CASE III.

236. To extract the cube root of polynomials.

237. The method of extracting the cube root of a polynomial is derived from the law for the cube of a polynomial.

1. Find the cube root of $a^3+3a^2b+3ab^2+b^3$.

OPERATION.

$$\begin{array}{l|l} & a^3+3a^2b+3ab^2+b^3 \;(a+b \\ & a^3 \\ 3a^2 & 3a^2b+3ab^2+b^3 \\ 3a^2+3ab+b^2 & 3a^2b+3ab^2+b^3 \end{array}$$

SOLUTION. The first term, a^3, is the cube of the first term of the root (Art. 220); hence, the first term of the root is the cube root of a^3, or a. Cubing a and subtracting it from the polynomial, we have $3a^2b+3ab^2+b^3$, which equals *three times the square of the first term of the root into the second, plus,* etc.; hence, if we divide the first term of the remainder by *three* times *the first term* of the root, we can ascertain the *second term.* Three times a squared equals $3a^2$, the *trial divisor;* dividing $3a^2b$ by $3a^2$, we obtain b, the *second term* of the root. Adding to the trial divisor *three times the product of the first term of the root by the last term,* and the *square of the last term,* we have for a complete divisor $3a^2+3ab+b^2$; multiplying this by b, we have $3a^2b+3ab^2+b^3$ subtracting this, nothing remains. Hence the cube root of the given polynomial is $a+b$.

Rule.—I. *Arrange the terms of the polynomial with reference to the powers of some letter.*

II. *Extract the cube root of the first term; write the result as the first term of the root; subtract its cube from the given polynomial by bringing down the next three terms for a dividend.*

III. *Take three times the square of the root found for a trial divisor; divide the first term of the dividend by it, and write the quotient for the next term of the root.*

IV. *Add to the trial divisor three times the product of the first and last terms of the root, and the square of the last term, for a complete divisor. Multiply the complete divisor by the last term of the root, and subtract the product from the dividend.*

V. *If there are other terms of the polynomial remaining, proceed in a similar manner until the work is completed.*

NOTES.—1. Arrange the terms of each new dividend, when necessary, with reference to the powers of the leading letter of the root.

2 When there are three terms in the root, to obtain the third, we use the first two terms as we did the first term in obtaining the second.

EXAMPLES.

Find the cube root—

2. Of $a^3-3a^2x+3ax^2-x^3$. *Ans.* $a-x$.
3. Of $a^3+6a^2b+12ab^2+8b^3$. *Ans.* $a+2b$.
4. Of $x^6-6x^4+12x^2-8$. *Ans.* x^2-2.
5. Of $a^3+3a^2b+3ab^2+b^3+3a^2c+6abc+3b^2c+3ac^2+3bc^2+c^3$. *Ans.* $a+b+c$.
6. Of $a^6-3a^5+5a^3-3a-1$. *Ans.* a^2-a-1.
7. Of $a^6-6a^5+15a^4-20a^3+15a^2-6a+1$. *Ans.* a^2-2a+1.
8. Of $m^6+6m^5-40m^3+96m-64$. *Ans.* m^2+2m-4.
9. Of $x^6-3ax^5+5a^3x^3-3a^5x-a^6$. *Ans.* x^2-ax-a^2.

CUBE ROOT OF NUMBERS.

238. The method of extracting the *Cube Root of Numbers* is most satisfactorily explained by means of Algebra.

PRINCIPLES.

1. *The cube of a number consists of three times as many figures as the number, or of three times as many less one or two.*

$1^3=1$
$10^3=1000$
$100^3=1,000,000$

Any integral number between 1 and 10 consists of one figure, and any integral number between their cubes, 1 and 1000, consists of one, two or three figures; hence the *cube* of a number of *one* figure is a number of *one, two* or *three* figures. Any number between 10 and 100 consists of *two* figures, and any number between their cubes, 1000 and 1,000,000, consists of *four, five* or *six* figures; hence the cube of a number of *two* figures consists of *three times two* figures, or *three times two,* less *one* or *two* figures. Therefore, etc.

2. *If a number be pointed off into periods of three figures each, beginning at units' place, the number of full periods, together with the partial period at the left, if there be one, will equal the number of figures in the root.*

This is evident from Prin. 1, since the cube of a number contains *three times* as many places as the number, or *three times* as many, *less one or two.*

3. *If we represent the units by* u, *the tens by* t, *the hundreds by* h, etc., *we will have the following formulas:*

1. $(t+u)^3 = t^3 + 3t^2u + 3tu^2 + u^3$;
2. $(h+t+u)^3 = h^3 + 3h^2t + 3ht^2 + t^3 + 3(h+t)^2u + 3(h+t)u^2 + u^3$.

1. Extract the cube root of 15625.

SOLUTION. Separating the number into periods of three figures each, we find there are two figures in the root (Prin. 2); hence the root consists of *tens* and *units*, and the number equals the $tens^3 + 3 \times tens^2 \times units + 3 \times tens \times units^2 + units^3$.

OPERATION.

$t^3+3t^2u+3tu^2+u^3=$		15·625	(20
$t^3=$	$20^3=$	8000	5
$3t^2u+3tu^2+u^3=$		7625	25
$3t^2=$	$3\times20^2=1200$		
$3tu=$	$3\times20\times5=\ 300$		
$u^2=$	$5^2=\ \ 25$		
$(3t^2+3tu+u^2)u$	$1525\times5=$	7625	

The greatest number of *tens* whose cube is contained in 15625 is 2 *tens*; cubing the tens and subtracting, we have 7625, which equals $3 \times tens^2 \times units + 3 \times tens \times units^2 + units^3$. Now, since $3 \times tens^2 \times units$ is generally greater than $3 \times tens \times units^2 +$, etc., 7625 consists principally of $3 \times tens^2 \times units$; hence, if we divide 7625 by $3 \times tens^2$, we can ascertain the *units*. $3 \times tens^2 = 20^2 \times 3 = 1200$; dividing 7625 by 1200, we find the *units* to be 5. We then find $3 \times tens \times units$ equals $3 \times 20 \times 5 = 300$, and $units^2$ equals $5^2 = 25$; taking their sum, we have $3t^2 + 3tu + u^2 = 1525$; and multiplying by the *units*, we have $(3t^2 + 3tu + u^2)u = 1525 \times 5$, or 7625; and subtracting, nothing remains. Hence the cube root of 15625 is 25.

AS IN PRACTICE.

		15·625 (25
$2^3=$	8	
$3\times20^2=$	1200	7625
$3\times20\times5=$	300	
$5^2=$	25	
	1525	7625

NOTE.—In practice we omit the naughts and abbreviate as is seen in the margin.

2. Extract the cube root of 14706125.

SHOWN BY LETTERS.

		14·706·125 (245
$h^3=200^3=$	8 000 000	
$3h^2=3\times200^2=$	120000	6 706 125
$3ht=3\times200\times4=$	24000	
$t^2=\quad 40^2=$	1600	
	145600	5 824 000
$3(h+t)^2u+3(h+t)u^2+u^3=$		882125
$3(h+t)^2=3\times240^2=$	172800	
$3(h+t)u=3\times240\times5=$	3600	
$u^2=\quad 5^2=$	25	
	176425	882125

AS IN PRACTICE.

		14·706·125 (245
$2^3=$	8	
$20^2\times3=$	1200	6706
$20\times4\times3=$	240	
$4^2=$	16	
	1456	5824
		882125
$240^2\times3=$	172800	
$240\times5\times3=$	3600	
$5^2=$	25	
	176425	882125

NOTE.—In practice, abbreviate by omitting ciphers and using periods instead of the whole number each time.

Rule.—I. *Separate the number into periods of three figures each, beginning at units' place.*

II. *Find the greatest number whose cube is contained in the left-hand period; place it at the right and subtract its cube from the period, and annex the next period to the remainder for a dividend.*

III. *Take 3 times the square of the first term of the root regarded as tens for a* TRIAL DIVISOR; *divide the dividend by it, and place the quotient as the second term of the root.*

IV. *Take 3 times the last term of the root multiplied by the preceding part regarded as tens; write the result under the trial divisor, and under this write the square of the last term of the root; their sum will be the* COMPLETE DIVISOR.

V. *Multiply the* COMPLETE DIVISOR *by the last term of the root; subtract the product from the dividend, and to the remainder annex the next period for a new dividend. Take 3 times the square of the root now found, regarded as tens, for a trial divisor, and find the third term of the root as before; and thus continue until all the periods have been used.*

NOTES.—1. If the product of the complete divisor by any term of the root exceeds the dividend, that term must be diminished by a unit.

2. When a dividend will not contain a trial divisor, place a cipher in the root and two ciphers at the right of the trial divisor, and proceed as before.

3. The cube root of a fraction equals the cube root of the numerator

and denominator. When these are not perfect cubes, reduce to a decimal and then extract the root.

4. From the work in the margin we see that the cube root of a decimal of *one*, *two* or *three* places is a decimal of *one* place; of *four*, *five* or *six* places, is a decimal of *two* places, etc.; hence, to extract the cube root of a decimal, we point off in periods of three figures each, commencing at units' place and counting to the right.

$\sqrt[3]{1}=1$
$\sqrt[3]{.001}=.1$
$\sqrt[3]{.000001}=.01$
etc., etc.

EXAMPLES.

Find the cube root of—

3. 24389.	*Ans.* 29.	11. 41063.625.	*Ans.* 34.5.
4. 300763.	*Ans.* 67.	12. 130.323843.	*Ans.* 5.07.
5. 405224.	*Ans.* 74.	13. 95256152263.	*Ans.* 4567.
6. 18191447.	*Ans.* 263.	14. $34\frac{21}{64}$.	*Ans.* $3\frac{1}{4}$.
7. 44361864.	*Ans.* 354.	15. 6.	*Ans.* 1.8171+.
8. 82881856.	*Ans.* 436.	16. 7.	*Ans.* 1.9129+.
9. 66923416.	*Ans.* 406.	17. 9.	*Ans.* 2.08008+.
10. 1879080904.	*Ans.* 1234.	18. 10.	*Ans.* 2.15443+.

NEW METHOD OF CUBE ROOT.

239. The method here presented is supposed to be simpler and more convenient in its application than those usually given.

240. In the operation, indicate the *trial divisor* by T. D., and the *complete divisor* by C. D., and use dots, thus, . . , to indicate the local value of the figures.

1. Extract the cube root of 14706125

SOLUTION. We find the number of figures in the root, and the first term of the root, as in the preceding method.

We write 2, the first term of the root, at the left at the head of Col. 1st; three times its square with two dots annexed at the head of Col. 2d; its cube under the first period; then subtract and annex the next period for a dividend;

OPERATION.

1ST COL.	2D COL.	14·706·125 (245
2	12 . . t. d.	8
4	256	6706
64	1456 C. D.	
8	16	5824
725	1728 . . t. d.	882125
	3625	
	176425 C. D.	882125

and divide by the number in Col. 2d as a *trial divisor*, for the second term of the root.

We then take 2 times 2, the first term, and write the product, 4, in Col. 1st, under the 2, and add; then annex the second term of the root to the 6 in Col. 1st, making 64, and multiply 64 by 4 for a *correction*, which we write under the trial divisor; and adding the *correction* to the *trial divisor*, we have the *complete divisor*, 1456. We then multiply the complete divisor by 4, subtract the product from the dividend, and annex the next period for a new dividend.

We then square 4, the second figure of the root, write the *square* under the *complete divisor*, and add the *correction*, the *complete divisor* and the *square* for the next *trial divisor*, which we find to be 1728. Dividing by the trial divisor, we find the next term of the root to be 5.

We then take 2 times 4, the second term, write the product 8 under the 64, add it to 64, and annex the third term of the root to the sum, 72, making 725, and then multiply 725 by 5, giving us 3625 for the next *correction*. We then find the *complete divisor* by adding the *correction* to the *trial divisor*; multiply the true divisor by 5, and subtract and have no remainder.

SHOWN BY ALGEBRA.

1ST COL.		2D COL.	
2	h	12	$3h^2$
4	$2h$	256	$3ht+t^2$
64	$3h+t$	1456	$3h^2+3ht+t^2$
8	$2t$	16	t^2
72	$3h+3t$	1728	$3h^2+6ht+3t^2$
725	$3h+3t+u$	3625	$3hu+3tu+u^2$
		176425	$3h^2+6ht+3t^2+3hu+3tu+u^2$

Rule.—I. *Separate the number into periods of three figures each; find the greatest number whose cube is contained in the first period, and write it in the root.*

II. *Write the first term of the root at the head of 1st Col., 3 times its square, with two dots annexed, for a* TRIAL DIVISOR, *at the head of 2d Col., and its cube under the first period; subtract and annex the next period to the remainder for a dividend; divide the dividend by the trial divisor, and place the quotient as the second term of the root.*

III. *Add twice the first term of the root to the number in the first column; annex the second term of the root, multiply the result by the second term, and write the product under the trial divisor* *for a* CORRECTION; *add the* CORRECTION *to the* TRIAL DIVISOR, *and the result will be the* COMPLETE DIVISOR; *multiply the* COMPLETE DIVISOR *by the last term of the root, subtract the product from the dividend, and annex the next period to the result for a new dividend.*

IV. *Square the last term of the root, and take the sum of this* SQUARE, *the last* COMPLETE DIVISOR *and the last* CORRECTION, *and annex two dots, for a new* TRIAL DIVISOR; *divide the dividend by it and obtain the next term of the root.*

V. *Add twice the second term of the root to the last number in the first column; annex the last term to the sum, multiply the result by the last term, and write the product under the last trial divisor for a* CORRECTION; *add the* CORRECTION *to the* TRIAL DIVISOR, *and the result will be the* COMPLETE DIVISOR; *use this as before, and thus continue until all the periods have been used.*

NOTE.—A part of this method can be easily remembered by means of the following formulas, which show the formation of the trial and *complete divisors:*

1. Trial Divisor + Correction = Complete Divisor.
2. Correction + Complete Divisor + Square = Trial Divisor.

2. Extract the cube root of 41673648563.

```
1ST COL.   2D COL.              41·673·648·563 ( 3467
  3        27.. t.d.            27
  6         376                 14673
 ----      ----
 94        3076 C.D.
  8          16                 12304
 ----      ----                 -----
 1026      3468.. t.d.           2369648
   12        6156
 ----      ------
 10387     352956 C.D.
               36                2117736
           ------                -------
           359148.. t.d.          251912563
             72709
           --------
           35987509 C.D.          251912563
                                  ---------
```

NOTE.—Let the pupils apply this to the problems under the **previous method.**

RADICALS.

241. A **Radical** is an indicated root of a quantity; as $\sqrt{a}$, $a^{\frac{1}{3}}$, $\sqrt{24}$, $3\sqrt{(a-x)}$, etc.

242. The **Coefficient** of a radical is the quantity which indicates the number of times it is taken. Thus, in $3\sqrt{(a^3x)}$ and $m(a^2-x^2)^{\frac{1}{2}}$, 3 and m are the coefficients.

243. The **Degree** of a radical is indicated by the index of the radical sign, or by the denominator of the fractional exponent. Thus,

$\sqrt{a}$; $b^{\frac{1}{2}}$; $(2a^2x^3)^{\frac{1}{2}}$ are radicals of the *second* degree.

$\sqrt[3]{a^2}$; $b^{\frac{1}{3}}$; $(a^2x^3)^{\frac{1}{3}}$ are radicals of the *third* degree.

$\sqrt[n]{a^3}$; $(2b)^{\frac{1}{n}}$; $(a-b)^{\frac{1}{n}}$ are radicals of the nth degree.

244. **Similar Radicals** are those which have the same quantity under the same radical sign; as $\sqrt{(a^2c)}$ and $4\sqrt{(a^2c)}$; also $a\sqrt[n]{c}$ and $bc^{\frac{1}{n}}$.

NOTE.—A quantity whose root cannot be expressed without a radical sign or fractional exponent is called an *irrational quantity* or a *surd;* when the root expressed can be obtained exactly, it is called a *rational quantity.*

REDUCTION OF RADICALS.

245. **Reduction of Radicals** is the process of changing their forms without altering their values.

246. The reduction of radicals depends upon the following principles:

PRIN. 1. *Any root of the product of two or more quantities equals the product of the same roots of those quantities.*

Thus, $\sqrt{(ab)} = \sqrt{a} \times \sqrt{b}$. For, $\sqrt{(ab)}$ equals $a^{\frac{1}{2}}b^{\frac{1}{2}}$ (Art. 231), and $a^{\frac{1}{2}}b^{\frac{1}{2}}$ equals $\sqrt{a} \times \sqrt{b}$. In the same way we may prove that $\sqrt[n]{(ab)} = \sqrt[n]{a} \times \sqrt[n]{b}$.

PRIN. 2. *Fractional exponents may be added, subtracted, multiplied, and divided the same as integral exponents.*

This may be readily inferred from the relation of integers and fractions; it will be rigidly demonstrated in the article on the *Theory of Exponents.*

CASE I.

247. To reduce a radical to its simplest form.

248. A radical is in its *simplest form* when the radical part is integral, and contains no factor of which the given root can be extracted.

1. Reduce $\sqrt{(48a^3x)}$ to its simplest form.

SOLUTION. Resolving the quantity under the radical sign into two factors, one of which, $16a^2$, is a *perfect square*, and the other, $3ax$, not a perfect square, we find $\sqrt{(48a^3x)}$ equals $\sqrt{(16a^2 \times 3ax)}$, which (Prin., Art. 246) equals $\sqrt{(16a^2)} \times \sqrt{(3ax)}$; which, extracting the square root of $16a^2$, equals $4a\sqrt{(3ax)}$.

OPERATION.

$$\sqrt{(48a^3x)} = \sqrt{(16a^2 \times 3ax)}$$
$$= \sqrt{(16a^2)}\sqrt{(3ax)}$$
$$= 4a\sqrt{(3ax)}$$

Rule.—I. *Separate the quantity under the radical into two factors, one of which shall contain all the perfect powers of the same degree as the radical.*

II. *Extract the root of the rational part, and prefix it as a coefficient to the other part placed under the radical sign.*

EXAMPLES.

Reduce the following radicals to their simplest form:

2. $\sqrt{(4a^2x)}$. *Ans.* $2a\sqrt{x}$.
3. $\sqrt{(9a^3c)}$. *Ans.* $3a\sqrt{(ac)}$.
4. $\sqrt{(8a^2x^3)}$. *Ans.* $2ax\sqrt{(2x)}$.
5. $3\sqrt{(12a^5)}$. *Ans.* $6a^2\sqrt{(3a)}$.
6. $a\sqrt{(48a^3b)}$. *Ans.* $4a^2\sqrt{(3ab)}$.
7. $\sqrt{(32a^6b^4c^5)}$. *Ans.* $4a^3b^2c^2\sqrt{(2c)}$.
8. $\sqrt{\{4a^2(a-b)\}}$. *Ans.* $2a(a-b)^{\frac{1}{2}}$.
9. $2\sqrt{\{9(a^3-a^2c)\}}$. *Ans.* $6a\sqrt{(a-c)}$.
10. $a\sqrt[3]{(x^3-x^3z)}$. *Ans.* $ax\sqrt[3]{(1-z)}$.
11. $2(ax^2-bx^3)^{\frac{1}{2}}$. *Ans.* $2x(a-bx)^{\frac{1}{2}}$.
12. $5a\sqrt[3]{(54a^2b^3c^4)}$. *Ans.* $15abc\sqrt[3]{(2a^2c)}$.

13. $(75a^3x^5y)^{\frac{1}{2}}$. *Ans.* $5ax^2(3axy)^{\frac{1}{2}}$.
14. $(24a^4b^5c^2)^{\frac{1}{3}}$. *Ans.* $2ab(3ab^2c^2)^{\frac{1}{3}}$.
15. $\sqrt[4]{\{162a^4(b^5-b^4c)^{\frac{1}{4}}\}}$. *Ans.* $3a\sqrt[4]{\{2b(b-c)^{\frac{1}{4}}\}}$.
16. $(a+b)(a^3-2a^2b+ab^2)^{\frac{1}{2}}$. *Ans.* $(a^2-b^2)\sqrt{a}$.
17. $(m-n)(2am^2+4amn+2an^2)^{\frac{1}{2}}$. *Ans.* $(m^2-n^2)\sqrt{2a}$.
18. $2a\sqrt{(8x^3y+16x^2y^2+8xy^3)}$. *Ans.* $4a(x+y)\sqrt{(2xy)}$.

WHEN THE RADICAL IS FRACTIONAL.

249. A **Fractional Radical** is reduced to its simplest form by changing it to an entire quantity.

1. Reduce $2\sqrt{\frac{2}{3}}$ to its simplest form.

SOLUTION. Multiplying both terms of the radical by 3 to make the denominator a perfect square, we have $2\sqrt{\frac{6}{9}}$; factoring, we have $2\sqrt{\frac{1}{9}\times 6}$, which equals $2\sqrt{\frac{1}{9}} \times \sqrt{6}$ (Art. 246), which, extracting the square root of $\frac{1}{9}$ and multiplying, equals $\frac{2}{3}\sqrt{6}$.

OPERATION

$$2\sqrt{\tfrac{2}{3}} = 2\sqrt{\frac{2\times 3}{3\times 3}} = 2\sqrt{\tfrac{6}{9}}$$
$$= 2\sqrt{\tfrac{1}{9}\times 6} = 2\sqrt{\tfrac{1}{9}} \times \sqrt{6}$$
$$= 2 \times \tfrac{1}{3}\sqrt{6} = \tfrac{2}{3}\sqrt{6}$$

Rule.—I. *Multiply both terms of the fraction by such a quantity as will render its denominator a perfect power of the degree indicated.*

II. *Resolve the quantity under the radical into two factors, one of which is a fraction and a perfect power of the degree indicated, and proceed as before.*

EXAMPLES.

Reduce the following to their simplest forms:

2. $\sqrt{\frac{4}{5}}$. *Ans.* $\frac{2}{5}\sqrt{5}$.
3. $\sqrt{\frac{7}{8}}$. *Ans.* $\frac{1}{4}\sqrt{14}$.
4. $\sqrt{\frac{45}{49}}$. *Ans.* $\frac{3}{7}\sqrt{5}$.
5. $2\sqrt{\frac{4a^3}{5}}$. *Ans.* $\frac{4a}{5}\sqrt{5a}$.
6. $2\sqrt{\frac{5}{8}}$. *Ans.* $\frac{1}{2}\sqrt{10}$.
7. $2\sqrt[3]{\frac{4a^4}{9}}$. *Ans.* $\frac{2a}{3}\sqrt[3]{12a}$.
8. $4b\sqrt[3]{\frac{3a^4}{4b^2}}$. *Ans.* $2a\sqrt[3]{6ab}$.
9. $6z^2\left(\frac{8x^5y^7}{27z^3}\right)^{\frac{1}{4}}$. *Ans.* $2xz(24xy^2z)^{\frac{1}{4}}$.

CASE II.

250. To reduce a rational quantity to the form of a radical.

1. Reduce $2a^2x$ to the form of the cube root.

SOLUTION. Since any quantity is equal to the cube root of its cube, $2a^2x$ is equal to the cube root of $(2a^2x)^3$, which equals the cube root of $8a^6x^3$.

OPERATION.

$$2a^2x = \sqrt[3]{(2a^2x)^3}.$$
$$= \sqrt[3]{(8a^6x^3)}$$

Rule.—*Raise the quantity to the power indicated by the given root, and place the result under the corresponding radical sign.*

NOTE.—The coefficient of a radical or any factor of a coefficient may be placed under the radical sign by raising it to the power indicated by the radical, and multiplying the quantity under the sign by the result.

EXAMPLES.

2. Reduce $2a^2c$ to the form of the square root. *Ans.* $\sqrt{(4a^4c^2)}$.
3. Reduce $3a^2b^3$ to the form of the cube root. *Ans.* $\sqrt[3]{(27a^6b^9)}$.
4. Reduce $a+2b$ to the form of the square root. *Ans.* $\sqrt{(a^2+4ab+4b^2)}$.
5. Reduce $2a^{\frac{1}{2}}b^{\frac{1}{3}}c^{\frac{1}{4}}$ to the form of the fourth root. *Ans.* $\sqrt[4]{(16a^2b^{\frac{4}{3}}c)}$.
6. Express $2a\sqrt{b}$ without a coefficient. *Ans.* $\sqrt{(4a^2b)}$.
7. Express $3a^2\sqrt[3]{(2ac^2)}$ without a coefficient. *Ans.* $\sqrt[3]{(54a^7c^2)}$.
8. Express $(x+y)\sqrt{z}$ without a coefficient. *Ans.* $\sqrt{z(x+y)^2}$.
9. Express $\frac{2}{3}\sqrt{(3ax)}$ without a coefficient. *Ans.* $\sqrt{\frac{4ax}{3}}$.
10. Express $6a\sqrt{(cx)}$ with a coefficient of 2. *Ans.* $2\sqrt{(9a^2cx)}$.
11. Express $\frac{2a}{b}\sqrt{(2ab^3)}$ without a literal coefficient. *Ans.* $2\sqrt{(2a^3b)}$.
12. Express $\frac{3c}{4}\sqrt[3]{\frac{4ax}{9c^2}}$ without a coefficient. *Ans.* $\sqrt[3]{\frac{3acx}{16}}$.

CASE III.

251. To reduce radicals of different degrees to a common radical index.

1. Reduce $a^{\frac{1}{2}}$ and $b^{\frac{1}{3}}$ to a common index.

SOLUTION. $\frac{1}{2}$ equals $\frac{3}{6}$; hence $a^{\frac{1}{2}}=a^{\frac{3}{6}}$, which equals $(a^3)^{\frac{1}{6}}$, which equals $\sqrt[6]{a^3}$. $\frac{1}{3}$ equals $\frac{2}{6}$; hence $b^{\frac{1}{3}}=b^{\frac{2}{6}}$, which equals $(b^2)^{\frac{1}{6}}$, which equals $\sqrt[6]{b^2}$.

OPERATION.

$$a^{\frac{1}{2}}=a^{\frac{3}{6}}=(a^3)^{\frac{1}{6}}, \text{ or } \sqrt[6]{a^3}$$
$$b^{\frac{1}{3}}=b^{\frac{2}{6}}=(b^2)^{\frac{1}{6}}, \text{ or } \sqrt[6]{b^2}$$

Rule.—I. *Reduce the exponents to a common denominator.*

II. *Raise each quantity to the power indicated by the numerator of its reduced exponent, and indicate the root denoted by the common denominator.*

EXAMPLES.

2. Reduce $a^{\frac{1}{2}}$ and $c^{\frac{2}{3}}$ to a common index. *Ans.* $\sqrt[6]{a^3}$; $\sqrt[6]{c^4}$.
3. Reduce $4^{\frac{1}{2}}$ and $6^{\frac{3}{4}}$ to a common index. *Ans.* $\sqrt[4]{16}$; $\sqrt[4]{216}$.
4. Reduce $a^{\frac{1}{n}}$ and $b^{\frac{1}{m}}$ to a common index. *Ans.* $\sqrt[mn]{a^m}$; $\sqrt[mn]{b^n}$.
5. Reduce $\sqrt{a}$, $\sqrt[3]{b^2}$ and $\sqrt[4]{c^3}$ to a common index. *Ans.* $\sqrt[12]{a^6}$; $\sqrt[12]{b^8}$; $\sqrt[12]{c^9}$.
6. Reduce $\sqrt{3}$, $\sqrt[3]{4}$ and $\sqrt[4]{5}$ to a common index. *Ans.* $\sqrt[12]{729}$; $\sqrt[12]{256}$; $\sqrt[12]{125}$.
7. Reduce $2\sqrt{6}$, $3\sqrt[3]{9}$ and $5\sqrt[6]{8}$ to a common index. *Ans.* $2\sqrt[6]{216}$; $3\sqrt[6]{81}$; $5\sqrt[6]{8}$.
8. Reduce $\sqrt{2a}$, $\sqrt[4]{3a^3}$ and $\sqrt[6]{4a^3}$ to a common index. *Ans.* $\sqrt[12]{(64a^6)}$; $\sqrt[12]{(27a^9)}$; $\sqrt[12]{(16a^6)}$.

ADDITION OF RADICALS.

252. Addition of Radicals is the process of finding the sum of two or more radical quantities.

1. What is the sum of $2\sqrt{8}$ and $3\sqrt{18}$?

SOLUTION. Factoring and reducing, we have $2\sqrt{8}$ equal to $4\sqrt{2}$, and $3\sqrt{18}$ equal to $9\sqrt{2}$; 9 times $\sqrt{2}$ plus 4 times $\sqrt{2}$ equals 13 times $\sqrt{2}$.

OPERATION.

$$2\sqrt{8} = 2\sqrt{(4\times 2)} = 4\sqrt{2}$$
$$3\sqrt{18} = 3\sqrt{(9\times 2)} = \underline{9\sqrt{2}}$$
$$13\sqrt{2}$$

2. What is the sum of $\sqrt{(2a^3x)}$ and $\sqrt{(8a^3x)}$.

SOLUTION. Reducing the radicals to their simplest form, we have $\sqrt{(2a^3x)}=a\sqrt{(2ax)}$; and $\sqrt{(8a^3x)}=2a\sqrt{(2ax)}$; $2a\sqrt{(2ax)}$ plus $a\sqrt{(2ax)}$ equals $3a\sqrt{(2ax)}$.

OPERATION.

$$\sqrt{(2a^3x)}=\sqrt{(a^2\times 2ax)}=a\sqrt{(2ax)}$$
$$\sqrt{(8a^3x)}=\sqrt{(4a^2\times 2ax)}=\underline{2a\sqrt{(2ax)}}$$
$$3a\sqrt{(2ax)}$$

Rule.—I. *Reduce the radicals to their simplest form.*

II. *If the radicals are then similar, add their coefficients and annex the common radical.*

III. *If the radicals are not similar, indicate their sum by the proper signs.*

EXAMPLES.

Find the sum—

3. Of $\sqrt{12}$ and $\sqrt{27}$. *Ans.* $5\sqrt{3}$.
4. Of $\sqrt{20}$ and $\sqrt{45}$. *Ans.* $5\sqrt{5}$.
5. Of $b\sqrt{(a^2b)}$ and $a\sqrt{b^3}$. *Ans.* $2ab\sqrt{b}$.
6. Of $\sqrt{(a^2b)}$ and $a\sqrt{b^3}$. *Ans.* $(a+ab)\sqrt{b}$.
7. Of $3\sqrt{(3a^3x)}$ and $a\sqrt{(27ax)}$. *Ans.* $6a\sqrt{(3ax)}$.
8. Of $a\sqrt{(18ax^2)}$ and $x\sqrt{(32a^3)}$. *Ans.* $7ax\sqrt{(2a)}$.
9. Of $a\sqrt{(ac^3)}$ and $c\sqrt{(a^3c)}$. *Ans.* $2ac\sqrt{(ac)}$.
10. Of $2\sqrt[3]{(16a)}$ and $3\sqrt[3]{(54a)}$. *Ans.* $13\sqrt[3]{(2a)}$.
11. Of $\sqrt{50}$, $\sqrt{72}$ and $\sqrt{128}$. *Ans.* $19\sqrt{2}$.
12. Of $\sqrt{(28a^2c^3)}$ and $c\sqrt{(112a^2c)}$. *Ans.* $6ac\sqrt{(7c)}$.
13. Of $\sqrt{2}$ and $2\sqrt{\frac{1}{2}}$. *Ans.* $2\sqrt{2}$.
14. Of $2\sqrt{3}$ and $3\sqrt{\frac{1}{3}}$. *Ans.* $3\sqrt{3}$.
15. Of $4\sqrt[3]{\frac{1}{4}}$ and $6\sqrt[3]{\frac{1}{32}}$. *Ans.* $\frac{7}{2}\sqrt[3]{2}$.
16. Of $\sqrt{(2a^2x)}$ and $\sqrt{(2b^2x)}$. *Ans.* $(a+b)\sqrt{2x}$.
17. Of $\sqrt{(a^2m)}$ and $\sqrt{(a^2n)}$. *Ans.* $a(\sqrt{m}+\sqrt{n})$.
18. Of $\sqrt{(a^4c)}$, $2\sqrt{(a^2b^2c)}$ and $\sqrt{(b^4c)}$. *Ans.* $(a+b)^2\sqrt{c}$.
19. Find the sum of $2x\sqrt{(50a^3c)}$, $3\sqrt[3]{(24a^4x^3)}$, $\frac{1}{3}a\sqrt{(72acx^2)}$ and $2x\sqrt[3]{(81x^4)}$. *Ans.* $12ax(\sqrt{2ac}+\sqrt[3]{3a})$.

SUBTRACTION OF RADICALS.

253. Subtraction of Radicals is the process of finding the difference between two radicals.

1. Subtract $3\sqrt{8}$ from $2\sqrt{32}$.

SOLUTION. Reducing the radicals to their simplest form, we have $2\sqrt{32}=8\sqrt{2}$, and $3\sqrt{8}=6\sqrt{2}$. $6\sqrt{2}$ subtracted from $8\sqrt{2}$ leaves $2\sqrt{2}$.

OPERATION.

$$2\sqrt{32}=2\sqrt{(16\times 2)}=8\sqrt{2}$$
$$3\sqrt{8}\ =3\sqrt{(4\ \times 2)}=\underline{6\sqrt{2}}$$
$$2\sqrt{2}$$

Rule.—I. *Reduce the radicals to their simplest form.*

II. *If the radicals are then similar, subtract the coefficient of the subtrahend from the coefficient of the minuend, and annex the common radical.*

III. *If the radicals are not similar, indicate their difference by the proper sign.*

EXAMPLES.

2. From $\sqrt{(20a)}$ take $\sqrt{(5a)}$. *Ans.* $\sqrt{(5a)}$.
3. From $\sqrt{(49ax^3)}$ take $\sqrt{(25ax^3)}$. *Ans.* $2x\sqrt{(ax)}$.
4. From $3\sqrt{(12a^3)}$ take $a\sqrt{(27a)}$. *Ans.* $3a\sqrt{(3a)}$.
5. From $\sqrt[3]{(125a^2)}$ take $\sqrt[3]{(8a^2)}$. *Ans.* $3\sqrt[3]{a^2}$.
6. From $2\sqrt{(a^2c)}$ take $a\sqrt{c^3}$. *Ans.* $(2a-ac)\sqrt{c}$.
7. From $\sqrt{(12a)}$ take $2\sqrt{\frac{3a}{4}}$. *Ans.* $\sqrt{3a}$.
8. From $\sqrt[3]{(250a^4x)}$ take $\sqrt[3]{(54a^4x)}$. *Ans.* $2a\sqrt[3]{(2ax)}$.
9. From $3\sqrt{\frac{3}{4}}$ take $2\sqrt{\frac{1}{3}}$. *Ans.* $\frac{5}{6}\sqrt{3}$.
10. From $4\sqrt[4]{32}$ take $4\sqrt[4]{\frac{1}{8}}$. *Ans.* $6\sqrt[4]{2}$.
11. From $\sqrt{(a^3+a^2x)}$ take $\sqrt{(9ab^2+9b^2x)}$. *Ans.* $(a-3b)\sqrt{(a+x)}$.
12. From $\sqrt{(2a^3+4a^2b+2ab^2)}$ take $\sqrt{(2a^3-4a^2b+2ab^2)}$. *Ans.* $2b\sqrt{(2a)}$.
13. Find the value of $7b\sqrt{(a^3x)}-a\sqrt{(9ab^2x)}+5\sqrt{(a^3c^2x)}-3c\sqrt{(9a^3x)}$. *Ans.* $4a(b-c)\sqrt{ax}$.

MULTIPLICATION OF RADICALS.

254. Multiplication of Radicals is the process of finding the product of two or more radicals.

255. Principle.—*The product of the same root of two quantities is equal to the same root of their product.*

For (Art. 246, Prin.), $\sqrt{ab}=\sqrt{a}\times\sqrt{b}$; hence, transposing, $\sqrt{a}\times\sqrt{b}=\sqrt{(ab)}$. Similarly, we may prove that $\sqrt[n]{a}\times\sqrt[n]{b}=\sqrt[n]{(ab)}$.

CASE I.

256. To multiply radicals of the same degree.

1. Multiply $2a\sqrt{(ab)}$ by $3\sqrt{(ac)}$.

OPERATION.

$$2a\sqrt{(ab)}\times3\sqrt{(ac)}=6a\sqrt{(a^2bc)}=6a^2\sqrt{(bc)}$$

Solution. We multiply the coefficients together, and the radical parts together. $2a$ multiplied by 3 equals $6a$; $\sqrt{(ab)}$ multiplied by $\sqrt{(ac)}$ equals $\sqrt{(a^2bc)}$ (Art. 255); hence, the product is $6a\sqrt{(a^2bc)}$, which, reduced, equals $6a^2\sqrt{(bc)}$.

Rule.—*Multiply the coefficients together for the coefficient of the product, and the quantities under the radical sign for the radical part of the product.*

Note.—We may reduce to the form of fractional exponents and add the exponents of the similar letters as in simple multiplication.

EXAMPLES.

2. Multiply $3\sqrt{(ac)}$ by $4\sqrt{(acx)}$. *Ans.* $12ac\sqrt{x}$.
3. Multiply $2\sqrt{6}$ by $3\sqrt{8}$. *Ans.* $24\sqrt{3}$.
4. Multiply $2\sqrt{x}$ by $3\sqrt{(ax)}$. *Ans.* $6x\sqrt{a}$.
5. Multiply $3\sqrt{(2a)}$ by $2\sqrt{(3c)}$. *Ans.* $6\sqrt{(6ac)}$.
6. Multiply $2\sqrt{(3a)}$ by $\sqrt{(6ax)}$. *Ans.* $6a\sqrt{(2x)}$.
7. Multiply $2\sqrt{27}$ by $3\sqrt{3}$. *Ans.* 54.
8. Multiply $5\sqrt[3]{(4a)}$ by $3\sqrt[3]{(2a)}$. *Ans.* $30\sqrt[3]{a^2}$.
9. Multiply $2\sqrt{\frac{1}{2}}$ by $2\sqrt{\frac{5}{8}}$. *Ans.* $\sqrt{5}$.

10. Multiply $3\sqrt{\frac{a}{3}}$ by $2\sqrt{\frac{a}{6}}$. *Ans.* $a\sqrt{2}$.
11. Multiply $3\sqrt[n]{(a^2x)}$ by $\sqrt[n]{(ax^3)}$. *Ans.* $3\sqrt[n]{(a^3x^4)}$.
12. Multiply $\sqrt[n]{(a^{n-1}c^{n+1})}$ by $\sqrt[n]{(ac^n)}$. *Ans.* $ac^2\sqrt[n]{c}$.
13. Multiply $a\sqrt[n]{(b^{n+1}c^{n-1})}$ by $\sqrt[n]{(a^{n+1}c^3)}$. *Ans.* $a^2bc\sqrt[n]{(abc^2)}$.
14. Multiply $\sqrt{a}+\sqrt{b}$ by $\sqrt{a}-\sqrt{b}$. *Ans.* $a-b$.
15. Multiply $\sqrt{(a+b)}$ by $\sqrt{(a-b)}$. *Ans.* $\sqrt{(a^2-b^2)}$.
16. Multiply $(m+n)^{\frac{2}{3}}$ by $(m-n)^{\frac{2}{3}}$. *Ans.* $(m^2-n^2)^{\frac{2}{3}}$.
17. Multiply $(a+c)^{\frac{1}{n}}$ by $(a-c)^{\frac{1}{n}}$. *Ans.* $(a^2-c^2)^{\frac{1}{n}}$.
18. Multiply $(a+b)^{\frac{n}{2}}$ by $(a+b)^{\frac{n}{2}}$. *Ans.* $(a+b)^n$.

CASE II.

257. To multiply radicals of different degrees.

1. Multiply $\sqrt{a}$ by $\sqrt[3]{b}$.

OPERATION.

$$\sqrt{a}=\sqrt[6]{a^3}$$
$$\sqrt[3]{b}=\sqrt[6]{b^2}$$
$$\sqrt[6]{(a^3b^2)}$$

Solution. We reduce the radicals to a common index, and then multiply. $\sqrt{a}$ equals $\sqrt[6]{a^3}$; $\sqrt[3]{b}$ equals $\sqrt[6]{b^2}$; $\sqrt[6]{a^3}$ multiplied by $\sqrt[6]{b^2}$ equals $\sqrt[6]{(a^3b^2)}$. (Case I.)

Rule.—I. *Reduce the radicals to a common index, and multiply as in Case I.;* or,

II. *Reduce the radicals to the form of fractional exponents, and multiply as in simple multiplication.*

EXAMPLES.

2. Multiply $a^{\frac{1}{3}}$ by $c^{\frac{1}{2}}$. *Ans.* $a^{\frac{1}{3}}c^{\frac{1}{2}}$, or $\sqrt[6]{(a^2c^3)}$.
3. Multiply $\sqrt{a}$ by $\sqrt[3]{a}$. *Ans.* $a^{\frac{5}{6}}$, or $\sqrt[6]{a^5}$.
4. Multiply $3a^{\frac{1}{3}}$ by $4(ab)^{\frac{2}{3}}$. *Ans.* $12ab^{\frac{2}{3}}$.
5. Multiply $a\sqrt[n]{b}$ by $b\sqrt[m]{c}$. *Ans.* $ab\sqrt[mn]{(b^mc^n)}$.
6. Multiply $3\sqrt[m]{a}$ by $4\sqrt[n]{a}$. *Ans.* $12a^{\frac{m+n}{mn}}$.
7. Multiply $a\sqrt[n]{c^{2n}}$ by $a\sqrt[m]{(c^{3m}x)}$. *Ans.* $a^2c^5\sqrt[m]{x}$.
8. Multiply $\sqrt{(a+c)}$ by $\sqrt[3]{(a+c)}$. *Ans.* $\sqrt[6]{(a+c)^5}$.
9. Multiply $2\sqrt[n]{(a-c)}$ by $3\sqrt{a}$. *Ans.* $6\sqrt[2n]{\{a^n(a-c)^2\}}$.
10. Multiply $\sqrt{(m+n)}$ by $\sqrt[3]{(m-n)}$. *Ans.* $\sqrt[6]{(m^2-n^2)^2(m+n)}$.

DIVISION OF RADICALS.

258. Division of Radicals is the process of dividing when one or both terms are radicals.

259. Principle.—*The quotient of the same roots of two quantities is equal to the same root of their quotient.*

For, by Art. 231, $\sqrt[n]{\frac{a}{b}}=\frac{\sqrt[n]{a}}{\sqrt[n]{b}}$;

hence, transposing, $\frac{\sqrt[n]{a}}{\sqrt[n]{b}}=\sqrt[n]{\frac{a}{b}}$.

CASE I.

260. To divide radicals of the same degree.

1. Divide $6\sqrt{(24ax)}$ by $2\sqrt{3a}$.

Solution. We first divide the coefficients, and then the radical parts. 6 divided by 2 equals 3; $\sqrt{(24ax)}$ divided by $\sqrt{3a}$ equals $\sqrt{8x}$, according to the principle above; hence the entire quotient is $3\sqrt{8x}$, which, reduced, equals $6\sqrt{2x}$.

OPERATION.

$$\frac{6\sqrt{(24ax)}}{2\sqrt{3a}}=3\sqrt{8x}=6\sqrt{2x}$$

Rule.—I. *Divide the coefficient of the dividend by the coefficient of the divisor, and the radical part of the dividend by the radical part of the divisor.*

II. *Annex the latter quotient to the former, and reduce the result to its simplest form.*

EXAMPLES.

2. Divide $6\sqrt{(ax)}$ by $2\sqrt{a}$. *Ans.* $3\sqrt{x}$.
3. Divide $4\sqrt{27}$ by $2\sqrt{3}$. *Ans.* 6.
4. Divide $5\sqrt{(27ac)}$ by $3\sqrt{(3a)}$. *Ans.* $5\sqrt{c}$.
5. Divide $6\sqrt{54a}$ by $3\sqrt{27}$. *Ans.* $2\sqrt{(2a)}$.
6. Divide $3\sqrt{(72ab)}$ by $2\sqrt{(6b)}$. *Ans.* $3\sqrt{(3a)}$.
7. Divide $6\sqrt{(20a)}$ by $2\sqrt{(30a)}$. *Ans.* $\sqrt{6}$.

8. Divide $2\sqrt{a^{\frac{7}{3}}}$ by $\sqrt{2a^{\frac{1}{3}}}$. *Ans.* $a\sqrt{2}$.
9. Divide $15(a^3b^5)^{\frac{1}{2}}$ by $3(ab^2)^{\frac{1}{2}}$. *Ans.* $5ab\sqrt{b}$.
10. Divide $5\sqrt[3]{(16a^2x^4)}$ by $2\sqrt[3]{(2ax)}$. *Ans.* $5x\sqrt[3]{a}$.
11. Divide $\frac{3}{4}\sqrt{\frac{1}{3a}}$ by $\frac{1}{2}\sqrt{\frac{3a}{5}}$. *Ans.* $\frac{1}{2a}\sqrt{5}$.
12. Divide $(1-a^2)^{\frac{1}{2}}$ by $(1-a)^{\frac{1}{2}}$. *Ans.* $(1+a)^{\frac{1}{2}}$.

CASE II.

261. To divide radicals of different degrees.

1. Divide $4\sqrt{(ax)}$ by $2\sqrt[3]{a}$.

Solution. We reduce the radicals to a common index, and then divide $4\sqrt{(ax)}$ equals $4\sqrt[6]{(a^3x^3)}$; $2\sqrt[3]{a}$ equals $2\sqrt[6]{a^2}$; $4\sqrt[6]{(a^3x^3)}$ divided by $2\sqrt[6]{a^2}$ equals $2\sqrt[6]{(ax^3)}$, according to Case I.

OPERATION.

$$\frac{4\sqrt{(ax)}}{2\sqrt[3]{a}}=\frac{4\sqrt[6]{(a^3x^3)}}{2\sqrt[6]{a^2}}=2\sqrt[6]{(ax^3)}$$

Rule.—I. *Reduce the radical parts of the dividend and divisor to a common index, and divide as in Case I.*; or,

II. *Reduce the radicals to the form of fractional exponents, and divide as in simple division.*

EXAMPLES.

2. Divide $3\sqrt[3]{(a^2c)}$ by $\sqrt{a}$. *Ans.* $3\sqrt[6]{(ac^2)}$.
3. Divide $6\sqrt[3]{(a^2x^4)}$ by $2\sqrt{(ax)}$. *Ans.* $3\sqrt[6]{(ax^5)}$.
4. Divide $2\sqrt[2]{(ab)}$ by $2\sqrt[4]{(ab)}$. *Ans.* $\sqrt[4]{(ab)}$.
5. Divide $8\sqrt{(ax)}$ by $4\sqrt{(ax^2)}$. *Ans.* $2\sqrt{x^{-1}}$.
6. Divide $6\sqrt{3}$ by $3\sqrt[3]{3}$. *Ans.* $2\sqrt[6]{3}$.
7. Divide 12 by $\sqrt{3}$. *Ans.* $4\sqrt{3}$.
8. Divide $4\sqrt[3]{(cz)}$ by $6\sqrt{(ac)}$. *Ans.* $\frac{2}{3}\sqrt[6]{\frac{z^2}{a^3c}}$.
9. Divide $a\sqrt[n]{x}$ by $c\sqrt[m]{x}$. *Ans.* $\frac{a}{c}\sqrt[mn]{x^{m-n}}$.
10. Divide $\frac{4}{5}\sqrt{\frac{a}{c}}$ by $\frac{2}{3}\sqrt[3]{\frac{a}{c}}$. *Ans.* $\frac{6}{5}\sqrt[6]{\frac{a}{c}}$.