting, $(a+b)^2+2(a+b)(c+d)+(c+d)^2$.

Expanding, $a^2+2ab+b^2+2ac+2bc+c^2+2ad+2bd+2cd+d^2$.

2. Expand $(a-b+c)^2$. *Ans.* $a^2-2ab+b^2+2ac-2bc+c^2$.

3. Expand $(a+b-c)^3$.
Ans. $a^3+3a^2b+3ab^2+b^3-3a^2c-6abc-3b^2c+3ac^2$
$+3bc^2-c^3$.

4. Expand $(a+b+c+d)^3$.
Ans. $a^3+3a^2b+3ab^2+b^3+3a^2c+6abc+3b^2c+3ac^2+3bc^2$
$+c^3+$, etc.

MISCELLANEOUS EXAMPLES.

1. Prove that the square of the sum of two numbers exceeds the square of their difference by 4 times their product.

2. Show how much the square of the sum of two numbers exceeds the product of their sum and difference.

3. What is the difference between one-half the square of a number and the square of one-half a number?

4. Prove that the difference between the squares of two consecutive numbers equals twice the less number, plus 1.

5. If two numbers differ by unity, prove that the difference of their squares equals the sum of the numbers.

6. Of three consecutive numbers, what is the difference between the square of the second and the product of the first and third?

7. Prove that the sum of the cubes of three consecutive numbers is divisible by the sum of the numbers.

EVOLUTION.

223. **Evolution** is the process of extracting the root of a quantity.

224. A **Root** of a quantity is *one* of its *several equal* factors; or, it is a quantity of which the given quantity is a power.

225. The **Square Root** of a quantity is *one* of its *two equal* factors; the *cube root* is *one* of its *three equal* factors, etc.

226. The **Symbol** $\sqrt{\ }$, called the *radical sign*, indicates the root of a quantity; thus, $\sqrt{4}$ indicates the square root of 4; $\sqrt[3]{8}$ indicates the cube root of 8.

227. The **Index** of the root is the figure placed above the radical sign to denote what root is required.

228. A **Fractional Exponent** is also used to indicate a root of a quantity; thus, $a^{\frac{1}{2}}$ indicates the square root of a; $8^{\frac{1}{3}}$ indicates the cube root of 8.

229. In fractional exponents the numerator indicates a power and the denominator a root of the quantity; thus,

$a^{\frac{2}{3}} = \sqrt[3]{a^2}$, or the cube root of a squared;

$a^{\frac{m}{n}} = \sqrt[n]{a^m}$, or the nth root of the mth power of a.

230. A **Perfect Power** is a quantity whose required root can be exactly obtained. An *Imperfect Power* is a quantity whose required root cannot be exactly obtained.

PRINCIPLES.

1. *The odd roots of a positive quantity are positive.*

For, the odd powers of a positive quantity are positive, while the odd powers of a negative quantity are negative. Thus, $\sqrt[3]{8} = +2$, and $\sqrt[5]{a^5} = +a$.

2. *The even roots of a positive quantity are either positive or negative.*

For, the *even* powers of either a positive or a negative quantity are positive; thus, $(+3)^2 = 9$ and $(-3)^2 = 9$; hence, $\sqrt{9} = +3$ or -3; also, $\sqrt{a^2} = +a$ or $-a$, since $(+a)^2$ and $(-a)^2$ both equal a^2.

NOTE.—The symbol $\pm$ means *plus* or *minus*; thus, $\sqrt{a^2} = \pm a$ is read the square root of a^2 is *plus* or *minus* a.

3. *The odd roots of a negative quantity are negative.*

For, the odd powers of a negative quantity are negative, while all the powers of a *positive* quantity are positive. Thus, $\sqrt[3]{-8} = -2$ and $\sqrt[5]{-a^5} = -a$.

4. *The even roots of a negative quantity are impossible.*

For, the *even* powers of both positive and negative quantities are positive; hence no quantity raised to an even power can produce a negative quantity. Thus, $\sqrt{-4}$, $\sqrt[4]{-16}$ and $\sqrt{-a^2}$, are all impossible.

NOTE.—The expression of the even root of a negative quantity is called an *Imaginary Quantity*.

CASE I.

231. To extract any root of a monomial.

232. The methods of extracting the roots of monomials are derived from those of involution, the one being the reverse of the other.

PRINCIPLES.

1. *To find any root of a monomial, we extract the root of the numeral coefficient and divide the exponents of the letters by the index of the root.*

This is evident, since in raising a monomial to any power we raise the coefficient to the required power and multiply the exponents of the letters by the index of the power. (Art. 210.)

2. *To find any root of a fraction, we extract the root of both terms.*

This is evident, since to raise a fraction to any power we raise both terms to the required power. (Art. 211.)

1. Find the square root of $9a^2b^4$.

OPERATION.

$\sqrt{9a^2b^4} = \pm 3ab^2$

SOLUTION. Since to square a monomial we square the coefficient and multiply the exponents of the letters by 2, to find the square root of a monomial we reverse the process, and extract the square root of the coefficient and divide the exponents of the letters by 2. Doing this, we have $3ab^2$, and this is *plus* or *minus* (Prin. 2, Art. 230). Hence, $\sqrt{9a^2b^4}$ equals $\pm 3ab^2$.

2. Find the cube root of $8a^6b^3$.

OPERATION.

$$\sqrt[3]{8a^6b^3}=2a^2b$$

SOLUTION. Since extracting the cube root of a quantity is the reverse of raising it to the third power, we extract the cube root of the coefficient 8, and divide the exponents of a^6 and b^3 by 3, and we have $2a^2b$; and the sign is plus (Prin. 1, Art. 230).

Rule.—I. *Extract the root of the numeral coefficient and divide the exponents by the index of the root.*

II. *Prefix the double sign, $\pm$, to all even roots, and the minus sign to odd roots of negative quantities.*

EXAMPLES.

3. Square root of $16a^4b^6$. *Ans.* $\pm 4a^2b^3$.
4. Square root of $25x^6y^2$. *Ans.* $\pm 5x^3y$.
5. Cube root of $27a^6b^9$. *Ans.* $3a^2b^3$.
6. Cube root of $-64a^3x^6$. *Ans.* $-4ax^2$.
7. Fifth root of $-32a^5c^{10}$. *Ans.* $-2ac^2$.
8. Fifth root of $-243a^{10}x^{15}z^{20}$. *Ans.* $-3a^2x^3z^4$.
9. Cube root of $216a^3x^6y^9$. *Ans.* $6ax^2y^3$.
10. Square root of $144a^{2n}b^{4n}c^6$. *Ans.* $\pm 12a^nb^{2n}c^3$.
11. Square root of $64ab^3c^6$. *Ans.* $\pm 8a^{\frac{1}{2}}b^{\frac{3}{2}}c^3$.
12. Cube root of $125a^2x^4z^6$. *Ans.* $5a^{\frac{2}{3}}x^{\frac{4}{3}}z^2$.
13. Value of $\sqrt[5]{(a^2c^3x^5)}$. *Ans.* $a^{\frac{2}{5}}c^{\frac{3}{5}}x$.
14. Value of $\sqrt[4]{(16a^{4n}b^{8n})}$. *Ans.* $\pm 2a^nb^{2n}$.
15. Value of $(a^{2n}x^{3n})^{\frac{1}{n}}$. *Ans.* a^2x^3.
16. Value of $(\frac{4}{9}a^4x^6)^{\frac{1}{2}}$. *Ans.* $\pm\frac{2}{3}a^2x^3$.
17. Square root of $\frac{16a^3}{25x^5}$. *Ans.* $\pm\frac{4a^{\frac{3}{2}}}{5x^{\frac{5}{2}}}$.
18. Cube root of $\frac{8a^3}{27m^6}$. *Ans.* $\frac{2a}{3m^2}$.
19. Cube root of $\frac{64a^{3n}}{125c^{6n}}$. *Ans.* $\frac{4a^n}{5c^{2n}}$.
20. Value of $\sqrt[n]{(a^nc^{2n}x^3z^m)}$. *Ans.* $ac^2x^{\frac{3}{n}}z^{\frac{m}{n}}$.

CASE II.

233. To extract the square root of a polynomial.

234. The method of extracting the square root of a polynomial is derived from the law for the square of a polynomial.

1. Find the square root of $a^2+2ab+b^2$.

OPERATION.

$$\begin{array}{r|l} & a^2+2ab+b^2 \;\lfloor a+b \\ & a^2 \\ 2a & 2ab+b^2 \\ 2a+b & 2ab+b^2 \end{array}$$

SOLUTION. The first term, a^2, is the square of the first term of the root, hence the first term of the root is the square root of a^2, which is a. Squaring a and subtracting it from the polynomial, we have $2ab+b^2$, which equals *twice the first term, plus the second term, multiplied by the second term* (Art. 214); hence, if we divide by twice the first term of the root, we will obtain the second term. Dividing by $2a$, we have b, the second term of the root; adding b to $2a$, the *trial divisor*, we have $2a+b$; multiplying by b and subtracting, there is no remainder. Hence, the square root of $a^2+2ab+b^2$ is $a+b$.

2. Extract the square root of $a^2+2ab+2b^2+2ac+2bc+c^2$.

OPERATION.

$$\begin{array}{r|l} & a^2+2ab+b^2+2ac+2bc+c^2 \;\lfloor a+b+c \\ & a^2 \\ 2a & 2ab+b^2 \\ 2a+b & 2ab+b^2 \\ 2a+2b & 2ac+2bc+c^2 \\ 2a+2b+c & 2ac+2bc+c^2 \end{array}$$

SOLUTION. Proceeding as in Prob. 1, we obtain the first two terms of the root, $a+b$, with a remainder $2ac+2bc+c^2$. This remainder equals twice the sum of the first and second terms, plus the third term, multiplied by the third term (Art. 215); hence, if we divide by twice the sum of the first and second terms, we will obtain the third term. Twice $a+b$ are $2a+2b$; dividing the remainder by $2a+2b$, we obtain c, the third term. Adding c to the *trial divisor*, we have $2a+2b+c$; multiplying by c and subtracting, nothing remains. Hence, etc.

Rule.—I. *Arrange the terms of the polynomial with reference to the powers of some letter.*

II. *Extract the square root of the first term; write the result as the first term of the root; subtract the square from the given polynomial, and bring down the next two terms for a dividend.*

III. *Double the root already found, for a trial divisor; divide the first term of the dividend by the result; annex the quotient to the root and also to the trial divisor for a complete divisor. Multiply the complete divisor by the second figure of the root, and subtract the product from the dividend.*

IV. *If there are other terms of the polynomial remaining, proceed in a similar manner until the work is completed.*

NOTES.—1. If the first term of any remainder, when properly arranged, is not divisible by double the first term of the root, the polynomial is not a perfect square.

2. When the trial divisor consists of two or more terms, it is necessary to divide only the first term of the dividend by the first term of the divisor.

EXAMPLES.

Find the square root—

3. Of a^2+4a+4. *Ans.* $a+2$.

4. Of $a^2-4ac+4c^2$. *Ans.* $a-2c$.

5. Of $4a^2-12ax+9x^2$. *Ans.* $2a-3x$.

6. Of $x^2-xy+\frac{1}{4}y^2$. *Ans.* $x-\frac{1}{2}y$.

7. Of $a^{2n}+2a^nb^n+b^{2n}$. *Ans.* a^n+b^n.

8. Of $a^2+b^2+c^2+2ab+2ac+2bc$. *Ans.* $a+b+c$.

9. Of $a^2+x^2+z^2-2ax-2az+2xz$. *Ans.* $a-x-z$.

10. Of $a^2+4ab+4b^2-2ac-4bc+c^2$. *Ans.* $a+2b-c$.

11. Of $a^2-4ac+4c^2+6a-12c+9$. *Ans.* $a-2c+3$.

12. Of $a-2a^{\frac{3}{2}}b+a^2b^2+2a^{\frac{1}{2}}b^{\frac{1}{3}}-2ab^{\frac{4}{3}}+b^{\frac{2}{3}}$. *Ans.* $a^{\frac{1}{2}}-ab+b^{\frac{1}{3}}$.

13. Of $m^4+4m^3n+10m^2n^2+12mn^3+9n^4$. *Ans.* $m^2+2mn+3n^2$.

14. Of $4x^2-16x+16+12xy-24y+9y^2$. *Ans.* $2x-4+3y$.

15. Of $10n^2+25n^4-20n^3+1-4n+16n^6-24n^5$. *Ans.* $1-2n+3n^2-4n^3$.

16. Of $a^6+4a^4x^2+4a^2x^4+x^6-4a^5x+4a^4x^2-2a^3x^3-8a^3x^3+4a^2x^4-4ax^5$. *Ans.* $a^3-2a^2x+2ax^2-x^3$.

SQUARE ROOT OF NUMBERS.

235. The method of extracting the *Square Root of Numbers* is most satisfactorily explained by Algebra.

PRINCIPLES.

1. *The square of a number consists of twice as many figures as the number, or of twice as many less one.*

$1^2=1$
$10^2=100$
$100^2=10{,}000$
$1000^2=1{,}000{,}000$

Any integral number between 1 and 10 consists of one figure, and any number between their squares, 1 and 100, consists of *one* or *two* figures; hence the square of a number of *one* figure is a number of *one* or *two* figures. Any number between 10 and 100 consists of *two* figures, and any number between their respective squares, 100 and 10,000, consists of *three* or *four* figures; hence, the square of a number of *two* figures is a number of *three* or *four* figures, etc. Therefore, etc.

2. *If a number be pointed off into periods of two figures each, beginning at units' place, the number of full periods, together with the partial period at the left, if there be one, will equal the number of places in the square root.*

This is evident from Prin. 1, since the square of a number contains *twice* as many places as the number, or *twice* as many, *less one.*

3. *If we represent the units by* u, *the tens by* t, *the hundreds by* h, *the thousands by* T, *we will have the following formulas:*

$(t+u)^2=t^2+2tu+u^2$;
$(h+t+u)^2=h^2+2ht+t^2+2(h+t)u+u^2$;
$(T+h+t+u)^2=T^2+2Th+h^2+2(T+h)t+t^2+2(T+h+t)u+u^2$.

1. Extract the square root of 2025.

OPERATION.

$t^2+2tu+u^2=$		20·25 (	40
$t^2=40^2$	$=$	1600	5
$2tu+u^2=$		425	45
$2t=40\times2=80$			
$u=$	5		
$(2t+u)u=85\times5=425$			

SOLUTION. Separating the number into periods of two figures each, we find there are two figures in the root (Prin. 2); hence the root consists of *tens* and *units*, and the number equals the *square of the tens, plus twice the tens into the units, plus the square of the units.*

The greatest number of *tens* whose square is contained in 2025 is 4 *tens;* squaring the *tens* and subtracting, we have

425, which equals *twice the tens into the units, plus the square of the units* Now, since *twice the tens into the units* is generally greater than the *units into the units*, 425 consists principally of *twice the tens into the units;* hence if we divide by *twice the tens*, we can ascertain the *units*. *Twice the tens* equals 40×2, or 80; dividing 425 by 80, we find the units to be 5; then $(2t+u)u$, which equals $2tu+u^2$, equals (80+5)×5, or 425; subtracting, nothing remains. Hence the square root of 2025 is 4 tens and 5 units, or 45.

2. Extract the square root of 104976.

SOLUTION. Separating the number into periods of two figures each, beginning at the right, we find there are three figures in the root, and the root consists of *hundreds, tens* and *units*, and the number equals $h^2+2ht+t^2+(2h+t)u+u^2$.

OPERATION.

$h^2+2ht+t^2+2(h+t)u+u^2=$		10·49·76	(300
$h^2=$	$300^2=$	9 00 00	20
$2ht+t^2+2(h+t)u+u^2$		14976	4
$2h=$	300×2=600		324
$t=$	20		
$(2h+t)t=$	620 × 20 =	12400	
$2(h+t)u+u^2=$		2576	
$2(h+t)=$	320×2=640		
$u=$	4		
$[2(h+t)+u]u=$	(644) × 4 =	2576	

The greatest number of *hundreds* whose square is contained in 104976 is 3 *hundreds;* squaring and subtracting, we have 14976 remaining, which equals $2ht+t^2+$, etc. Now, since *twice the hundreds into the tens* is generally much greater than t^2+, etc., 14976 must consist principally of *twice the hundreds into the tens;* hence, if we divide by *twice the hundreds*, we can ascertain the *tens*. *Twice the hundreds* equals 300×2=600; dividing by 600, we find the tens to be 2, etc.

NOTE.—In practice we abbreviate the work by omitting the ciphers and condensing the other parts, preserving merely the *trial* and *true* divisors, as is indicated in the margin. Here 6 is the first *trial divisor* and 62 the first *true divisor;* 64 is the second *trial*, and 644 the second *true divisor.*

OPERATION AS IN PRACTICE

	10·49·76(324
3	9
62	149
644	124
	2576
	2576

From these explanations we derive the following rule:

Rule.—I. *Commence at units' place and separate the number into periods of two figures each.*

II. *Find the greatest number whose square is contained in the left-hand period, place it at the right as a quotient and its square under the left-hand period, subtract and annex the next period to the remainder for a dividend.*

III. *Double the root found, and place it at the left for a trial divisor, divide the dividend exclusive of the right-hand figure by it; the quotient will be the second term of the root.*

IV. *Annex the second term of the root to the trial divisor for the true divisor, multiply the result by the second term of the root, subtract and bring down the next period for the next dividend.*

V. *Double the root now found, find the third term of the root as before, and thus proceed until all the periods have been used.*

NOTES.—1. If the product of any true divisor by the corresponding term of the root exceeds the dividend, the term of the root must be diminished by a unit.

2. When a dividend, exclusive of the right-hand figure, will not contain the trial divisor, place a cipher in the root and at the right of the trial divisor, then bring down the next period and proceed as before.

3. To extract the square root of a decimal, *we point off the decimal into periods of two figures each, counting from units' place, and proceed as with whole numbers;* the reason of which may be easily seen. When a number is not a perfect square, annex ciphers and find the root on to decimals.

4. To find the square root of a common fraction, it is evident that we extract the square root of both numerator and denominator. When these terms are not perfect squares, the shortest way is to reduce the fraction to a decimal and extract the root.

EXAMPLES.

Extract the square root—

3. 576.	*Ans.* 24.	11. 5503716.	*Ans.* 2346.
4. 1296.	*Ans.* 36.	12. 4137156.	*Ans.* 2034.
5. 2401.	*Ans.* 49.	13. 11594025	*Ans.* 3405.
6. 56644.	*Ans.* 238.	14. $\frac{196}{625}$.	*Ans.* $\frac{14}{25}$.
7. 119025.	*Ans.* 345.	15. $164\frac{50}{72}$.	*Ans.* $12\frac{5}{6}$.
8. 207936.	*Ans.* 456.	16. 12.96.	*Ans.* 3 6.
9. 321489.	*Ans.* 567.	17. .0064.	*Ans.* .08.
10 6421156.	*Ans.* 2534.	18. 10.6929.	*Ans.* 3.27.